SCRIPTORES LOGARITHMICI.

SCRIPTORES LOGARITHMICI;

OR,

A COLLECTION

OF

SEVERAL CURIOUS TRACTS

ON THE

NATURE AND CONSTRUCTION

OF

L O G A R I T H M S,

MENTIONED IN

DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW EDITION OF SHERWIN'S MATHEMATICAL TABLES:

TOGETHER WITH

SOME TRACTS ON THE BINOMIAL THEOREM AND OTHER SUBJECTS CONNECTED WITH THE DOCTRINE OF LOGARITHMS.

VOLUME II.

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XXII.

E R R A T A.

In the figure in page 3 the line XZ ought to pass through the point Y.

In page 4, line 6, instead of "NCDH," read "NODH."

In the same page 4, line 9, instead of "*infinitas,*" read "*infinitis.*"

In the same page 4, line 15, instead of "*aequalis,*" read "*aequales.*"

In page 55, lines 9, 13, 14, 15, 16, instead of "*trapezia,*" read "*trapezium.*"

In page 67, line 17, instead of "*to his doctrine,*" read "*according to his doctrine.*"

In page 84, line 17 from the bottom, instead of "*conntryman,*" read "*countryman.*"

In page 85, the last line, instead of "*infinite* infinite," read "*infinitè* infinite."

In page 88, line 10 from the bottom, instead of "*demonstratiou,*" read "*demonstration.*"

In page 100, line 11, instead of "*the ratio of* 22 × 4," read "*the ratio of* 22 × 24."

In page 105, line 8 from the bottom, instead of "*Suppose* b *to be,*" read "*Suppose* k *to be.*"

In page 110, line 6, instead of "$\frac{1\,l}{1 - \frac{1}{2}\,l}$," read "$\frac{a\,l}{1 - \frac{l}{2}}$."

And in the same page 110, in the last line but one and the last line, instead of "*the second* $l - \frac{b\,l^2}{2}$ *&c,*" read "*the series* $b\,l - \frac{b\,l^2}{2}$ *&c.*"

In page 111, note ix, line 3, instead of "$b - \frac{b\,l}{\frac{1}{m} +}$," read "$b - \frac{b\,l}{\frac{1}{m} + \frac{1}{2}\,l}$."

In page 112, line 4, instead of "$\frac{a\,d}{\frac{1}{m} - \frac{1}{\ }\,d}$," read "$\frac{a\,d}{\frac{1}{m} - \frac{1}{2}\,d}$."

And in the same page 112, line 15, instead of "$\frac{b\,d}{\frac{1}{m} + \frac{1}{\ }\,d}$," read "$\frac{b\,d}{\frac{1}{m} + \frac{1}{2}\,d}$."

And in the same page 112, line 10 from the bottom, instead of "$\overset{l}{2}$," read "$\frac{l}{2}$."

And in the same page 112, line 8 from the bottom, dele the mark) of a parenthesis after $\frac{a\,d}{\frac{1}{m} - \frac{d}{2}}$, and substitute a semicolon; in its stead.

In page 154, line 2, instead of "or to $\frac{1}{m} \times \frac{m-1}{2} \times \frac{m-2}{3}$," read "or to $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$."

4 In

E R R A T A.

In the fame page 154, line 3, inftead of "*fevaral*," read "*feveral*."

In the fame page, line 14, inftead of "$1 \times am$," read "$1 \times a^m$."

In page 156, line 11 from the bottom, inftead of "or am" read "or a^m."

In page 157, line 5, inftead of "$\overline{a + b}|m$," read "$\overline{a + b}|^m$."

In the fame page 157, line 7, inftead of "$\overline{1 + 1}|m$," read "$\overline{1 + 1}|^m$."

In page 158, line 5, after the words "to do which" place a comma.

In page 164, line 7 from the bottom, inftead of "$\frac{m-3}{4}$" read "$\frac{n-3}{4}$."

In page 190, line 7th from the bottom, after "G, H, I, K, L, &c" infert the word "*equal*."

And in the fame page 190, line 6th from the bottom, after the word "refpectively" infert the mark of a parenthefis,).

In page 197, line 8th from the bottom, after the word "refpectively," infert the mark of a parenthefis,).

In page 202, article 11, line 1ft, inftead of "$1\ x$" read "$1 + x$."

In page 226, line 8, inftead of "$\overset{1}{\overline{n}}\, y$" read "$\frac{1}{n} y$."

In page 322, line 4th from the bottom, in the numerator of the laft fraction in the faid line, inftead of "48,000" read "48,000 x".

In page 341, line 6, inftead of "$1\ \frac{120y}{720}$," read "$- \frac{120y}{720}$."

In page 363, line 10th from the bottom, inftead of "$\overline{1 - x}|^{\frac{1}{n}}$" read "$\overline{1 + x}|^{\frac{1}{n}}$."

In page 388, article 9, line 6, inftead of "other following terms," read "other following even terms."

In page 396, line 3d from the bottom, after $\frac{e^{13}}{s^{12}}$, &c, dele the word *which*.

In page 418, the laft line, inftead of "the fign" read "the fign +."

In page 439, the laft line, inftead of "$\sqrt{^3}\,\overline{a - \sqrt{bb}}$" read "$\sqrt{^3}\,\overline{a - \sqrt{-bb}}$."

In page 449, in the running title, inftead of "THE RESOLUTION OF CUBICK EQUATIONS," read "RESOLVING THE CUBICK EQUATION."

In page 584, line 11, inftead of "$\frac{2\sqrt{q}}{\sqrt{q}}$," read "$\frac{2\sqrt{q}}{\sqrt{3}}$."

And in the fame page 584, line 20, inftead of, "if $x^3 - qx = r$, or r," read "if $x^3 - qx$, or r."

EXTRACTS

FROM

MR. JAMES GREGORY'S

EXERCITATIONES GEOMETRICÆ,

RELATING TO

LOGARITHMS.

PRINTED AT LONDON IN 1668.

N. MERCATORIS

QUADRATURA HYPERBOLES

GEOMETRICE DEMONSTRATA.

PROP. I.

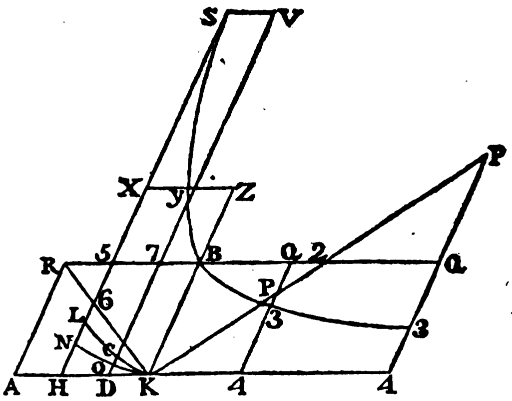
SI fuerint quantitates continuè proportionales A, B, C, D, E, F, &c, numero infinitæ, quarum prima & maxima A; erit $A - B$ ad A ut A ad ſummam omnium; hoc enim paſſim demonſtratur apud geometras.

PROP. II.

Eiſdem poſitis quæ antecedente; dico $A + B$ eſſe ad A ut A eſt ad exceſſum omnium A, C, E, G, &c, in locis imparibus, ſupra omnes B, D, F, &c, in locis paribus: eſt enim dictus exceſſus ſumma ſeriëi infinitæ continuè proportionalium in ratione A ad C, nempe $A - B$, $C - D$, $E - F$, &c, & ideo (ex præcedente) ut $A - C$ ad A vel $A^2 - B^2$ ad A^2, ita $A - B$ ad ſummam dicte ſeriëi, quam vocamus Z; & priores analogiæ terminos applicando ad $A - B$, $A + B$ eſt ad $\frac{A^2}{A - B}$ ut $A - B$ ad Z, & ideo $AZ + BZ = A^2$, & proinde $A + B$ eſt ad A ut A ad Z exceſſum omnium A, C, E, G, &c, ſupra omnes B, D, F, &c. Quod demonſtrare oportuit.

PROP.

PROP. III.



Sit hyperbola SB3, cujus vertex B, aſymptotæ AR, A4, aſymptotæ RA ducatur parallela BK, & altera ad libitum inter rectas BK, RA, utrique parallela YD; dico YD eſſe ſummam infinitæ ſeriëi continuè proportionalium, cujus primus terminus BK = KA & ſecundus KD; eſt enim BK — KD = AD ad BK ut BK ad DY; & ideo ex hujus prima patet propoſitum.

Eiſdem rectis RA, BK, ultra punctum K fiat parallela 34; dico rectam 34 æqualem eſſe exceſſui omnium terminorum imparium ſupra omnes terminos pares infinitæ ſeriëi cujus primus terminus KB, & ſecundus K4: eſt enim KB + K4 = A4 ad BK ut BK ad 34; & ideo ex hujus 2 patet propoſitum.

PROP. IV.

Sit SBKH ſpatium hyperbolicum, contentum à curva hyperbolica SB, aſymptotæ portione HK, & rectis SH, BK, alteri aſymptotæ parallelis, poſito B hyperbolæ vertice: ſit parallelogrammum BKHS, et producatur B5 in R, jungatúrque KR quæ 5H ſecet in 6: deinde continuetur ſeries infinita continuè proportionalium nempe 5H, 6H, LH, NH, & ſic deinceps; ſitque 5BKH parallelogrammum, K6H triangulum KLH trilineum quadraticum, KNH trilineum cubicum, & ita deinceps in infinitum. Dico ſpatium hyperbolicum, BKHS æquale eſſe dicto parallelogrammo, dicto triangulo, unà cum infinitis illis trilineis, quorum omnium ſummam vocamus ω. Si figura BKHS & ω non ſunt æquales, ſit inter illas differentia α; & dividatur recta HK in tot partes æquales à rectis aſymptotæ RA parallelis, ut rectangula (ab illis & portionibus rectæ KH contenta) figuræ BKHS circumſcripta, nempè VH, ZD, differant à rectangulis figuræ BKHS inſcriptis, nempè YH, BD, minore intervallo quàm α; hoc enim fieri poteſt ab indefinita diviſione rectæ KH. Quoniam B eſt hyperbolæ vertex, parallelogrammum BK AR eſt æquilaterum; & proinde recta 6H ad libitum eſt æqualis rectæ HK, cùmque 5H, 6H, LH, NH, &c, ſint continuè proportionales in infinitum, ex hujus 3 erit recta SH æqualis ſummæ omnium, & parallelogrammum SD æquale ſummæ omnium parallelogrammorum 5D, 6D, LD, ND, &c, in infinitum; atque ſumma omnium parallelogrammorum 5D, 6D, LD, ND, &c, in infinitum, major eſt parallelogrammo 5D unà cum portione trianguli 6FDH unà cum portione trilinei quadratici LCDH unà cum portione trilinei cubici NCDH, &c, in infinitum, quoniam prædictæ portiones dictis parallelogrammis inſcribuntur, & ideo parallelogrammum SD majus eſt parallelogrammo 5D unà cum dictis portionibus; eodem modo demonſtratur parallelogrammum YK majus eſſe rectangulo 7K unà cum infinitis numero portionibus FKD, CKD, OKD, &c, & proinde rectilineum SVYZKH majus eſt quam ω. Deindê

indè recta FD est æqualis rectæ DK; atque 7D, FD, CD, OD, &c sunt rectæ continuè proportionales in infinitum, & igitur recta YD est æqualis ipsarum summæ, & parallelogrammum XD æquale parallelogrammis 7H, FDH, CDH, ODH, &c, at summa parallelogrammorum 7H, FDH, CDH, ODH, &c minor est quàm rectangulum 7H unà cum portione trianguli 6FDH unà cum portione trilinei quadratici LCDH unà cum portione trilinei cubici NCDH, &c, quoniam dicta parallelogramma dictis portionibus inscribuntur, & ideo parallelogrammum YH minus est parallelogrammo 7H unà cum dictis portionibus; eodem modo demonstratur parallelogrammum BD minus esse parallelogrammo BD unà cum infinitas numero portionibus FKD, CKD, OKD, & ideo rectilineum XY7BKH minus est quàm ω: quatuor igitur sunt quantitates, quarum maxima & minima sunt rectilinea SVYZKH, XY7BKH, intermediæ autem ω & spatium hyperbolicum SBKH; & ideo differentia intermediarum, nempè α minor est quàm differentia extremarum, quod est absurdum, ponitur enim major; nulla igitur est differentia inter figuram SBKH & ω, & ideo æqualis sunt, quod demonstrandum erat.

Conf. 1. Et proinde si fuerit series infinita quantitatum continuè proportionalium in ratione KB ad KH = 6H, cujus primus terminus est parallelogrammum BH; erit primus terminus + $\frac{1}{2}$ secundi + $\frac{1}{3}$ tertii + $\frac{1}{4}$ quarti + $\frac{1}{5}$ quinti + &c in infinitum = spatio hyperbolico SBKH, hoc enim evidenter sequitur ex quadratura trilineorum.

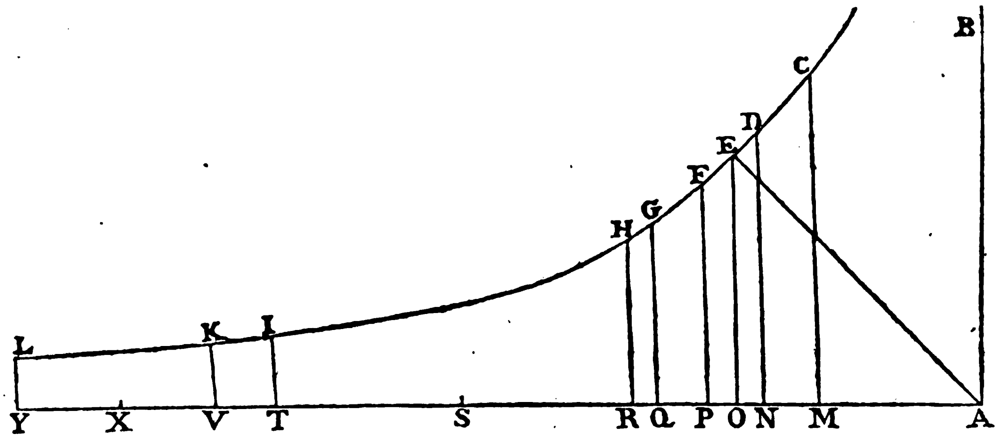
Conf. 2. Si autem ultra K sumatur spatium B34K, positâ 34 parallela rectæ BK, & sit series infinita quantitatum in continua ratione BK ad K4, cujus primus terminus est parallelogrammum KQ; erit primus terminus − $\frac{1}{2}$ secundi + $\frac{1}{3}$ tertii − $\frac{1}{4}$ quarti + &c in infinitum = spatio hyperbolico B34K: poterit hoc consectarium eodem ferè modo demonstrari geometricè ex secunda conclusione hujus tertiæ, quo antecedens ex ejusdem conclusione priore; utrumque autem ex methodo indivisibilium Cavallerianâ nullo negotio demonstratur; sed quoniam magni sunt momenti, placuit methodum rigorosam adhibere.

Conf. 3. Quod si K4 = KH, & fuerit series infinita quantitatum in continua ratione BK ad K4 = KH, cujus primus terminus est BH = B4; erit excessus spatii SBKH supra spatium B34K = toti secundo termino + $\frac{1}{2}$ quarti + $\frac{1}{3}$ sexti + $\frac{1}{4}$ octavi + &c in infinitum: nam ex primo consectario, spatium SBKH = primo termino + $\frac{1}{2}$ secundi + $\frac{1}{3}$ tertii + $\frac{1}{4}$ quarti + &c in infinitum: & ex secundo consectario spatium B34K = primo termino − $\frac{1}{2}$ secundi + $\frac{1}{3}$ tertii − $\frac{1}{4}$ quarti + &c; at manifestum est horum differentiam = toti secundo + $\frac{1}{2}$ quarti + $\frac{1}{3}$ sexti + $\frac{1}{4}$ octavi + &c; & ideo patet propositum.

Conf. 4. Eisdem positis quæ in antecedente consectario, manifestum est spatium hyperbolicum SH43 = duplo primi termini + $\frac{2}{3}$ tertii + $\frac{2}{5}$ quinti + $\frac{2}{7}$ septimi + $\frac{2}{9}$ noni + &c in infinitum.

PROP. V.

Sit hyperbola CEL, cujus vertex E, & asymptotæ AB, AY; in qua sumantur duo spatia hyperbolica ad libitum HITR, KLYV, contenta à curvâ hyperbolicâ, unâ asymptotâ & rectis alteri asymptotæ parallelis: dividantur rectæ RT, VY, bifariam in S & X punctis. Dico spatium HITR esse ad spatium KLYV, ut $\frac{RS \times AO}{AS} + \frac{RS^3 \times AO}{3AS^3} + \frac{RS^5 \times AO}{5AS^5} + \frac{RS^7 \times AO}{7AS^7}$ + &c in infinitum ad $\frac{VX \times AO}{AX}$ +



$\frac{VX^3 \times AO}{3AX^3} + \frac{VX^5 \times AO}{5AX^5} + \frac{VX^7 \times AO}{7AX^7} +$ &c in infinitum. Asymptotæ AB fiat recta parallela EO; fitque ut AS ad AR ita AO = EO ad AM; & ut AS ad AT ita AO ad AQ; fimiliter, fit ut AX ad AV ita AO ad AN, & AO ad AP ut AX ad AY; manifeftum eft MO = OQ & NO = OP. Ductis rectis MC, ND, PF, QG, evidens eft (ex hyperbolæ proprietatibus) fpatium CGQM effe æquale fpatio HITR & fpatium DFPN fpatio KLYV; at patet (ex confectario 4 antecedentis) fpatium CGQM effe ad fpatium DFPN, ut $MO + \frac{MO^3}{3AO^2} + \frac{MO^5}{5AO^4} + \frac{MO^7}{7AO^6} +$ &c ad $NO + \frac{NO^3}{3AO^2} + \frac{NO^5}{5AO^4} + \frac{NO^7}{7AO^6} +$ &c, quæ analogia eadem eft cum propofita, quod demonftrandum erat.

Conf. Hinc manifeftum eft (ob analogiam inter fpatia hyperbolica & logarithmos) differentiam inter logarithmos numerorum A, B, effe ad differentiam inter logarithmos numerorum D, E, (pofito C medio arithmetico inter A, B, & F medio arithmetico inter D, E, item N differentiâ inter C & A, & O differentiâ inter F & D) ut $\frac{N}{C} + \frac{N^3}{3C^3} + \frac{N^5}{5C^5} + \frac{N^7}{7C^7} +$ &c ad $\frac{O}{F} + \frac{O^3}{3F^3} + \frac{O^5}{5F^5} + \frac{O^7}{7F^7} +$ &c, & ideo fi ponatur A = D = 1, hinc patet methodus inveniendi logarithmum quemcunque ex uno dato, abfque ulla hyperbolæ confideratione, fed calculo plerumque nimis laboriofo. Quod fi ponatur A = 999, B = 1001, cum datis logarithmis, item D, E, numeri dati majores, unitate vel parvo aliquo intervallo differentes, nullo negotio invenietur differentia logarithmorum numeris D, E, debitorum; hâc methodo non difficulter computatur integra logarithmorum tabula ad quotvis notas.

N A
C B
O D
F E

Facilè quoque deducitur (ex 3 confect. antecedentis prop.) differentiam fecundam logarithmorum numerorum in ratione arithmetica G, H, I, effe ad differentiam fecundam logarithmorum numerorum in ratione arithmetica K, L, M, (pofito P differentia inter G, H, & Q differentia inter K, L) ut $\frac{P^2}{H^2} + \frac{P^4}{2H^4} + \frac{P^6}{3H^6} +$ &c ad $\frac{Q^2}{L^2} + \frac{Q^4}{2L^4} + \frac{Q^6}{3L^6} +$ &c, non diffimili ferè methodo determinantur differentiæ logarithmorum tertiæ, quartæ, quintæ, & fic deinceps; fed non eft operæ pretium illas profequi: primæ enim differentiæ compofitioni logarithmorum abundè fufficiunt.

G
P H
I
K
Q L
M

ANALOGIA

A N A L O G I A

INTER

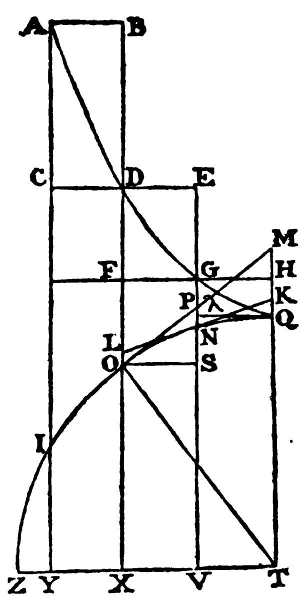
LINEAM MERIDIANAM PLANISPHERII NAUTICI,

E T

TANGENTES ARTIFICIALES,

GEOMETRICE DEMONSTRATA, &c.

PROP. I. THEOREMA.



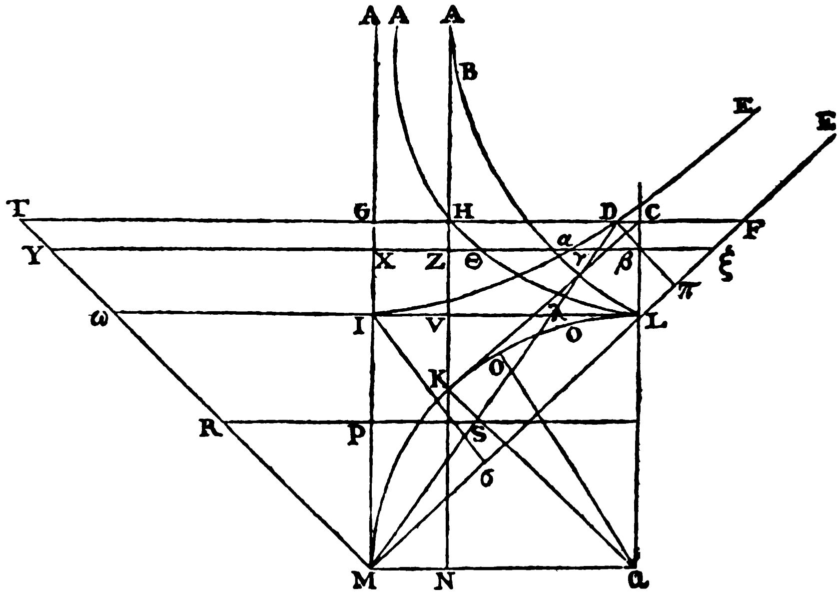
SIT circuli quadrans TQZ, cujus pars sit arcus QI: super arcu QI imaginetur portio superficiei cylindrici recti talis naturæ, ut (sumpto in arcu QI quolibet puncto O) perpendicularis ad planum TZQ ex puncto O ad summitatem portionis superficiei cylindricæ excitata semper fiat æqualis secanti arcus OQ. Deinde sit mixtilineum AIYTQ talis naturæ, ut (ducta in eo recta XD radio QT parallela & arcum quadrantis secante in puncto ad libitum O) recta DX, secans arcus OQ, & radius TQ sint continuè proportionales. Dico mixtilineum AYTQ esse æquale dictæ portioni superficiei cylindricæ: si prædictæ figuræ non sunt æquales, sit inter illas differentia α, & dividatur recta YT in tot partes æquales à rectis BX, EV, radio QT parallelis, ut (completis rectangulis AX, DV, GT, CX, FV, λT,) differentia rectilineorum ABDEGHTY, CDFGλQTY, sit minor quàm α; manifestum est enim hoc fieri posse ab indefinita divisione recte YT. Ducatur in puncto O recta tangens OPM, sitque in rectam

EV

XV perpendicularis OS: triangula OPS, TMO, rectangula ad S & O sunt similia ob æqualitatem angulorum POS, OTM; nam ab angulis æqualibus nempe rectis POT, SOX, auferendo eundem angulum SOT, relinquuntur anguli æquales POS, XOT, atque ob parallelas OX, QT, angulus XOT æqualis est angulo OTM, & ideo æqualis sunt anguli POS, OTM; & proinde ut OS ad OP, ita OT ad TM, ut autem OT ad TM ita TM ad XD, & igitur rectangulum OS in XD, nempe rectangulum DV, æquale est rectangulo TM in OP, sed rectangulum TM in OP majus est portione superficiei cylindricæ rectæ insistente curvæ ON, quoniam (TM existente maxima ejusdem portionis altitudine) recta PO major est quàm curva NO, ut patet ex prop. I. Geomet. part. universalis; & proinde rectangulum DV majus est quàm portio superficiei cylindricæ rectæ insistens curvæ ON: eodem modo demonstratur GT majus portione ejusdem superficiei insistente curvæ NQ & AX majus portione super curva IO, & ideo rectilineum ABDEGHTY majus est integrâ propositâ portione superficiei cylindricæ; idem quoque rectilineum majus est mixtilineo AQTY. Tangens in puncto N ducatur LNK, à parallelis proximis utrinque terminata in L, K; ex prop. I. Geomet. part. univer. patet rectam LN seu NK minorem esse arcu ON, sed (ut hactenus) demonstratur rectangulum GT vel FV æquale esse rectangulo NK vel NL in KT; atque rectangulum NL in KT minus est portione superficiei cylindricæ rectæ insistente curvæ ON, quoniam (TK existente minima ejusdem portionis altitudine) recta LN minor est curva ON: & proinde rectangulum FV minus etiam est quàm prædicta portio: eodem modo demonstratur rectangulum λT minus esse portione super curva NQ & rectangulum CX minus portione super curva IO, & ideo rectilineum CDFGλQTY minus est integrâ propositâ superficiei cylindricæ rectæ portione, idem quoque rectilineum minus est mixtilineo AQTY: ex prædictis ergo manifestum est quatuor esse quantitates, quarum maxima & minima sunt rectilinea ABDEGHTY, CDFGλQTY, intermediæ autem superficiei cylindricæ portio proposita & mixtilineum AQTY; & ideo differentia intermediarum minor est quàm differentia maximæ & minimæ, differentia autem maximæ & minimæ ex constructione minor est quàm α, & proinde differentia intermediarum nempe superficiei cylindricæ & mixtilinei AQTY multo minor est quàm α, quod est absurdum, ponitur enim α, non igitur differunt mixtilineum AQTY & proposita superficiei cylindricæ portio; & ideo æqualia sunt, quod demonstrare oportuit.

PROP. II. THEOREMA.

Sit circuli quadrans QML, sitque mixtilineum AHLQM talis naturæ, ut (ducta recta ad libitum HN radio LQ parallela & quadrantis arcum secante in K) recta HN æqualis sit secanti arcus LK, sitque mixtilineum ABLQMA talis naturæ, ut (productâ arbitrariâ NH in B) rectæ LQ, NH, NB, sint continuè proportionales: deinde sit semihyperbola IDE cujus axis MA, vertex I & asymptoton MLE: ducatur ad libitum radio LQ parallela recta NB, curvas LKM, LHA, LBA, secans in punctis K, H, B; & per punctum H ducatur radio MQ recta parallela HF hyperbolæ occurrens in puncto D. Dico sectorem hyperbolicum IMD æqualem esse semissi figuræ BLQN, quæ figura (ut in antecedente demonstratum est) æqualis est



eſt ſuperficiei cylindricæ conflatæ ex omnibus ſecantibus arcuum infinitorum OL plano LMQ in debitis ſuis punctis O normaliter inſiſtentibus. Ex *Gregorii* à *S. Vincentio* doctrina ductuum, truncus cylindrici recti ſuper figura LHNQ reſectus à plano baſin ſeminormaliter ſecante in recta QM, æqualis eſt ſemiſſi cylindrici recti cujus baſis eſt figura LBNQ & altitudo recta LQ, quoniam LQ ſemper eſt ad NH ut NH ad NB: manifeſtum autem eſt eundem truncum oriri ex ductu trianguli rectanguli iſoſcelis TGM in figuram LHNQ, hoc eſt ex ductu trianguli rectanguli iſoſcelis ωIM in rectangulum VLQN & ex ductu trapezii TGIω in figuram HLV; atque ſolidum factum ex ductu trianguli rectanguli ωIM in rectangulum VLQN æquale eſt priſmati cujus altitudo LQ & baſis triangulum IλM, quod ſic probo; $MG^2 : MI^2 :: MI^2 : KN^2$, & per converſionem rationis $MG^2 : MG^2 - MI^2 = GD^2 :: MI^2 : MI^2 - KN^2 = NQ^2$, & ideo $MG : GD :: MI : NQ$, ut autem MG ad DG, ita quælibet $MP = PR$ ad ductam PS ipſi GD parallelam, & ideo ut PR ad PS ita MI ad NQ, cúmque hoc ſemper fiat, manifeſtum eſt priſma cujus baſis λIM & altitudo IM æquale eſſe ſolido ex ductu trianguli ωIM in rectangulum VLQN: item ſolidum factum ex ductu trapezii TGIω in figuram HLV æquale eſt cylindrico cujus baſis eſt figura IDλ & altitudo IM, quod ſic probo; $GM : GD :: MI : QN$, & per converſionem rationis $GM : DF :: MI : GH$; ducatur recta ad libitum Yξ radio MQ parallela & curvas ſecans ut in figura, eritque $GM : DF :: XM : \gamma\xi$, & ideo $XM = XY : \gamma\xi :: MI : GH = XZ$, & ideo $XY \times XZ = MI \times \gamma\xi$; ut autem antè demonſtratum eſt GM eſſe ad DF ut MI ad GH, eodem modo nunc demonſtrari poteſt $XM = XY$ eſſe ad αξ ut MI ad Xθ, & ideo $XY \times X\theta = MI \times \alpha\xi$, cúmque $XY \times XZ = MI \times \gamma\xi$, ſi ab æqualibus æqualia auferantur, $XY \times Z\theta = MI \times \alpha\gamma$, cúmque hæc æqualitas ſemper contingat, manifeſtum eſt trapezium TGIω ductum in figuram HLV efficere ſolidum æquale cylindrico cujus baſis IDλ & altitudo IM; & proinde totus truncus ſuper figura HLQN ortus ex ductu trianguli TGM

in eandem figuram HL QN eſt æqualis cylindrico cujus baſis eſt ſector hyperbolicus MID & altitudo MI, atque idem truncus æqualis eſt dimidio cylindrici cujus baſis eſt figura BLQN & altitudo eadem IM, & ideo ſemiſſis cylindrici ſuper figura BLQN cum altitudine IM æqualis eſt cylindrico cujus baſis MID & altitudo eadem IM, eſt igitur figura BLQN dupla ſectoris hyperbolici MID, quod demonſtrare oportuit.

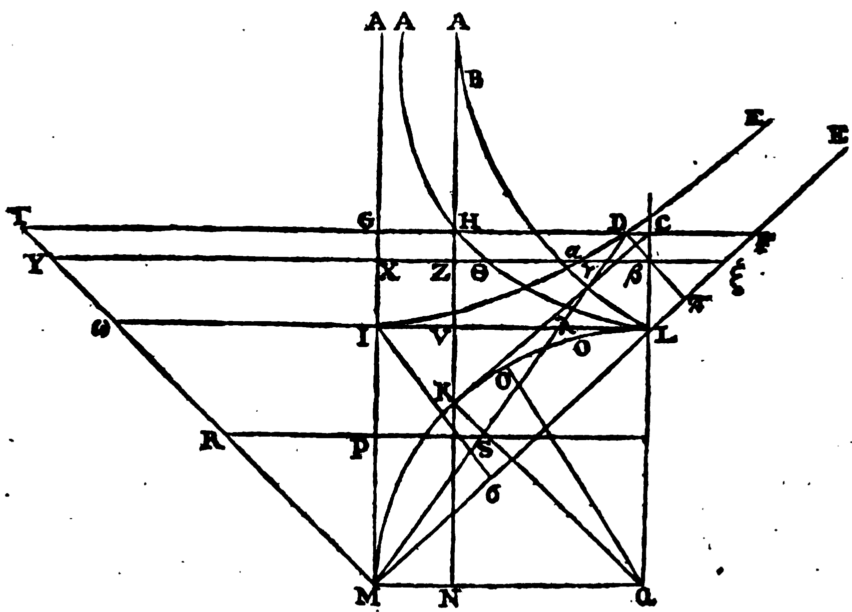
CONSECTARIUM.

Hinc ſequitur, quòd figura BLQN ſemper ſit dupla logarithmi differentiæ inter tangentem & ſecantem arcus KL, poſito radio LI loco unitatis, quod ſic probo. Ex punctis I, D, in aſymptoton ME demittantur perpendiculares $I\sigma$, $D\pi$; ex demonſtratis in Circ. & Hyperb. Quad. manifeſtum eſt ſectorem MID eſſe æqualem figuræ $ID\pi\sigma$, item figuram $ID\pi\sigma$ eſſe logarithmum rectæ $D\pi$ poſitâ $I\sigma$ unitate; ut autem $I\sigma$ ad $D\pi$ ita IL radius ad DF differentiam inter tangentem & ſecantem, & ideo poſita IL unitate erit idem ſector MID logarithmus rectæ DF, nempe exceſſus quo ſecans arcus KL ſuperat ejuſdem tangentem.

PROP. III. THEOREMA.

Linea Meridiana Planiſphærii Nautici *eſt Scala Logarithmorum Exceſſuum, quibus* Secantes *Latitudinum ſuperant earundem* Tangentes, *poſito Radio Loco Unitatis.*

Suppono ex ſcriptoribus nauticis in eorum planiſphærio arcum LK in æquatore eſſe ad eundem LK in latitudine, ut rectangulum ex LQ in KL ad portionem ſuperficiëi cylindricæ conflatæ ex omnibus ſecantibus arcuum infinitorum OL



 plano

plano LMQ in debitis ſuis punctis O normaliter inſiſtentibus, ſeu figuram BLQN; demonſtratum autem eſt in antecedente ſectorem hyperbolicum MID ſeu logarithmum rectæ DF (poſita QL unitate) differentiæ inter tangentem & ſecantem arcus KL eſſe ſemiſſem figuræ BLQN, patet quoque ſectorem circularem QKL eſſe ſemiſſem rectanguli LQ in KL; & ideo LK in æquatore eſt ad LK in latitudine ut ſector QKL ad logarithmum exceſſus quo ſecans arcus KL ſuperat ejuſdem tangentem; & ideo ſolidum ex LK in logarithmum dicti exceſſus æquale eſt ſolido ex LK in latitudine in ſectorem QKL, & utrumque ſolidum applicando ad LK, logarithmus dicti exceſſus æqualis eſt rectangulo ex LK in latitudine $\frac{QL}{2}$; eodem modo demonſtratur logarithmum exceſſus, quo ſecans arcus cujuſcunque OL excedit ejuſdem tangentem, æqualem eſſe rectangulo ex OL in latitudine in $\frac{QL}{2}$; & proinde logarithmus exceſſus quo ſecans arcus KL ſuperat ejuſdem tangentem, eſt ad rectangulum ex LK in latitudine in $\frac{QL}{2}$, ut logarithmus exceſſus quo ſecans arcus OL ſuperat ejuſdem tangentem ad rectangulum ex OL in latitudine in $\frac{QL}{2}$, & permutando, & utrumque rectangulum applicando ad $\frac{QL}{2}$; logarithmus exceſſus quo ſecans arcus KL ſuperat ejuſdem tangentem, eſt ad logarithmum exceſſus quo ſecans arcus OL ſuperat ſuam tangentem, ut arcus KL in latitudine ad arcum OL in latitudine ab æquatore planiſphærii nautici in linea recta computatis; et ideo linea meridiana planiſphærii nautici analoga eſt logarithmis exceſſuum, quibus ſecantes latitudinum ſuperant ſuas tangentes, quod demonſtrare oportuit.

SCHOLIUM.

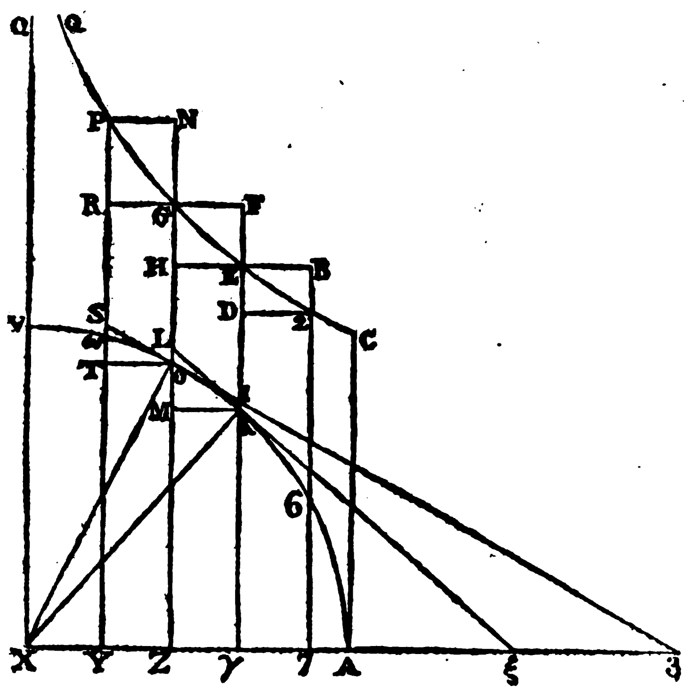
Ex hoc theoremate evidens eſt methodus deſcribendi integram meridianam, etiam ignota arcus dati in æquatore menſura; quam menſuram ex hujus 2 tali praxe invenimus. Sit QL radius 10000000000, DF 1000000000, & proinde ex noſtra Circl. & Hyper. Quadratura prop. 32 invenietur ſector hyperbolicus MID 1151292546497022812008, qui eandem habet rationem ad ſectorem QKL quam arcus KL in latitudine ad eundem in æquatore, & dividendo utrumque ſectorem per $\frac{QL}{2}$, erit ut 2302585092994|045624 ad arcum KL ita KL in latitudine ad eundem in æquatore; qualis autem ſit arcus KL ita invenimus, ut DF exceſſus ſecantis arcus KL ſupra ejuſdem tangentem ad LQ radium, ita LQ ad 100000000000 ſummam tangentis & ſecantis ejuſdem arcus KL, erit ergo KC tangens arcus KL 49500000000, CQ ejuſdem ſecans 50500000000, & NQ ejuſdem ſinus $9801980198\frac{2}{101}$, è quibus non difficilè erit invenire ipſum arcum ope noſtræ Quadr. Circ. & Hyp. prop. 30, vel (ſi quis rudiore calculo contentus fuerit) è tabula ſinuum. Ad datam autem arcûs dati in æquatore menſuram, meridianam nauticam conſtruere, praxis eſſet hujus inverſa, quæ nullo negotio ex hac colligitur.

Ex prædictis manifeſtè patet lineam meridianam planiſphærii nautici eſſe ſcalam tangentium artificialium arcuum, qui ſunt ſemiſſes complementorum

latitudinum, posito radio loco unitatis, quoniam (ut patet ex trigonometria) predictæ differentiæ sunt eædem cum dictis tangentibus. Si autem complementorum secantes radio insistant, erit figura ex illis conflata nempe AHLQMA ad quadratum radii ut quadrans circumferentiæ ad radium, ut patet ex nostræ Geom. part. universali prop. 2dâ.

PROP. IV. THEOREMA.

Sit circuli quadrans VOA, cujus sit pars 6ω: super arcu 6ω imaginetur superficies cylindrici recti talis naturæ, ut (sumpto in arcu 6ω quolibet puncto O) perpendicularis ad planum VAX ex puncto O ad summitatem superficiei cylindricæ excitata semper fiat æqualis tangenti arcus OA. Sit hyperbola CPQ cujus vertex C (nempe suppositâ AC parallelâ & æquali radio XV) & asymptotæ XA, XV; ducantur rectæ PωY, 267, radio VX parallelæ: dico spatium hyperbolicum 2PY7 æquale esse dictæ superficiei cylindricæ; si dictæ figuræ non sint æquales, sit inter illas differentia α, & dividatur recta 7Y in tot partes æquales à planis



rectæ 7Y perpendicularibus & spatium hyperbolicum in rectis Eγ, GZ, secantibus, item superficiem cylindricam in diversas portiones dividentibus, ut omnia rectangula cylindrica simul hisce portionibus inscripta differant ab omnibus rectangulis cylindricis simul eisdem circumscriptis minore quantitate quàm α, hoc enim absque dubio fieri potest ab indefinita divisione rectæ 7Y; intelligo autem rectangulum cylindricum portioni inscriptum esse superficiem cylindricam rectam

cujus

cujus eadem est basis cum portione cui inscribitur, & altitudo ubique eadem cum minima altitudine portionis, item circumscriptum cui eadem etiam est basis cum portione, & altitudo eadem cum portionis altitudine maxima. Compleantur rectangula 2γ, Bγ, EZ, FZ, GY, NY; & ducatur in puncto K tangens LKξ, sitque in rectam LZ perpendicularis KM. Triangula LKM, Kξx, rectangula ad K & M sunt similia ob angulos æquales Kξx, LKM; & ideo ut KM ad KL, ita Kξ ad xξ, & ideo $KL \times K\xi = KM \times x\xi$, atque γE est æqualis rectæ xξ, quod sic probo; XY est ad XK vel AC, ut XK ad xξ, sed ob hyperbolam Xγ est ad AC ut AC ad γE, sunt ergo æquales γE, xξ, & ideo rectangulum ex KL in Kξ æquale est rectangulo ex KM in Eγ nempe EZ, atque rectangulum ex KL in Kξ majus est rectangulo cylindrico inscripto portioni super KO, quoniam eandem cum illo habens altitudinem nempe Kξ basem habet majorem (est enim recta KL major quam curva KO) & ideo rectangulum EZ majus est rectangulo cylindrico inscripto portioni super KO; eodem modo probatur rectangulum GY majus esse rectangulo cylindrico inscripto portioni super Oω, & rectangulum 2γ majus esse rectangulo cylindrico inscripto portioni super 6K; & proinde rectilineum 72DEHGRY majus est omnibus rectangulis cylindricis inscriptis simul sumptis, & ideo spatium hyperbolicum 2PY7 eisdem rectangulis cylindricis multò majus est. In puncto O ducatur tangens 3OS; demonstratur ut antè rectangulum OS vel OI in O3 æquale esse rectangulo GY seu FZ, atque rectangulum O3 in OI minus est rectangulo cylindricò circumscripto portioni super KO, quoniam eandem cum illo habens altitudinem, basem habet minorem (est enim recta OI minor quam curva KO) & proinde rectangulum FZ minus est rectangulo cylindrico circumscripto portioni super KO; eodem modo probatur rectangulum NY minus esse rectangulo cylindrico portioni super Oω circumscripto, & rectangulum Bγ minus esse rectangulo cylindrico portioni super 6K circumscripto; & ideo rectilineum 7BEFGNPY minus est omnibus rectangulis cylindricis circumscriptis simul sumptis; & ideo spatium hyperbolicum 2PY7 eisdem rectangulis cylindricis multo minus est. Quatuor igitur sunt quantitates, quarum maxima & minima sunt rectangula cylindrica circumscripta simul sumpta, & rectangula cylindrica inscripta simul sumpta, intermediæ autem spatium hyperbolicum 2PY7 & superficies cylindrica super curva 6ω, & ideo differentia intermediarum minor est quam differentia maximæ & minimæ; at differentia maximæ & minimæ ex constructione minor est quam α, & ideo differentia intermediarum nempe spatii hyperbolici & superficiei cylindricæ est multo minor quam α, quod est absurdum, ponitur enim α; non igitur differunt quantitates intermediæ, & ideo æquales sunt, quod demonstrandum erat.

Quòd si superficies cylindrica producatur usque ad terminum quadrantis A: dico adhuc illam esse æqualem spatio hyperbolico correspondenti CPYA; si non sunt æquales, sit superficies cylindrica super Aω major spatio hyperbolico CPYA, & abscindatur plano B67 rectæ AX normali portio superficiei cylindricæ, ita ut relicta nempe superficies cylindrica super curva 6ω sit æqualis spatio hyperbolico CPYA, atque superficies super 6ω æqualis est spatio hyperbolico 2PY7, ex hactenus demonstratis; spatia igitur hyperbolica CPYA, 2PY7, sunt æqualia, quod est absurdum; superficies ergo cylindrica super Aω non est major spatio hyperbolico CDYA; sit (si fieri potest) minor & à spatio hyperbolico CPYA auferatur rectâ 27 ipsi CA parallelâ spatium C27A, ita ut relictum 2PY7 fiat æquale superficiei

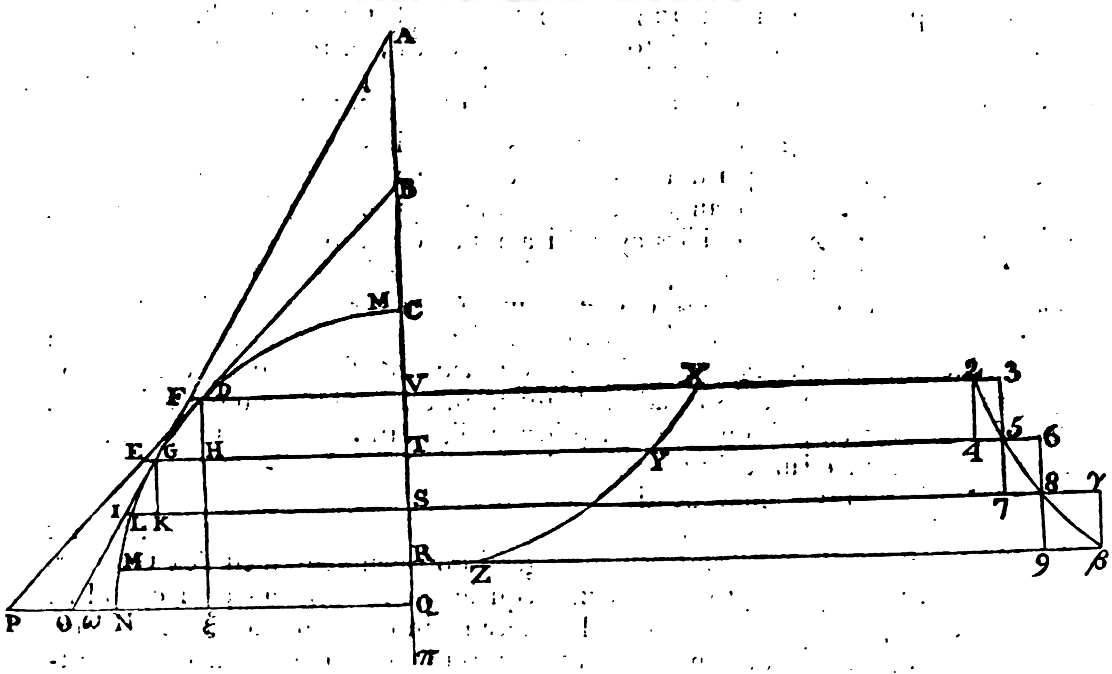
ficiei cylindricæ ſuper Aω, atque ſpatium hyperbolicum 2PY7 æquale eſt ſuperficiei cylindricæ ſuper 6ω; & ideo ſuperficies cylindricæ ſuper 6ω & ſuper Aω inter ſe ſunt æquales, quod eſt abſurdum; & ideo ſuperficies cylindrica ſuper Aω non eſt minor ſpatio hyperbolico CPYA, ſed (ut ante demonſtratum eſt) nec eſt major; reſtat igitur, ut illi ſit æqualis, quod demonſtrandum erat.

CONSECTARIUM.

Hinc manifeſtum eſt ſuperficiem cylindricam ex tangentium ſumma conflatam (*v. g.* ſuper curva Aω) eſſe ſecantem artificialem ejuſdem arcus Aω, poſito radio loco unitatis; nam (ut demonſtratur in noſtra Circuli & Hyp. Quadr.) poſita CA ſeu radio loco unitatis, erit ſpatium hyperbolicum ACPY logarithmus rectæ YP ſecantis arcus Aω.

PROP. V. THEOREMA.

Sit circuli quadrans QCω biſariam diviſus in puncto D. Sit arcus DM minor quam Dω, ſuper quo imaginetur ſuperficies cylindrici recti talis naturæ, ut (ſumpto in arcu DM quolibet puncto G) perpendicularis ad planum QCω ex puncto G ad ſummitatem ſuperficiei cylindricæ excitata ſemper fiat æqualis ſecanti arcus GC. In radium QC demittantur rectæ perpendiculares DV, MR,



quæ producantur in 2 & β; fiatque mixtilineum V2βR talis naturæ, ut (ductâ rectâ quâcunque GT5 rectæ DV parallelâ) tangentes arcuum GC, Gω, ſimul, nempe recta AO, æquales ſint rectæ T5. Dico mixtilineum V2βR æquale eſſe ſuperficiei cylindricæ ſuper curva DM: ſi prædictæ figuræ non ſint æquales, ſit inter

inter illas differentia α, & dividatur recta VR in tot partes æquales planis rectæ VR perpendicularibus, & mixtilineum V2βR in rectis T5, s8, secantibus & superficiem cylindricam in diversas portiones dividentibus, ut omnia rectangula cylindrica simul, hisce portionibus inscripta, differant ab omnibus rectangulis cylindricis simul, eisdem circumscriptis minore quantitate quàm α; hoc enim absque dubio fieri potest ab indefinita divisione rectæ VR. Compleantur rectangula V4, V5, T7, T8, s9, sβ, & ducatur in puncto G tangens AGO, sitque in rectam LS perpendicularis GK. Triangula IGK, OAQ, sunt similia, & ideo ut GK vel VT ad GI vel GE, ita QA ad AO seu T5, & proinde rectangulum VT in T5 nempe V5 æquale est rectangulo EG in AQ, atque rectangulum EG in AQ minus est rectangulo cylindrico circumscripto portioni super GD, quoniam eandem cum illo habens altitudinem nempe AQ, basem habet minorem; & ideo rectangulum V5 minus est rectangulo cylindrico circumscripto portioni super GD; eodem modo demonstratur rectangulum T8 minus esse rectangulo cylindrico circumscripto portioni super GI, & rectangulum sβ minus esse rectangulo cylindrico circumscripto portioni super LM; & ideo rectilineum V35687βR minus est omnibus rectangulis cylindricis circumscriptis simul sumptis, & igitur mixtilineum V2βR eisdem rectangulis cylindricis multo minus est. Ob similia triangula DFH, BPQ; DH vel VT est ad DF ut BQ ad BP vel V2, & ideo rectangulum V4 æquale est rectangulo DF in BQ; at rectangulum DF in BQ majus est rectangulo cylindrico portioni super DG inscripto, quoniam eandem cum illo habens altitudinem BQ, basem habet majorem; & ideo rectangulum V4 majus est rectangulo cylindrico inscripto portioni super DG; eodem modo probatur rectangulum T7 majus esse rectangulo cylindrico inscripto portioni super GL, & rectangulum s9 majus esse rectangulo cylindrico inscripto portioni super LM, & ideo rectilineum V245789R majus est omnibus rectangulis cylindricis inscriptis simul sumptis; est ergo mixtilineum V2βR eisdem rectangulis cylindricis multo majus. Quatuor igitur sunt quantitates, quarum maxima & minima sunt rectangula cylindrica circumscripta simul, & rectangula cylindrica inscripta simul; intermediæ autem mixtilineum VTβR & superficies cylindrica super curva DM, & ideo differentia maximæ & minimæ est major differentiâ intermediarum, differentia autem maximæ & minimæ ex constructione est minor quam α, & ideo differentia intermediarum est multo minor quam α, quod est absurdum, ponitur enim α; non igitur differunt quantitates intermediæ & ideo æquales sunt, quod, &c. Quod si arcus DM sumeretur versus C, adhuc staret prop. demonstratio autem esset paulo diversa, quæ tamen nullo negotio ex priore colligeretur. Si punctum M in ipso C caderet, verum etiam esset *theorema*, sed negativè per duas positiones demonstrandum, sicut in fine antecedentis.

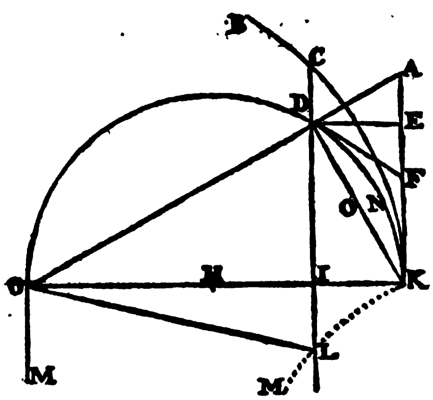
Sit mixtilineum VXZR talis naturæ, ut (ducta recta quacunque GTY rectæ DX parallela) recta TY semper æqualis fiat tangenti GO; manifestum est (ex Geomet. part. univer. prop. 4.) mixtilineum VXZR æquale esse superficiei trunci cylindrici recti resectæ à plano basem seminormaliter secante in recta PQ, atque (ex Geom. part. univers. prop. 3.) evidens est eandem trunci superficiem æqualem esse rectangulo Nξ in CQ, & ideo mixtilineum VXZR eidem rectangulo est æquale. Ex prædictis evidens est mixtilineum X2βZ talis esse naturæ ut Y5 semper sit æqualis tangenti GA; & proinde mixtilineum X2βZ unà cum portione quadrantis DVRM æquale est spatio conchoïdali resecto à rectis DV, MR, cujus conchoïdis

choïdis vertex eſt C, norma PQ, polus π, CQ = Qπ: ex hac prop. & hujus 2 evidens eſt ſequens conſectarium ſpatio conchoïdali quadrando ſatis expeditum.

CONSECTARIUM.

Si in prædicta conchoïde accipiatur ſpatium contentum à curva conchoidali & rectis CT, T6; erit predictum ſpatium equale duplo ſpatii hyperbolici (cujus aſymptotæ QA, QP, ſemiaxis QC) contenti à curva hyperbolica una aſymptotâ & rectis QC, QA − GA, alteri aſymptotæ parallelis, una cum ſemiſegmento circulari CGT dempto rectangulo QC in GT. Aliarum conchoïdeon ſpatia (nempe quando vertex & polus non æqualiter diſtant à normali) poſſunt menſurari per Analogiam à *Walliſio* in Epiſt. Com. pag. 171 demonſtratam.

PROP. VI. THEOREMA.



Sit ciſſois KLM cujus aſymptoton GM, ſemicirculus GDK. Diametro GK ſit normalis DIL, & jungantur rectæ DK, GL; dico ſpatium ciſſoidale GKL triplum eſſe ſegmenti circularis DNK. Ducatur curva KCB talis naturæ, ut (ductâ ad libitum à puncto G recta GDA quæ tangenti occurrat in A) CDI perpendicularis rectæ GK, fiat æqualis rectæ KA. Recta DF ſemicirculum tangat in puncto D, & ideo æquales ſunt rectæ FD, FK; cúmque angulus KDA ſit rectus, patet FA, FD, eſſe æquales; cúmque CI ſemper ſit æqualis duplæ ipſius DF, manifeſtum eſt (ex Geom. part. univer. prop. 4) mixtilineum CKI eſſe duplum ſuperficiei trunci cylindrici recti ſuper curva DNK reſectæ à plano baſem ſeminormaliter ſecante in recta KA; atque eadem ſuperficies trunci æqualis eſt rectangulo ex curva DNK in radium KH ablato rectangulo ex DI in HK (ut ſatis patet verſatis in ſuperficierum ſtudio) hoc eſt duplo ſegmenti circularis DNK; eſt igitur mixtilineum CKI quadruplum ſegmenti circularis DNK. Ut DI ad IK ita eſt IK vel DE ad EA vel CD, & proinde æquales ſunt rectæ CD, IL, cúmque hoc ſemper fiat, patet ſpatium ciſſoidale IKL æquale eſſe mixtilineo DNKC, & idem utrinque addendo nempe ſemiſegmentum circulare DIKN, mixtilineum CKI, ſeu quadruplum ſegmenti circularis DNK, æquale eſt mixtilineo DNKL, & utrinque auferendo ſegmentum circulare DNK; triplum ſegmenti circularis DNK æquale eſt mixtilineo DOKL, ſeu (ob æqualitatem triangulorum DKI, GIL) ipſi ſpatio ciſſoidali propoſito GKL, quod demonſtrandum erat.

Hic ſupponitur arcus DNK quadrante minor, quod ſi quadrante eſſet major, nullo negotio variari poteſt demonſtratio ut illi inſerviat: at KND exiſtente ſemicircumferentiâ, multò facilior eſſet demonſtratio, nempe quòd ſpatium ciſſoidale infinitè extenſum æquetur ſemicirculi triplo.

METHODUS

METHODUS

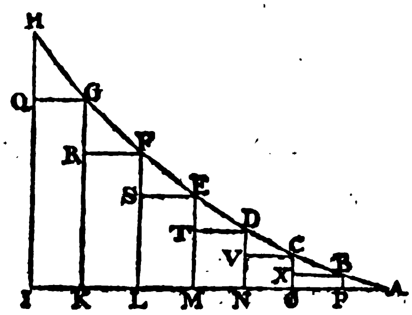
COMPONENDI

Tabulas Tangentium & Secantium Artificialium

EX

Tabulis Tangentium & Secantium Naturalium

EXACTISSIME ET MINIMO CUM LABORE.



SIT AI arcus quadrantis in lineam rectam extensus, sitque figura AHI conflata ex tangentibus naturalibus singulorum arcuum à puncto A, in debitis suis punctis, rectis AI normaliter insistentibus: sit AP pars minima, in cui æquales dividitur quadrans, nempe $\frac{1}{60}$ vel $\frac{1}{100}$ gradus, sintque illi æquales PO, ON, NM, &c, & ducantur rectæ AI perpendiculares PB, OC, ND, ME, &c, manifestum est ex consectario 4 hujus, figuram ABP esse secantem artificialem arcus AP item figuram ACO esse secantem artificialem arcus AO (positâ cyphrâ loco radii artificialis), &c, manifestum est rectangula BO, CN, DM, &c, inveniri ex multiplicatione minimæ partis quadrantis AP in singulas tangentes naturales; at in mensurandis figuris ABP, BCX, CDV, pauló major est difficultas; primo igitur si tangentes conveniant in differentiis primis, non differunt lineæ AB, BC, CD, &c, à rectis, & ideo figuræ ABP, BCX, CDV, &c, erunt triangula rectangula, & proinde, *e. g.*

GHQ

$GHQ = \frac{HQ \times QG}{2}$: quod si differentiæ secundæ fuerint æquales, erunt dictæ figuræ portiones trilineorum quadraticorum, *e. g.* erit GHQ portio trilinei quadratici, cujus axi HQ est parallela, differentiæ illæ inter se æquales sint z; & proinde erit $GHQ = \frac{HQ \times GQ}{2} - \frac{z \times GQ}{12}$; si autem differentiæ tertiæ fuerint æquales, erunt dictæ figuræ portiones trilineorum cubicorum, eritque *e. g.* $GHQ = \frac{HQ \times GQ}{2} - \sqrt{q}\,(\frac{HQ \times z \times GQ^2}{72} - \frac{z^2 \times GQ^2}{1728})$: quando differentiæ quartæ sunt æquales, erunt dictæ figuræ portiones trilineorum Q-quadraticorum, & differentiæ quartæ erunt æquales 24^tuplo^ Q-quadrati ipsius GQ diviso per cubum lateris recti, item quando differentiæ quintæ sunt æquales, erunt dictæ figuræ portiones trilineorum surfolidorum, & differentiæ quintæ erunt æquales 120^plo^ surfolidi ipsius GQ diviso per Q-quadratum lateris recti, & sic in infinitum. Quæ hic diximus de compositione secantium artificialium ex tangentibus naturalibus eodem modo intelligi velim de compositione tangentium artificialium ex secantibus naturalibus secundum hujus tertiam. Animadvertendum tangentes & secantes artificiales suprà computari, posito o logarithmo unitatis, 1000000000000 radio, & 2302585092994045624017870 logarithmo denarii: facilius autem (nimirum solâ additione) posito $\frac{1}{60}$ grad. $= GQ = AP = 1$, computabimus tangentes & secantes artificiales ad 7915704467897819 denarii logarithmum; nam hac ratione $\frac{HQ \times GQ}{2} = GHQ = \frac{HQ}{2}$, item $\frac{HQ \times GQ}{2} - \frac{z \times GQ}{12} = GHQ = \frac{HQ}{2} - \frac{z}{12}$, item $\frac{HQ \times GQ}{2} - \sqrt{q}\,(\frac{HQ \times z \times GQ^2}{72} - \frac{z^2 \times GQ^2}{1728}) = HGQ = \frac{HQ}{2} - \sqrt{q}\,(\frac{HQ \times z}{72} - \frac{z^2}{1728})$: & tandem unicâ solâ divisione invenimus tangentes & secantes artificiales ad logarithmum denarii 1000000000000000, posito semper radio unitatis loco, quæ sunt differentiæ tangentium & secantium artificialium in tabula ab ipso radio artificiali; & proinde divisoris multiplices, ad facilitandam operationem, in tabella subsidiaria hic reponimus. Quod si $\frac{1}{100}$ grad. $= GQ$, tangentes & secantes artificiales debentur logarithmo denarii 13192840779829703, cujus etiam multiplices in subjecta tabella notantur. Quod, si quis velit hos numeros potius repræsentare radium artificialem quam denarii logarithmum, addatur cyphra & habebit intentum. Notandum hos numeros convenire radio naturali 1000000000000, hinc enim patebit numerus notarum in adscriptis artificialibus.

1	7915704467897819	13192840779829703
2	15831408935795638	26385681559659406
3	23747113403693457	39578522339489109
4	31662817871591276	52771363119318812
5	39578522339489095	65964203899148515
6	47494226807386914	79157044678978218
7	55409931275284733	92349885458807921
8	63325635743182552	105542726238637624
9	71241340211080371	118735567018467327

Ex Epistola D. Jacobi Gregorii *ad D.* Collins, 15 *Feb. Anno* $167\frac{0}{1}$ *data, cujus habetur Autographon.*

EX quo epistolam ad te dedi, tres a te accepi, unam *Dec.* 15, alteram *Dec.* 24, tertiam 21 *Januarii* nuper elapsi datam.

Quod attinet *Newtoni* methodum universalem, aliqua ex parte, ut opinor, mihi innotescit, tam quoad geometricas quam mechanicas curvas. Nihilo tamen minus ob series ad me missas gratias habeo, quas ut remunerem mitto quæ sequuntur.

Sit radius $= r$, arcus $= a$, tangens $= t$, secans $= s$,

$$\text{Et erit } a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} \text{ \&c.}$$

$$\text{Eritque } t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} \text{ \&c.}$$

$$\text{Et } \quad s = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} \text{ \&c.}$$

Sit nunc tangens artificialis $= t$, & secans artificialis $= s$, & integer quadrans $= q$,

$$\text{Erit } s = \frac{a^2}{2r} + \frac{a^4}{12r^3} + \frac{a^6}{45r^5} + \frac{17a^8}{2520r^7} + \frac{62a^{10}}{28350r^9} \text{ \&c.}$$

$$\text{Sit } 2a - q = e, \text{ \& erit } t = e + \frac{e^3}{6r^2} + \frac{e^5}{24r^4} + \frac{61e^7}{5040r^6} + \frac{277e^9}{72576r^8} \text{ \&c.}$$

Sit nunc secans artificialis 45 gr. $= s$, sitque $s + l$ secans artificialis ad libitum, erit ejus arcus $= \frac{1}{2}q + l - \frac{l^2}{r} + \frac{4l^3}{3r^2} - \frac{7l^4}{3r^3} + \frac{14l^5}{3r^4} - \frac{452l^6}{45r^5}$ &c,

$$\text{Eritque } 2a - q = t - \frac{t^3}{6r^2} + \frac{t^5}{24r^4} - \frac{61t^7}{5040r^6} + \frac{277t^9}{72576r^8} \text{ \&c.}$$

Hic

Hic animadvertendum eſt radium artificialem eſſe o; & ubi inveneris q majorem quam aa, ſive artificialem ſecantem 24 gr. majorem eſſe data ſecante, mutanda eſſe ſigna, & pergendum ſecundum vulgaris algebræ præcepta.

Sit ellipſis cujus alter ſemiaxium $= r$, alter $= c$; ex quolibet curvæ ellipticæ puncto demittatur in ſemiaxem r recta perpendicularis $= a$: erit curva elliptica perpendiculari a adjacens $= a + \frac{r^2a^3}{6c^4} + \frac{4r^2c^2a^5 - r^4a^5}{40c^8} + \frac{8c^4r^2a^7 + r^6a^7 - 4c^2r^4a^7}{112c^{12}} + \frac{64c^6r^2a^9 - 48c^4r^4a^9 + 24c^2r^6a^9 - 5r^8a^9}{1152c^{16}}$ &c.

Si determinetur ellipſeos ſpecies, ſeries hæc ſimplicior evadet. Ut ſi $c = 2r$, foret curva prædicta $= a + \frac{a^3}{96r^2} + \frac{3a^5}{2048r^4} + \frac{113a^7}{458752r^6} + \frac{3419a^9}{75497472r^8}$ &c. Reliquis vero manentibus, ſi curva prædicta eſſet hyperbola, predicta quoque ſeries ei inſerviret; ſi modo omnium terminorum partes affirmentur, & negentur totus terminus tertius, totus quintus, ſeptimus, &c, in locis imparibus.

END OF EXTRACTS FROM GREGORY.

OTHER

EXTRACTS

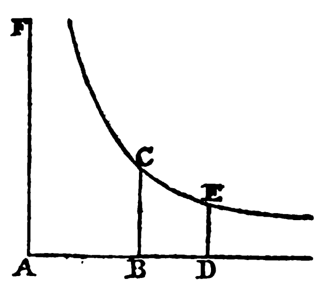
FROM

COLLINS'S COMMERCIUM EPISTOLICUM;

RELATING TO

LOGARITHMS.

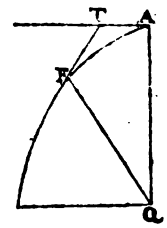
1. *An Extract from Mr. Isaac Newton's First Epistle to Mr. Henry Oldenburgh, Secretary to the Royal Society of London; with a Direction to communicate the Contents of it to Mr. Godfrey William Leibnitz; dated on the* 13*th Day of June, in the Year* 1676, *from Cambridge, where Mr. Newton (who was afterwards Sir Isaac Newton, Knight) was at that Time Professor of the Mathematicks upon Dr. Lucas's Foundation: containing a Discovery relating to Logarithms.*



PRÆTEREA, si sit CE hyperbola, cujus asymptoti AD, AF, rectum angulum FAD constituant; & ad AD erigantur utcunque perpendicula BC, DE occurrentia hyperbolæ in C & E: & AB dicatur a, BCb, & area BCED z; erit BD $= \frac{z}{b} + \frac{zz}{2abb} + \frac{z^3}{6aab^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} +$ &c. Ubi coefficientium denominatores prodeunt multiplicando terminos hujus arithmeticæ progressionis, 1, 2, 3, 4, 5, &c, in se continuo. Et hinc ex logarithmo dato potest numerus ei competens inveniri.

2. *An*

2. ***An Extract from an Epistle of Mr. Godfrey William Leibnitz to Mr. Henry Oldenburgh, Secretary of the Royal Society of London; with a Direction to communicate the Contents of it to Mr. Isaac Newton; dated the 27th Day of August, in the Year 1676: containing a Passage relating to Logarithms.***

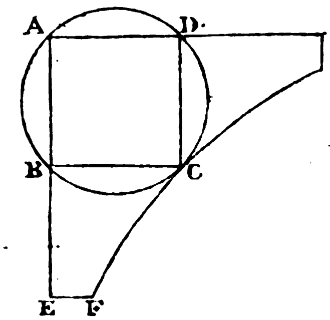


OMNIUM verò possibilium circuli, & sectoris conici centrum habentis cujuslibet, per series infinitas quadraturarum, simplicissimam hanc esse dicere ausim quam nunc subjicio.

Sit QAF sector, duabus rectis in centro Q concurrentibus & curvâ conicâ AF, ad verticem A sive axis extremum perveniente, comprehensus. Tangenti verticis AT occurrat tangens FT. Ipsum AT vocemus t; & rectangulum sub semi-latere recto in semi-latus transversum sit unitas. Erit sector hyperbolæ, circuli vel ellipseos, per semi-latus transversum divisus, $= \frac{t}{1} \pm \frac{t^3}{3} + \frac{t^5}{5} \pm \frac{t^7}{7}$ &c. Signo ambiguo $\pm$ valente $+$ in hyperbola, $-$ in circulo vel ellipsi. Unde, posito quadrato circumscripto 1, erit circulus $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ &c. Quæ expressio, jam triennio abhinc & ultrà à me communicata amicis, haud dubiè omnium possibilium simplicissima est maximeque afficiens mentem.

Unde duco harmonium sequentem *;

$$\begin{array}{cccccccccccl}
\frac{1}{3} + & \frac{1}{8} + & \frac{1}{15} + & \frac{1}{24} + & \frac{1}{35} + & \frac{1}{48} + & \frac{1}{63} + & \frac{1}{80} + & \frac{1}{99} + & \frac{1}{120} & \&c = \frac{3}{4} \\
\frac{1}{3} & & \frac{1}{15} & & \frac{1}{35} & & \frac{1}{63} & & \frac{1}{99} & & \&c = \frac{2}{4} \\
& \frac{1}{8} & & \frac{1}{24} & & \frac{1}{48} & & \frac{1}{80} & & \frac{1}{120} & \&c = \frac{1}{4} \\
\frac{1}{3} & & & & \frac{1}{35} & & & & \frac{1}{99} & & \&c \\
& \frac{1}{8} & & & & \frac{1}{48} & & & & \frac{1}{120} & \&c.
\end{array}$$



Ubi $\frac{1}{3} + \frac{1}{35} + \frac{1}{99}$ &c, exprimit aream circuli ABCD, & $\frac{1}{8} + \frac{1}{48} + \frac{1}{120}$ &c, aream hyperbolæ æquilateræ BCEF, cum sit BC dupla ipsius EF, & quadratum inscriptum $= \frac{1}{4}$. Numeri 3, 8, 15, 24, &c, sunt quadrati unitate minuti.

Vissicim, † ex seriebus regressuum pro hyperbola hanc inveni. Si sit numerus aliquis unitate minor $1 - m$, ejusque logarithmus hyperbolicus l. Erit $m = \frac{l}{1} - \frac{l^2}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} - \frac{l^4}{1 \times 2 \times 3 \times 4}$ &c. Si numerus sit major unitate, ut $1 + n$; tunc pro eo inveniendo mihi etiam † prodiit regula, quæ in *Newtoni* Epistolâ expressa est;

* Vide Acta Lipsica *Feb.* 1682.

† N. B. Methodum perveniendi ad has series *Leibnitius* à *Newtono* jam modo acceperat, idque ex ipsius rogatu. Imo series ipsas à *Newtono* una cum methodo perveniendi ad easdem jam modo acceperat, & pro hyperbola signum tantum mutavit; pro circulo sinum versum à *Newtono* acceptum subduxit à radio, ut haberet sinum complementi.

scilicet,

scilicet, erit $n = \frac{l}{1} + \frac{l^2}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^4}{1 \times 2 \times 3 \times 4}$ &c. Prior tamen celerius appropinquat. Ideoque efficio ut ea possim uti, etiam cum major est unitate numerus $1 + n$. Nam idem est logarithmus pro $1 + n$ & pro $\frac{1}{1+n}$. Unde, si $1 + n$ sit major unitate, erit $\frac{1}{1+n}$ minor unitate. Fiat ergo $1 - m = \frac{1}{1+n}$, ac inventa m, habebitur & $1 + n$ numerus quæsitus.

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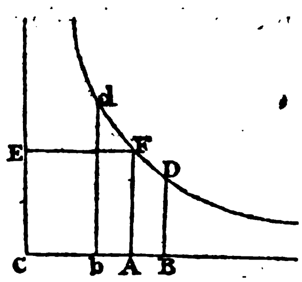
3. *An Extract from Mr. Isaac Newton's Second Epistle to Mr. Henry Oldenburgh, Secretary of the Royal Society of London; with a Direction to communicate the Contents of it to Mr. Godfrey William Leibnitz; dated the 24th Day of October, in the Year 1676: containing some Discoveries relative to Logarithms.*

EO tempore pestis ingruens (*quæ contigit annis* 1665, 1666), coegit me hinc fugere, & alia cogitare. Addidi tamen subinde condituram quandam logarithmorum ex area hyperbolæ, quam hic subjungo.



Sit dFD hyperbola, cujus centrum C, vertex F, & quadratum interjectum CAFE $= 1$. In AC cape AB, Ab hinc inde $= \frac{1}{10}$ seu 0.1: Et, erectis perpendiculis BD, bd ad hyperbolam terminatis, erit semi-summa spatiorum AD & Ad $= 0.1 + \frac{0.001}{3} + \frac{0.00001}{5} + \frac{0.0000001}{7}$ &c, & semi-differentia $= \frac{0.01}{2} + \frac{0.0001}{4} + \frac{0.000001}{6} + \frac{0.00000001}{8}$ &c. Quæ reductæ sic se habent,

0.1000000000000	0.0050000000000
3333333333	250000000
20000000	1666666
142857	12500
1111	100
9	1
0.1003353477310	0.0050251679267

Horum summa 0.1053605156577 est Ad; & differentia 0.0953101798043 est AD. Et eadem ratione positis AB, Ab hinc inde $= 0.2$, obtinebitur Ad $= 0.2231435513142$, & AD $= 0.1823215567939$. Habitis sic logarithmis hyperbolicis numerorum quatuor decimalium 0.8, 0.9, 1.1, & 1.2; cum sit $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = 2$; & 0.8 & 0.9, sint minores unitate: adde logarithmos eorum ad duplum logarithmi 1.2, & habebis 0.6931471805597 logarithmum hyperbolicum

1 bolicum

bolicum numeri 2. Cujus triplo adde log. 0.8 (siquidem sit $\frac{2 \times 2 \times 2}{0.8} = 10$) & habebis 2.302585092993 logarithmum numeri 10: Indeque per additionem simul prodeunt logarithmi numerorum 9 & 11: adeoque omnium primorum horum 2, 3, 5, 11 logarithmi in promptu sunt. Insuper, ex sola depressione numerorum superioris computi per loca decimalia & additione, obtinentur logarithmi decimalium 0.98, 0.99, 1.01, 1.02; ut & horum 0.998, 0.999, 1.001, 1.002. Et inde per additionem & subductionem prodeunt logarithmi primorum 7, 13, 17, 37, &c. Qui unà cum superioribus, per logarithmum numeri 10 divisi, evadunt veri logarithmi in tabulam inserendi. Sed hos postea propius obtinui.

Pudet dicere ad quot figurarum loca has computationes, otiosus eo tempore, produxi. Nam tunc sane nimis delectabar inventis hisce. Sed ubi prodiit ingeniosa illa * *Nicolai Mercatoris* Logarithmotechnia (quem suppono sua primum invenisse), cœpi ea minus curare; suspicatus, vel eum nosse extractionem radicum æque ac divisionem fractionum; vel alios saltem, divisione patefacta, inventuros reliqua, prius quam ego ætatis essem maturæ ad scribendum.

Eo ipso tamen tempore quo liber iste prodiit, communicatum est per amicum D. *Barrow* (tunc Matheseos Professorem *Cantab.*) cum D. *Collinio*, † compendium quoddam methodi harum serierum; in quo significaveram areas & longitudines curvarum omnium, & solidorum superficies & contenta, ex datis rectis; & vice versa, ex his datis rectas determinari posse: & methodum ibi indicatum illustraveram diversis seriebus.

Suborta deinde inter nos epistolari consuetudine; D. *Collinius*, vir in rem mathematicam promovendam natus, non destitit suggerere ut hæc publici juris facerem. Et ante annos quinque (1671) cum suadentibus amicis consilium ceperam edendi tractatum de Refractione Lucis, & Coloribus, quem tunc in promptu habebam; cœpi de his seriebus iterum cogitare; & ‡ tractatum de iis etiam conscripsi, ut utrumque simul ederem.

Sed, ex occasione telescopii catadioptrici, epistolâ ad te missâ qua breviter explicui conceptus meos de natura lucis, inopinatum quiddam effecit ut mei interesse sentirem ad te festinanter scribere de impressione istius epistolæ. Et subortæ statim per diversorum epistolas (objectionibus aliisque refertas) crebræ interpellationes me prorsus à consilio deterruerunt; & effecerunt ut me arguerem imprudentiæ, quod umbram captando, eatenus perdideram quietem meam, rem prorsus substantialem.

* Mathematici priores invenerunt hoc theorema, quod *summa terminorum progressionis geometricæ in infinitum pergentis est ad terminorum primum & maximum, ut hic terminus ad differentiam duorum terminorum primorum.* Idem demonstratur arithmetice multiplicando extrema & media. Demonstravit *Wallisius* dividendo rectangulum sub mediis per extremum ultimum. Vide *Wallisii* opus arithmeticum anno 1657 editum, cap. 33 § 68. Per *Wallisii* divisionem *Mercator* demonstravit & auxit quadraturam hyperbolæ à D. *Brounker* prius inventam. Et *Gregorius* idem demonstravit geometrice. Sed horum nemo methodum generalem quadrandi curvas per divisionem invenit. *Mercator* hoc nunquam professus est. *Gregorius* ejusmodi methodum, licet vir acutissimus & literis *Collinii* admonitus, vix tandem invenit. *Newtonus* invenit per interpolationem serierum, & postea divisionibus & extractionibus radicum, ut notioribus, usus est.

† Analysin intelligit per æquationes infinitas.

‡ Hujus tractatus meminit D. *Collins* in epistolis duabus impressis, pag. 101, 102, Commerc. Epistol. Et *Newtonus* in epist. impressa, p. 205 ibid.

Sub

Sub eo tempore *Jacobus Gregorius*, ex unica quadam ſerie è meis, quam D. *Collinius* ad eum tranſmiſerat, poſt multam conſiderationem (ut ad *Collinium* reſcripſit) pervenit ad eandem methodum, & tractatum de eâ reliquit quem ſperamus ab amicis ejus editum iri. Siquidem, pro ingenio quo pollebat, non potuit non adjicere de ſuo nova multa, quæ rei mathematicæ intereſt ut non pereant.

Ipſe autem tractatum meum non penitus abſolveram, ubi deſtiti à propoſito; neque in hunc diem mens rediit ad reliqua adjicienda. Deerat quippe pars ea qua decreveram explicare modum ſolvendi problemata, quæ ad quadraturas reduci nequeunt; licet aliquid de fundamentis ejus poſuiſſem. Cæterum in tractatu iſto, ſeries infinitæ non magnam partem obtinebant.

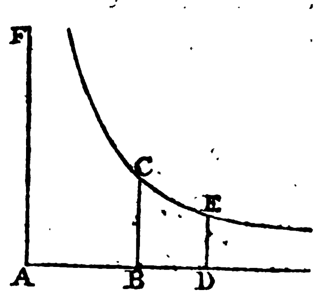
Neque majori labore eruitur area totius circuli ex ſegmento cujus ſagitta eſt quadrans diametri. Ejus computi ſpecimen, ſiquidem ad manus eſt, viſum fuit apponere; & unà adjungere aream hyperbolæ quæ eodem calculo prodit.

Poſito axe tranſverſo $= 1$, & ſinu verſo ſeu ſegmenti ſagitta $= x$; erit ſemi-ſegmentum $\left.\begin{matrix}\text{hyperbolæ}\\ \text{circuli}\end{matrix}\right\} = x^{\frac{1}{2}}$ in $\frac{2}{3}x \pm \frac{xx}{5} - \frac{x^3}{28} \pm \frac{x^4}{72}$ &c. Hæc autem ſeries ſic in infinitum producitur, ſit $2x^{\frac{3}{2}} = a$, $\frac{ax}{2} = b$, $\frac{bx}{4} = c$, $\frac{3cx}{6} = d$, $\frac{5dx}{8} = e$, $\frac{7ex}{10} = f$, &c. Et erit ſemi-ſegmentum $\left.\begin{matrix}\text{hyperbolæ}\\ \text{circuli}\end{matrix}\right\} = \frac{a}{3} \pm \frac{b}{5} - \frac{c}{7} \pm \frac{d}{9} - \frac{e}{11} \pm \frac{f}{13}$ &c. Eorumque ſemi-ſumma $\frac{a}{3} - \frac{c}{7} - \frac{e}{11} -$ &c, & ſemi-differentia $\frac{b}{5} + \frac{d}{9} + \frac{f}{13} +$ &c. His ita præparatis, ſuppono $x = \frac{1}{4}$, quadrantem nempe axis; & prodit a $(= \frac{1}{4}) = 0.25$; b $(= \frac{ax}{2} = \frac{0.25}{1 \times 8}) = 0.03125$: c $(= \frac{bx}{4} = \frac{0.03125}{2 \times 8}) = 0.001953125$: d $(= \frac{3cx}{6} = \frac{0.001953125}{8}) = 0.000244140625$. Et ſic procedo uſque dum venero ad terminum depreſſiſſimum, qui poteſt ingredi opus. Deinde hos terminos per 3, 5, 7, 9, 11, &c, reſpective diviſos diſpono in duas tabulas: ambiguos cum primo in unam; & negativos in aliam; & addo ut hic vides,

0.0833333333333333	0.0002790178571429
6250000000000	34679066051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	0.0002825719389575
1	
0.0896109885646618	

Tunc

Tunc à priori summa aufero posteriorem, & restat 0.0893284166257043 area semi-segmenti hyperbolici. Addo etiam eas summas, & aggregatum aufero a primo termino duplicato 0.1666666666666666, & restat 0.0767731061630473 area semi-segmenti circularis. Huic addo triangulum istud quo completur in sectorem, hoc est $\frac{1}{32}\sqrt{3}$, seu 0.0541265877365274, & habeo sectorem 60 graduum, 0.1308996938995747, cujus sextuplum 0.7853981633974482 est area totius circuli: quæ divisa per $\frac{1}{4}$ sive quadrantem diametri, dat totam peripheriam 3.1415926535897928. Si alias artes adhibuissem, potui per eundem numerum terminorum seriei pervenisse ad multo plura loca figurarum, puta viginti quinque aut amplius: sed animus fuit hic ostendere, quid per simplex seriei computum præstari posset. Quod sane haud difficile est, cum in omni opere multiplicatores ac divisores magna ex parte non majores quam 11, & nunquam majores quam 41 adhibere opus sit.



Neque observasse videtur [clarissimus *Leibnitius*] morem meum generaliter usurpandi literas pro quantitatibus cum signis suis + & − affectis, dum dividit hanc seriem $\frac{x}{b} + \frac{xx}{2abb} + \frac{x^3}{6aab^3} + \frac{x^4}{24a^3b^4} +$ &c. Nam cum area hyperbolica BE, hic significata per z, sit affirmativa vel negativa, prout jaceat ex una vel altera parte ordinatim applicatæ BC; si area illa in numeris data sit l, & l substituatur in serie pro z, orietur vel $\frac{l}{b} + \frac{ll}{2abb} + \frac{l^3}{6aab^3} + \frac{l^4}{24a^3b^4}$ &c, vel $-\frac{l}{b} + \frac{ll}{2abb} - \frac{l^3}{6aab^3} + \frac{l^4}{24a^3b^4}$ &c; prout l sit affirmativa vel negativa. Hoc est posito $a = 1 = b$, & l logarithmo hyperbolico; numerus ei correspondens erit $1 + \frac{l}{1} + \frac{ll}{2} + \frac{l^3}{6} + \frac{l^4}{24}$ &c, si l sit affirmativus; & $1 - \frac{l}{1} + \frac{ll}{2} - \frac{l^3}{6} + \frac{l^4}{24}$ &c, si l sit negativus. Hoc modo fugio multiplicationem theorematum, quæ alias in nimiam molem crescerent. Nam v. g. illud unicum theorema, quod supra posui pro quadratura curvarum, resolvendum esset in 32 theoremata, si pro signorum varietate multiplicaretur.

Præterea, quæ habet vir clarissimus de inventione numeri unitate majoris per datum logarithmum hyperbolicum, ope seriei $\frac{l}{1} - \frac{ll}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} - \frac{l^4}{1 \times 2 \times 3 \times 4}$ + &c, potius quam ope seriei $\frac{l}{1} + \frac{ll}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^4}{1 \times 2 \times 3 \times 4}$ + &c, nondum percipio. Nam si unus terminus adjiciatur amplius ad seriem posteriorem quam ad priorem, posterior magis appropinquabit. Et certe minor est labor computare unam vel duas primas figuras adjecti hujus termini, quam dividere unitatem per numerum prodeuntem ex logarithmo hyperbolico ad multa figurarum loca extensum, ut inde habeatur numerus quæsitus unitate major. Utraque igitur series (si duas dicere fas sit) officio suo fungatur. Potest tamen $\frac{l}{1} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^5}{1 \times 2 \times 3 \times 4 \times 5}$ &c, series, ex dimidia parte terminorum constans, optime adhiberi;

ſiquidem hæc dabit ſemi-differentiam duorum numerorum, ex qua et rectangulo dato uterque datur. Sic & ex ſerie $1 + \frac{ll}{1 \times 2} + \frac{l^4}{1 \times 2 \times 3 \times 4}$ &c, datur ſemi-ſumma numerorum, indeque etiam numeri. Unde prodit relatio ſerierum inter ſe, qua ex unâ datâ dabitur altera.

Conſtructionem logarithmorum non aliunde peti debere credetis forte, ex hoc ſimplici proceſſu qui ab iſtis pendet. Per methodum ſupra traditam quærantur logarithmi hyperbolici numerorum 10, 0.98, 0.99, 1.01, 1.02 : id quod fit ſpatio unius & alterius horæ. Dein diviſis logarithmis quatuor poſteriorum per logarithmum numeri 10, & addito indice 2, prodibunt veri logarithmi numerorum 89, 99, 100, 101, 102, in tabulam referendi. Hi per dena intervalla interpolandi ſunt, & exibunt logarithmi omnium numerorum inter 980 & 1020: & omnibus inter 980 & 1000 iterum per dena intervalla interpolatis, habebitur tabula eatenus conſtructa. Tunc ex his colligendi erunt logarithmi omnium primorum numerorum & eorum multiplicium, minorum quam 100: ad quod nihil requiritur præter additionem & ſubtractionem. Siquidem fit $\sqrt[10]{\frac{9984 \times 1020}{9945}} = 2$, $\sqrt[4]{\frac{8 \times 9963}{984}} = 3$, $\frac{10}{2} = 5$, $\sqrt{\frac{98}{2}} = 7$, $\frac{99}{9} = 11$, $\frac{1001}{7 \times 11} = 13$, $\frac{102}{6} = 17$, $\frac{988}{4 \times 13} = 19$, $\frac{9936}{16 \times 27} = 23$, $\frac{986}{2 \times 17} = 29$, $\frac{992}{32} = 31$, $\frac{999}{27} = 37$, $\frac{984}{24} = 41$, $\frac{989}{23} = 43$, $\frac{987}{21} = 47$, $\frac{9911}{11 \times 17} = 53$, $\frac{9971}{13 \times 13} = 59$, $\frac{9882}{2 \times 81} = 61$, $\frac{9949}{3 \times 49} = 67$, $\frac{994}{14} = 71$, $\frac{9928}{8 \times 17} = 73$, $\frac{9954}{7 \times 18} = 79$, $\frac{996}{12} = 83$, $\frac{9968}{7 \times 16} = 98$, $\frac{9894}{6 \times 17} = 97$. Et habitis ſic logarithmis omnium numerorum minorum quam 100, reſtat tantum hos etiam ſemel atque iterum per dena intervalla interpolare.

END OF EXTRACTS FROM COLLINS.

THE

THE

TWELFTH CHAPTER

OF

DR. WALLIS'S TREATISE OF ALGEBRA.

Of Logarithms: Their Invention and Use.

THE other improvement which I mentioned (as added to the algorifme of the Arabs fince we borrowed it from them), is that of Logarithms; an improvement of our own age and nation.

This was firft of all invented (without any example of any before him, that I know of) by John Neper, Baron of Merchifton in Scotland; and by him firft publifhed at Edinburgh, in the year 1614: and foon after by himfelf (with the affiftance of Henry Brigges, Profeffor of Geometry, firft at London, in Grefham College, and afterwards at Oxford) reduced to a better form, and perfected.

The invention was greedily embraced (and defervedly) by learned men.

Mr. Brigges, upon the firft publication of it, was fo pleafed with it, that he prefently repaired into Scotland, to confult the author, advife with him, and be affiftant to him in the perfecting of it, and in calculating tables for it; which was a work of great labour, as well as fubtile invention.

And it was embraced and promoted abroad by Benjamin Urfinus, John Kepler, Adrian Ulack, Petrus Crugerus, and others.

And at home by Henry Gellebrand, who perfected the Trigonometria Britannica, which Mr. Brigges began, but died before it was perfected.

So that, in a fhort time, it became generally known, and greedily embraced in all parts, as of unfpeakable advantage; efpecially for eafe and expedition in trigonometrical calculations.

The foundation of it is this:

If to a rank of continual proportionals in a geometrical progreſſion from 1: ſuppoſe

1. 2. 4. 8. 16. 32. 64. &c.

We accommodate a rank of exponents in an arithmetical progreſſion, from 0: ſuppoſe

0. 1. 2. 3. 4. 5. 6. &c.

It is manifeſt, that for every multiplication or diviſion of thoſe terms one by another, there is an anſwerable addition or ſubduction of the exponents.

For as (in the terms) 4 multiplied by 8 makes 32, ſo (in the exponents) if to 2 we add 3, it makes 5; and as 32 divided by 8, gives 4: ſo if from 5 we ſubduct 3, there remains 2: and ſo every where.

Terms.	1.	2.	4.	8.	16.	32.	64.
Exponents.	0.	1.	2.	3.	4.	5.	6.

$$4 \times 8 = 32. \qquad \tfrac{32}{8} = 4.$$
$$2 + 3 = 5. \qquad 5 - 3 = 2.$$

(Not much unlike to what we before ſhewed out of Archimedes's Arenarius, concerning his α, β, γ, δ, &c, in continual progreſſion geometrical from 1, attended by a ſeries of exponents in arithmetical progreſſion; the foundation of that and this being all one.)

And the ſame holds, if between any two of thoſe terms, interpoſe one or more means proportional; and between their exponents, as many arithmetical means.

As if between 4 and 8 (or between 2 and 16) we interpoſe a mean proportional $\sqrt{32}$, that is $4\sqrt{2}$; and between 2 and 3 (or 1 and 4) an arithmetical mean, $2\frac{1}{2}$; then as $4\sqrt{2}$ by 8 makes $32\sqrt{2}$ (a mean proportional between 32 and 64): ſo adding their exponents $2\frac{1}{2}$ and 3, makes $5\frac{1}{2}$, an arithmetical mean between 5 and 6; and ſo every where.

And univerſally, (whatever be the values of r. e.) ſuppoſing

The terms, $1.\ r.\ rr.\ rrr.\ r^4.\ r^5.\ r^6.$ &c.
Exponents, $0.\ e.\ 2e.\ 3e.\ 4e.\ 5e.\ 6e.$ &c.
Then, as $rr \times r^3 = r^5$, and $rr\sqrt{r} \times rrr = r^5\sqrt{r}$;
So $2e + 3e = 5e$, and $2\frac{1}{2}e + 3e = 5\frac{1}{2}e$.

And ſo every where.

And conſequently whatever term we interpoſe between any of thoſe continual proportionals; if we alſo interpoſe between their exponents, a like arithmetical mean, as that is a proportional mean (as if that be the firſt or ſecond of two means proportional, this accordingly is the firſt or ſecond of two means arithmetical; if that the ſecond of five means proportional, this the ſecond of as many

many arithmetical means, &c) then to every addition or ſubduction of theſe one with another, will anſwer a like multiplication or diviſion of thoſe.

And if for o, e, $2e$, $3e$, &c, (taking $e = 1$) we put o, 1, 2, 3, &c, then doth this exponent always give us the number of rations or dimenſions in the term to which it belongs.

$$1.\ r.\ rr.\ r^3.\ r^4.\ r^5.\ r^6.\ \&c.$$
$$0.\ 1.\ 2.\ 3.\ 4.\ 5.\ 6.\ \&c.$$

(as 3 in r^3, 6 in r^6, and ſo every where) or ſhews how many fold *(quam multiplicata)* the proportion (for inſtance) of r^6 to 1 is of r to 1. That is, how many rations or proportions of r to 1 are compounded in r^6 to 1, to wit 6; to which the name Logarithmus fitly anſwers, that is λόγων ἀριθμὸς, the number or proportions ſo compounded.

Now this foundation being laid, their deſign in the logarithms is this: Having ſelected (as moſt convenient) a rank of continual proportionals, in a decuple progreſſion; to wit,

1. 10. 100. 1000. 10000. 100000. 1000000. &c.

they fit hereunto (as their exponents) in arithmetical progreſſion,

0. 1. 2. 3. 4. 5. 6. &c.

(And conſequently the logarithm of any fractions leſs than 1 is to be a negative number.) And then for each of the numbers interpoſed between 1 and 10, between 10 and 100, and ſo of the reſt (as 2, 3, 4, &c, 11, 12, 13, &c); they ſeek out (between o and 1, between 1 and 2, &c) an exponent (to be expreſſed in decimal parts), which is ſuch a mean arithmetical as the other is a mean proportional.

And theſe exponents they call logarithms, which are artificial numbers, ſo anſwering to the natural numbers, as that the addition and ſubduction of theſe anſwers to the multiplication and diviſion of the natural numbers.

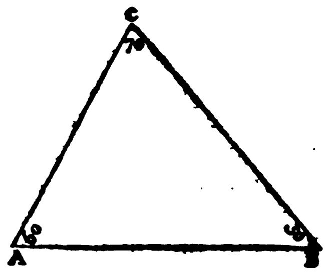
By this means (the tables being once made) the work of multiplication and diviſion is performed by addition and ſubduction; and conſequently that of ſquaring and cubing, by duplation and triplation, and that of extracting the ſquare and cubick root, by biſection and triſection; and the like in higher powers.

Of theſe logarithms we have printed tables, for all numbers as far as one hundred thouſand, ſo that if any two numbers (not exceeding 100,000) be propoſed to be multiplied or divided one by the other, the logarithms of thoſe numbers (to be found in thoſe printed tables) being accordingly added or ſubducted, will give the logarithm of that natural number (to be found by thoſe tables), which is the product or quotient of ſuch multiplication or diviſion. And the double or treble of ſuch logarithm, is the logarithm of its ſquare or cube. And the half or third part of it, is the logarithm of its quadratick or cubick root; and the like of higher powers, which, in large numbers, is matter of great expedition.

And

And (becauſe a main end of this deſign was to facilitate aſtronomical and other trigonometrical calculations) beſide thoſe logarithms for numbers in their natural order, we have alſo tables of artificial or logarithmical ſines, tangents, and ſecants; the addition and ſubduction of which anſwers to the multiplication and diviſion of the natural ſines, tangents, and ſecants: which is a very compendious advantage for expediting ſuch calculations, and is not leſs accurate than the operation by tables of natural ſines, tangents, and ſecants.

Thus in a plain triangle; ſuppoſing the angles given, A 60 degrees, B 50 degrees (and conſequently C 70 degrees), and the ſide AB 31323 paces: for finding the ſides AC, or BC, we have this proportion:



As the ſine of C, 70 degrees,	9396926	
To the ſine of B, 50 degrees,	7660444	
So is the ſide AB,	31323	paces.
To the ſide AC.	25535—	paces.

For finding which, we are to multiply 7660444 by 31323, and then divide by 9396926; which gives for the ſide AC (almoſt) 25535 paces.

And, as the ſine of C, 70 degrees,	9396926	
To the ſine of A, 60 degrees,	8660254	
So is the ſide AB,	31323	paces.
To the ſide BC.	28867½	paces.

For finding which, we are to multiply 8660254 by 31323, and divide by 9396926, which gives for the ſide BC, 28867½ paces, *proximè*.

Now (to prevent theſe tedious multiplications and diviſions) by logarithms, we proceed thus:

Log. ſine C, 70 degrees,	— 9.9729858
Log. ſine B, 50 degrees,	+ 9.8842540
Log. AB, num. 31323,	+ 4.4958633
Log. AC, num. 25535,	+ 4.4071315

where ſubducting the firſt logarithm from the ſum of the ſecond and third, gives the fourth, which (the tables tell us) anſwers to the number 25535, *ferè*. So many paces therefore is the ſide AC.

Again,	Log. ſine C, 70 degrees,	— 9.9729858
	Log. ſine A, 60 degrees,	+ 9.9375306
	Log. AB, num. 31323	+ 4.4958633
	Log. BC, num. 28867½	+ 4.4604081

Where

Where ſubducting the firſt logarithm from the ſum of the ſecond and third, gives the fourth; which (the table tells us) anſwers to the number $28867\frac{1}{2}$, *proximè*. So many paces therefore is the ſide BC; which operations are much more expeditious than multiplying and dividing ſuch large numbers.

And in like manner, in ſpherical triangles, ſave that there all the logarithms are to be taken out of the tables of ſines, tangents, and ſecants; which in this example are taken partly from thence, partly from the table of numbers; but the expedition is alike in both.

This was firſt publiſhed by the Lord Neper (the firſt inventor of it), in the year 1614, under the title of *Mirificus Logarithmorum Canon*, with its deſcription and uſe; but reſerving the manner of conſtruction, and its demonſtration, to be after publiſhed; this being but an eſſay ſet forth to ſee the judgment of learned men concerning this deſign, and how it was like to be received.

In this we have a canon or table of natural and logarithmical ſines for each degree and minute of the quadrant.

And whereas it was at his choice to give to what number he pleaſed the logarithm 0, and whether to proceed by way of increaſe or decreaſe, he choſe to make 0 the logarithm of the whole ſine 10000000, that ſo the multiplication or diviſion by the whole ſine (frequent in trigonometrical calculation) might be diſpatched without trouble, requiring here but the addition or ſubduction of 0.

And becauſe the uſe of leſſer ſines and numbers, leſs than the radius or whole ſine, were likely to be of more frequent uſe than of tangents, ſecants, and other numbers greater than the radius, he choſe to give to thoſe leſſer numbers affirmative logarithms (increaſing the logarithms from 0, as the ſines decreaſe), which he calls *abundantes:* and conſequently negative logarithms (which he calls *defectives)* to greater numbers. Deſigning thoſe by +, theſe by —.

And, by this means, he directs how this table of ſines (with the differences there inſerted) may ſerve alſo for a table of tangents and of ſecants: ſo that this canon is a complete canon of natural ſines, and of logarithmical ſines, tangents, and ſecants.

He ſhews alſo how this table may be applied to the logarithms of abſolute numbers: but becauſe with ſome trouble, he reſerves the fuller account hereof to a farther treatiſe.

In the year 1619, the Lord Neper being then dead, the ſame was again publiſhed by his ſon, Robert Neper, with ſome poſthumous treatiſes of his father, concerning the conſtruction of this Logarithmical Canon, and concerning his deſign (after communication had with Mr. Brigges) of changing the form of logarithms, making 0 to be the logarithm of 1 (of which he had before given notice in the preface to his *Rabdologia*, publiſhed in the year 1617), and concerning ſome things pertaining to trigonometry; with ſome lucubrations of Mr. Brigges on the ſame ſubject.

But, the Lord Neper being dead, the whole work was devolved on Mr. Brigges, who (according to their joint advice) making the logarithm of 1 to be 0, and of 10, 100, 1000, &c, to be 1, 2, 3, &c, which he calls *indices*, or *characteriſtics*, and which we may repute as integer numbers, with fourteen ciphers annexed, which we may repute as ſo many places of decimal fractions

7 below

below the place of units, or of the characteriftick: and between thefe he fits the intermediate logarithms for the intermediate numbers.

And confequently the logarithm of 1 being 0, the logarithm of fractions lefs than 1, or of numbers intermediate between 1 and 0, muft be negative numbers, or numbers lefs than 0 (which he calls defective logarithms), denoted by — (the note of negation) prefixed.

Now thefe defective logarithms may be two ways expreffed; either fo as that the note of negation fhall affect the whole logarithm, or fo as to affect only the characteriftick (leaving the reft of the logarithm to be underftood as affirmative).

As, for example, the fraction $\frac{3}{8}$, or (which is equivalent) 0.375. This fraction fuppofeth the numerator 3 to be divided by the denominator 8, which in logarithms is to be performed by fubtracting the logarithm of 8 from that of 3, and the remainder will be the logarithm of $\frac{3}{8}$, which will then be the negative number, —0.4259687.

Log. 3.	0.4771212
Log. 8.	0.9030899
Log. $\frac{3}{8}$.	— 0.4259687

Or thus; for as much as the logarithm of 375 (fuppofing it to be an integer number) is 2.5740312. And the depreffing this to the firft, fecond, or third, or further place of decimal fraction, doth (without altering the figures) divide the value by 10, 100, 1000, &c; which in logarithms is done by fubtracting 1, 2, 3, &c, from the characteriftick or place of integers (1, 2, 3, &c, in that place, being the logarithms of 10, 100, 1000, &c). Such alteration of the value (the figures remaining) is done by altering the characteriftick of the logarithm, without varying the other figures in this manner.

Log.	3750	3.5740312
Log.	375	2.5740312
Log.	37|5	1.5740312
Log.	3|75	$\underline{0}$.5740312
Log.	0|375	$\underline{1}$.5740312
Log.	0|0375	2.5740312

Which two forms, though they feem different, and fome may rather choofe the one, fome the other; or in fome cafes the one, in fome cafes the other; yet they are in fubftance or value the fame. For

$$\begin{array}{r} -\ 1.0000000 \\ +\ 0.5740312\ + \\ \hline \text{is} = -\ 0.4259687 \end{array}$$

And every one is left to his liberty, whether of the two ways (or what other equivalent thereunto) he fhall pleafe to ufe.

In this method Mr. Brigges hath calculated a table of logarithms (publifhed in the year 1624) for twenty chiliads of abfolute numbers (from 1 to 20,000); and again for 10 more (from 90,000 to 100,000), and one chiliad fupernumerary (to wit, the hundred and firft chiliad), that is 31 chiliads in all.

Before which is prefixed, a large account of the nature and conftruction of this Logarithmical Canon, and the ufes thereof; and direction how to fupply the

the intermediate chiliads which are here wanting. The whole intituled *Arithmetica Logarithmica.*

The ſame is again publiſhed in the year 1628, by Adrian Vlacq (or Flack), with a ſupplement (as Mr. Brigges directed) of the chiliads before omitted; that is, in all, of 100 chiliads, with one ſupernumerary. But in ſhorter numbers, extending but to 10 places below that of the integers, or the characteriſtick. And he ſubjoins alſo a logarithmical canon of ſines, tangents, and ſecants (for degrees and minutes of the quadrant) of as many places.

Mr. Brigges proceeded to calculate a trigonometrical canon logarithmical, ſuited to that for abſolute numbers, to the logarithms extending (as in that other) to 14 places, beſide the characteriſtick. And having before calculated a table of natural ſines, tangents, and ſecants (for degrees and centeſmes of degrees) in numbers extending to 15 places, he fitted thereunto a canon of logarithmical ſines and tangents (becauſe thoſe of ſecants might be ſpared); and a treatiſe prefixed concerning the conſtruction thereof, with other things pertinent thereunto; intending a farther treatiſe concerning the uſe of it.

But dying before this laſt was finiſhed, or the reſt publiſhed, Mr. Henry Gellibrand ſupplied this latter, and publiſhed the whole, with the title of *Trigonometria Britannica*, in the year 1633. To which is ſubjoined another canon of logarithmical ſines and tangents, by Adrian Vlacq, for degrees, minutes, and tenth ſeconds, extending (as his former did) to 10 places, beſide the characteriſtick; and Mr. Brigges's 20 chiliads for logarithms of abſolute numbers.

So that the whole doctrine of logarithms was by this time ſufficiently perfected, with convenient canons or tables fitted thereunto, in large numbers: of which alſo Petrus Crugerus gives an account in the preface to his *Trigonometria Logarithmica*, printed in the year 1634, with his logarithmical tables, but in ſhorter numbers.

And the tables of logarithms above mentioned (for 100 chiliads of abſolute numbers, and for ſines and tangents to degrees and centeſmes) were the ſame year, 1633, contracted into a leſſer form, and more manageable (but in ſhorter numbers, the former not extending to above ſeven places, beſide the characteriſtick, but the latter to 10); by Nathaniel Roe; with directions for the uſe of them (in trigonometry, geometry, aſtronomy, geography, and navigation), by Edmund Wingate.

In the mean time, Benjamin Urſinus did alſo publiſh tables of logarithms in the year 1618, and again in the year 1625, in his *Trigonometria*; and Johannes Keplerus alſo in the year 1624, in his *Chilias Logarithmorum* (which he applies alſo to his *Rudolphine Tables*, publiſhed in 1627), and Claudius Batſchius about the ſame time, or ſoon after; and Georgius Ludovicus Frobenius, in the year 1634, and perhaps ſome others. But all, or moſt of them, in ſhort numbers, and conformable to the Lord Neper's firſt deſign; not to that form, which, upon ſecond thoughts, he and Mr. Brigges agreed upon as moſt eligible, and which hath ſince been received in common practice.

Since which time, much hath not been added to the doctrine of logarithms; not was it neceſſary, that work having obtained ſufficient perfection.

But in case logarithms on any emergent occasion be desirable with greater exactness, and in larger numbers than those printed tables do afford, Mr. Nicholas Mercator, in a small treatise, called *Logarithmotechnia*, printed in the year 1668, shews (with great subtilty) how it may be effected, in numbers of whatever length desirable, with much more ease than heretofore.

Nor shall I need to add more concerning logarithms: those who desire farther, may find it in the authors above mentioned; especially Mr. Brigges's *Arithmetica Logarithmica* and *Trigonometria Britannica*, with Adrian Vlac's additions to both.

Without farther insisting, therefore, on the algorism by numeral figures (with the improvements thereof since we had them from the Arabs), I shall return to what doth more immediately concern Algebra.

END OF THE TWELFTH CHAPTER OF WALLIS'S ALGEBRA.

EXTRACT

EXTRACT

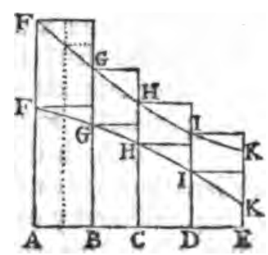
FROM THE

PHILOSOPHICAL TRANSACTIONS.

A Letter from the Reverend Dr. Wallis, Professor of Geometry in the University of Oxford, and Fellow of the Royal Society, London, to Mr. Richard Norris; concerning the Collection of Secants, and the true Division of the Meridians in the Sea Chart.

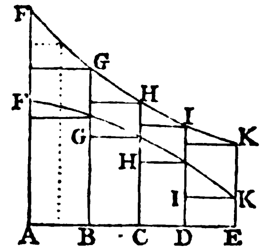
AN old inquiry (about the ſum or aggregate of ſecants) having been of late moved anew, I have thought fit to trace it from its original, with ſuch ſolution as ſeems proper to it; beginning firſt with the general preparation, and then applying it to the particular caſe.

GENERAL PREPARATION.



1. Becauſe curve lines are not ſo eaſily managed as ſtraight lines: the ancients, when they were to conſider of figures terminated (at leaſt on one ſide) by a curve line (convex or concave), as AFKE, did oft make uſe of ſome ſuch expedient as this following (but diverſely varied as occaſion required), namely,

2. By



2. By parallel ſtraight lines, as AF, BG, CH, &c (at equal or unequal diſtances, as there was occaſion), they parted it into ſo many ſegments as they thought fit; or ſuppoſed it to be ſo parted.

3. Theſe ſegments were *ſo many wanting one*, as was the number of thoſe parallels.

4. To each of theſe parallels wanting one, they fitted parallelograms, of ſuch breadths as were the intervals (equal or unequal) between each of them reſpectively, and the next following; which formed an adſcribed figure made up of thoſe parallelograms.

5. And if they began with the greateſt (and therefore neglected the leaſt) ſuch figure was circumſcribed (as fig. 1), and therefore bigger than the curvilinear propoſed.

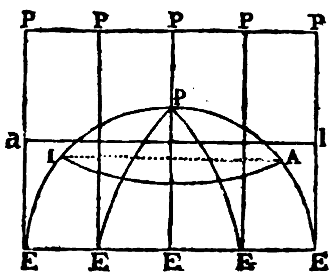
6. If with the leaſt (neglecting the greateſt), the figure was inſcribed (as fig. 2), therefore leſs than that propoſed.

7. But, as the number of ſegments was increaſed (and thereby their breadths diminiſhed), the difference of the circumſcribed from the inſcribed (and therefore of either from that propoſed) did continually decreaſe, ſo as at laſt to be leſs than any aſſigned.

8. On which they grounded their method of exhauſtions.

9. In caſes wherein the breadth of the parallelograms, or intervals of the parallels, is not to be conſidered, but their length only; or, which is much the ſame, where the intervals are all the ſame, and each reputed = 1; Archimedes (inſtead of inſcribed and circumſcribed figures) uſed to ſay, "All except the greateſt, and all except the leaſt:" as prop. 11 Lin. Spiral.

PARTICULAR CASE.



10. Though it be well known, that, in the terreſtrial globe, all the meridians meet at the pole, as EP, EP, whereby the parallels to the equator, as they be nearer to the pole, do continually decreaſe.

11. And hereby a degree of longitude in ſuch parallels, is leſs than a degree of longitude in the equator, or a degree of latitude.

12. And that, in ſuch proportion, as is the co-ſine of latitude (which is the ſemidiameter of ſuch parallel), to the radius of the globe, or of the equator.

13. Yet hath it been thought fit (for ſome reaſons) to repreſent theſe meridians, in the ſea chart, by parallel ſtraight lines, as EP, EP.

14. Whereby each parallel to the equator (as LA) was repreſented in the ſea chart (as *la*) as equal to the equator EE, and a degree of longitude therein as large as in the equator.

15. By

15. By this means each degree of longitude in ſuch parallels was increaſed beyond its juſt proportion, at ſuch rate as the equator, or its radius, is greater than ſuch parallel, or the radius thereof.

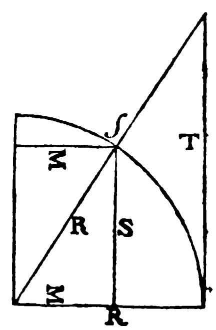
16. But in the old ſea charts, the degrees of latitude were yet repreſented (as they are in themſelves) equal to each other, and to thoſe of the equator.

17. Hereby, amongſt many other inconveniencies (as Mr. Edward Wright obſerves, in his *Correction of Errors in Navigation*, firſt publiſhed in the year 1599), the repreſentation of places remote from the equator was ſo diſtorted in thoſe charts, as that, for inſtance, an iſland in the latitude of 60 degrees (where the radius of the parallel is but half ſo great as that of the equator) would have its length, from Eaſt to Weſt, in compariſon of its breadth, from North to South, repreſented in a double proportion of what indeed it is.

18. For rectifying this in ſome meaſure (and of ſome other inconveniencies), Mr. Wright adviſeth, that (the meridians remaining parallel, as before) the degrees of latitude remote from the equator, ſhould, at each parallel, be protracted in like proportion with thoſe of longitude.

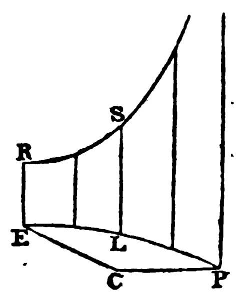
19. That is, as the co-ſine of latitude (which is the ſemi-diameter of the parallel) to the radius of the globe (which is that of the equator), ſo ſhould be a degree of latitude (which is every where equal to a degree of longitude in the equator) to ſuch a degree of latitude ſo protracted (at ſuch diſtance from the equator), and ſo to be repreſented in the chart.

20. That is, every where in ſuch proportion as is the reſpective ſecant (for ſuch latitude) to the radius. For as the co-ſine to the radius, ſo is the radius to the ſecant (of the ſame arch or angle), as $\Sigma . R :: R . \mathit{ſ}$.



21. So that, by this means, the poſition of each parallel in the chart ſhould be at ſuch diſtance from the equator, compared with ſo many equinoctial degrees or minutes (as are thoſe of latitude), as are all the ſecants (taken at equal diſtances in the arch) to ſo many times the radius.

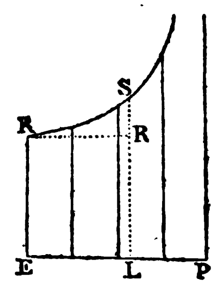
22. Which is equivalent (as Mr. Wright there notes) to the projection of the ſpherical ſurface (ſuppoſing the eye at the center) on the concave ſurface of a cylinder, erected at right angles to the plain of the equator.



23. And the diviſion of meridians, repreſented by the ſurface of a cylinder erected (on the arch of latitude) at right angles, to the plain of the meridian, or a portion thereof. The altitude of ſuch projection, or portion of ſuch cylindrick ſurface, being, at each point of ſuch circular baſe, equal to the ſecant of latitude anſwering to ſuch point.

24. This

24. This projection (or portion of the cylindrick ſurface) if expanded into a plain, will be the ſame with a plain figure, whoſe baſe is equal to a quadrantal arch extended, or a portion thereof, on which, as ordinates, are erected perpendiculars equal to the ſecants, anſwering to the reſpective points of the arch ſo extended; the leaſt of which (anſwering to the equinoctial) is equal to the radius, and the reſt continually increaſing till, at the pole, it be infinite.



25. So that as ERSL (a figure of ſecants erected at right angles on EL, the arch of latitude extended) to ERRL (a rectangle on the ſame baſe, whoſe altitude ER is equal to the radius), ſo is EL (an arch of the equator equal to that of latitude) to the diſtance of ſuch parallel (in the chart) from the equator.

26. For finding this diſtance anſwering to each degree and minute of latitude, Mr. Wright (as the moſt obvious way) adds all the ſecants (as they are found calculated in the Trigonometrical Canon) from the beginning, to the degree or minute of latitude propoſed.

27. The ſum of all which, except the greateſt (anſwering to the figure inſcribed), is too little; the ſum of all, except the leaſt (anſwering to the circumſcribed), is too great (which is that he follows; and it would be nearer to the truth than either, if (omitting all theſe) we take the intermediates; for min. $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c, or (the doubles of theſe) min. 1, 3, 5, 7, &c: which yet (becauſe on the convex ſide of the curve) would be ſomewhat too little.

28. But any of theſe ways are exact enough for the uſe intended, as creating no ſenſible difference in the chart.

29. If we would be more exact, Mr. Oughtred directs (and ſo had Mr. Wright done before him) to divide the arch into parts yet ſmaller than minutes, and calculate ſecants ſuiting thereunto.

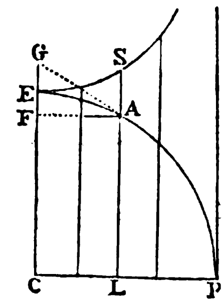
30. Since the arithmetick of infinites introduced, and, in purſuance thereof, the doctrine of infinite ſeries (for ſuch caſes as would not, without them, come to a determinate proportion), methods have been found for ſquaring ſome ſuch figures, and particularly the exterior hyperbola (in a way of continual approach), by the help of an infinite ſeries. As in the Philoſophical Tranſactions, Numb. 38, for the month of Auguſt, 1668; and my book, De Motu, cap. 5, prop. 31.

31. In imitation whereof, it hath been deſired (I find) by ſome, that a like quadrature for this figure of ſecants (by an infinite ſeries fitted thereunto) might be found.

32. In order to which, put we for the radius of a circle, R; the right ſine of an arch or angle, S; the verſed ſine, V; the co-ſine (or ſign of the complement) $\Sigma = R - V = \sqrt{:RQ - SQ}$: the ſecant, $ſ$; the tangent T. Fig. 4.

33. Then is $\Sigma . R :: R . ſ$, that is, $\Sigma)\ R^2\ (s = \frac{R^2}{\Sigma}$; the ſecant.

34. And $\Sigma . S :: R . T$, that is, $\Sigma)\ SR\ (T = \frac{SR}{\Sigma}$; the tangent.

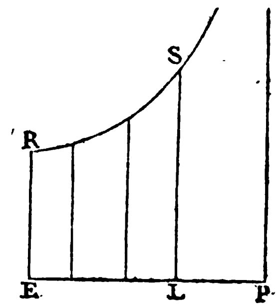


35. Now if we ſuppoſe the radius CP divided into equal parts (and each of them $= \frac{1}{100}$ R), and on theſe to be erected the co-ſines of latitude LA :

36. Then are the ſines of latitude in arithmetick progreſſion.

37. And the ſecants anſwering thereunto, $L\int = \frac{R^2}{\Sigma}$.

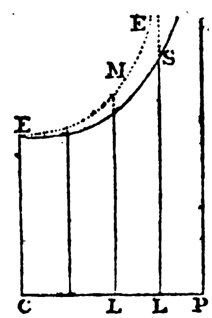
38. But theſe ſecants (anſwering to right ſines in arithmetical progreſſion) are not thoſe that ſtand at equal diſtance on the quadrantal arch extended. Fig. 6.



39. But ſtanding at unequal diſtances on the ſame extended arch ; namely, on thoſe points thereof whoſe right ſines (whilſt it was a curve) are in arithmetical progreſſion.

40. To find therefore the magnitude of RELſ, fig. 6, which is the ſame with this (ſuppoſing EL of the ſame length in both, however the number of ſecants therein may be unequal), we are to conſider the ſecants, though at unequal diſtances here, to be the ſame with thoſe at equal diſtances in the preceding figure, anſwering to ſines in arithmetical progreſſion.

41. Now theſe intervals, or portions of the baſe, are the ſame with the intercepted arches, or portions of the arch, in the preceding figure ; for this baſe is but that arch extended.



42. And theſe arches, in parts infinitely ſmall, are to be reputed equivalent to the portions of their reſpective tangents intercepted between the ſame ordinates. As in fig. 7 and 9.

43. That is, equivalent to the portions of the tangents of latitude.

44. And theſe portions of tangents are to the equal intervals in the baſe, as the tangent of latitude to its ſine.

45. To find therefore the true magnitude of the parallelograms, or ſegments of the figure, we muſt either protract the equal ſegments of the baſe (fig. 7), in ſuch proportion as is the reſpective tangent to the ſine, to make them equal to thoſe of fig. 8.

46. Or elſe, which is equivalent, retaining the equal intervals of fig. 7, protract the ſecants in the ſame proportion ; for either way the intercepted rectangles or parallelograms will be equally increaſed ; as LM, fig. 9.

47. Namely ; as the ſine of latitude to its tangent, ſo is the ſecant to a fourth ; which is to ſtand on the radius equally divided, inſtead of that ſecant.

$$S . \frac{SR}{\Sigma} (:: \Sigma . R) :: \frac{R^2}{\Sigma} . \frac{R^3}{\Sigma^2 = R^2 - S^2} = LM, \text{ fig. } 9.$$

48. Which therefore are as the ordinates in (what I call Arith. Infin. prop. 104) Reciproca Secundinorum, ſuppoſing Σ^2 to be ſquares in the order of ſecundanes.

49. This

$$\begin{array}{l} R^2 - s^2)\ R^3\ (R + \frac{s^2}{R} + \frac{s^4}{R^3} + \\ \quad R^3 - s^2R \\ \hline \quad\quad + s^2R \\ \quad\quad + s^2R - \frac{s^4}{R} \\ \hline \quad\quad\quad + \frac{s^4}{R} \\ \quad\quad\quad + \frac{s^4}{R} - \frac{s^6}{R^3} \\ \hline \quad\quad\quad\quad + \frac{s^6}{R^3} \end{array}$$

49. This becauſe of $\Sigma^2 = R^2 - s^2$, and the ſines s, in arithmetical progreſſion, is reduced by diviſion into this infinite ſeries:

$$R + \frac{s^2}{R} + \frac{s^4}{R^3} + \frac{s^6}{R^5} \text{ \&c.}$$

50. That is (putting $R = 1$), $1 + s^2 + s^4 + s^6$ &c.

51. Then (according to the arithmetick of infinites) we are to interpret s, ſucceſſively, by 1s, 2s, 3s, &c, till we come to s, the greateſt; which therefore repreſents the number of all.

52. And becauſe the firſt member doth repreſent a ſeries of equals, the ſecond of ſecundans, the third of quartans, &c; therefore the firſt member is to be multiplied by s, the ſecond by $\frac{1}{3}$s, the third by $\frac{1}{5}$s, the fourth by $\frac{1}{7}$s, &c.

53. Which makes the aggregate

$$s + \tfrac{1}{3}s^3 + \tfrac{1}{5}s^5 + \tfrac{1}{7}s^7 + \tfrac{1}{9}s^9 \text{ \&c} = \text{ECLM, fig. 9.}$$

54. This (becauſe s is always leſs than $R = 1$) may be ſo far continued, till ſome power of s become ſo ſmall as that it, and all which follow it, may be ſafely neglected.

55. Now (to fit this to the ſea chart, according to Mr. Wright's deſign) having the propoſed parallel of latitude given, we are to find, by the Trigonometrical Canon, the ſine of ſuch latitude, and take, equal to it, CL = s. And, by this, find the magnitude of ECLM, fig. 9; that is, of RELſ, fig. 8; that is, of RELſ, fig. 6. And then, as RRLE, or ſo many times the radius, to RELſ (the aggregate of all the ſecants), ſo muſt be a like arch of the equator (equal to the latitude propoſed) to the diſtance of ſuch parallel (repreſenting the latitude in the chart) from the equator; which is the thing required.

56. The ſame may be obtained, in like manner, by taking the verſed ſines in arithmetical progreſſion. For if the right ſines, as here, beginning at the equator, be in arithmetical progreſſion, as 1, 2, 3, &c, then will the verſed ſines beginning at the pole (as being their complements to the radius) be ſo alſo.

THE COLLECTION OF TANGENTS.

57. The ſame may be applied in like manner (though that be not the preſent buſineſs) to the aggregate of tangents, anſwering to the arch divided into equal parts.

58. For thoſe anſwering to the radius ſo divided, are $\frac{sR}{\Sigma}$; taking s in arithmetical progreſſion.

59. And

59. And then enlarging the base (as in fig. 8), or the tangent (as in fig. 9), in the proportion of the tangent to the sine.

$$s \,.\, \frac{sR}{\Sigma} \;(:: \Sigma \,.\, R) :: \frac{sR}{\Sigma} \,.\, \frac{sR^2}{\Sigma^2} = \frac{sR^2}{R^2 - s^2}.$$

60. We have by division this series,

$$s + \frac{s^3}{R^2} + \frac{s^5}{R^4} + \frac{s^7}{R^6} + \frac{s^9}{R^8} \text{ \&c.}$$

$$R^2 - s^2)\; sR^2 \;\left(s + \frac{s^3}{R^2} + \frac{s^5}{R^4} + \right.$$

$$\underline{sR^2 - s^3}$$

$$+ s^3$$

$$\underline{+ s^3 - \frac{s^5}{R^2}}$$

$$+ \frac{s^5}{R^2}$$

$$\underline{+ \frac{s^5}{R^2} - \frac{s^7}{R^4}}$$

$$+ \frac{s^7}{R^4}$$

61. That is (putting $R = 1$),

$$s + s^3 + s^5 + s^7 + s^9 \text{ \&c.}$$

62. Which (multiplying the respective members by $\frac{1}{2}s$, $\frac{1}{4}s$, $\frac{1}{6}s$, $\frac{1}{8}s$, $\frac{1}{10}s$, &c) becomes

$$\tfrac{1}{2}s^2 + \tfrac{1}{4}s^4 + \tfrac{1}{6}s^6 + \tfrac{1}{8}s^8 + \tfrac{1}{10}s^{10} \text{ \&c.}$$

Which is the aggregate of tangents to the arch, whose right sine is s.

63. And this method may be a pattern for the like process in other cases of like nature.

LOGARITHMOTECHNIA:

OR, THE

MAKING OF NUMBERS

CALLED

LOGARITHMS

TO TWENTY-FIVE PLACES,

FROM A

GEOMETRICAL FIGURE,

WITH SPEED, EASE, AND CERTAINTY.

By EUCLID SPEIDELL, Philomath.

PRINTED AT LONDON, IN THE YEAR 1688.

TO THE READER.

HAVING for ſome years paſt ſhewn to ſeveral perſons the *praxis* of the following Treatiſe, and alſo communicated to them ſomewhat of the doctrine leading thereunto, I was often deſired not to let them ſleep in oblivion, but to publiſh the ſame; which was firſt promoted by my honoured friend Mr. Peter Hoot, merchant, and ſeconded by my loving friend Mr. Reeve Williams; or elſe they had not ſeen the publick. What I have done therein I deſire thee to take in good part, being alſo at proportional charges myſelf, beſides my compoſing thereof, to make it communicable to thee, rather than ſuch an eaſy and certain way to make logarithm numbers (to ſo many places) ſhould not be known in our native tongue. I have called them Geometrical Logarithms, for that the firſt inventors of thoſe numbers had not adapted geometrical figures to them. But the ſcheme hereunto annexed having ſuch properties and affections as logarithm numbers have, hath made me ſo ſtyle them. What I have done herein is to gratify ſuch who have a curioſity to examine logarithm tables, and to make logarithm numbers to ſo ſmall radiuſes as are ſo often printed for common uſes with brevity and exactneſs. Two ſheets of the *praxis* hereof were printed ſome time before the reſt; which having found kind acceptance with divers, induced me alſo to let the remainder be publiſhed. And before the printing thereof, one was writing upon thoſe two ſheets, and was ſo fair to deſire my conſent to publiſh it; which I readily gave; for that I knew him able enough to do it, and when to be at leiſure myſelf to attend the publiſhing of the reſidue, I knew not. But that not being performed by him, I deſire thee to accept of what is done herein as time and leiſure hath permitted. I ſhall not need to write how needful logarithm numbers are in thoſe great and uſeful arts of Navigation, Aſtronomy, Dialling, Fortification and Gunnery, Surveying, Gauging, Intereſt, and Annuities, &c, when, as there are ſo many books written and publiſhed thereof, not only in our own language but in many others. And truly the firſt inventors thereof are not a little to be had in reverence for making and perfecting thoſe numbers with ſo much labour as thoſe methods by which they derived them did require. Here thou mayſt make a logarithm to 7 or 8 places readily and eaſily; but to 25 places would have been very difficult, if not impoſſible, for the firſt inventors to have produced after their ways. If any thing herein ſhall offer whereby thou mayſt make farther improvement, let the publick ſhare of the benefit thereof. Thus wiſhing thee good ſucceſs in all thy ſtudies, is moſt earneſtly deſired by

London,
March 26, 1688.

E. SPEIDELL.

GEOMETRICAL LOGARITHMS.

CHAP. I.

ABOUT the year $167\frac{5}{6}$, being in company with Michael Dairy, a citizen of London (who had for moſt part of his life-time addicted himſelf to mathematical ſtudies, and hath publiſhed divers practical pieces of ſeveral parts of the mathematicks, of good uſe and delight); and diſcourſing about making hyperbolical logarithms, I deſired him to give me a rule to make the hyperbolical logarithm of 10, from the conſideration of an hyperbola inſcribed within a right-angled cone, who gave me this rule following.

To the number propoſed, viz. 10, add an unit, and ſubtract from it an unit, and there will be a reſult of $\frac{9}{11}$; then divide 1, or 100000000 &c, by $\frac{9}{11}$, which is 818181818 &c, which cube *in infinitum*, and divide every one of them (which will be a rank of proportional numbers) by the proper indices of their reſpective powers, that is to ſay, by 3, 5, 7, 9, 11, &c, then the addition of thoſe quotes will make the logarithm of 10.

Finding, then, that 10 divided by $\frac{9}{11}$ maketh 818181818181 &c, and to cube it *in infinitum*, was very difficult, I rejected the rule, and thought it then not much more eaſy than Briggs's way: neither did he tell any reaſon or demonſtration for the ſaid rule; and becauſe in this example I found it ſo intricate, I did not much care to proſecute it, but neglected it. Not long after he departed this life, and ſince his death reſuming the ſaid thing, and trying if it were ſerviceable in any other part of the hyperbola, I ſoon found it a jewel, and could make the hyperbolical logarithm of 10 at twice, that is to ſay, from two parts numbered in any aſymptote, whoſe fact is 10, with eaſe, certainty, and delight, and have made the hyperbolical logarithm of 2 to 25 places, in order to ſee if the learned and laborious Henry Briggs's logarithms were true to 15 places, which were made after a moſt laborious and difficult way of extracting ſquare roots, and, as I have heard, was the work of eight perſons a whole year, and that without any proof but only if any two or more agreed in their extractions, line by

by line, ſtep by ſtep, it was taken *de bene eſſe*; which was a work of very great pains and uncertainty. However, they did effect it; and I do find they made the logarithm of 2 to 15 places very true, as by my operation, hereafter following, will appear, being done to 25 places, and afterwards from theſe hyperbolical logarithms deduced Briggs's logarithms; both which figurative operations were performed and examined by me in 8 hours' time. I took this pains to make the hyperbolical logarithm to 25 places, in order, alſo, to ſee if the moſt ingenious and laborious James Gregory's hyperbolical logarithm would agree with this of mine, which he hath, in his *Quadratura Circuli & Hyperbolæ*, printed at Padua; but I find that his logarithm of 2 correſponds with mine but to 17 places. I muſt confeſs, I did not take the pains to raiſe the logarithm of 2 to 25 places, according to the doctrine he hath delivered in that moſt learned piece, but am contented that this eaſy and certain way I deliver here, and by the operation thereof the hyperbolical logarithm of 2 to 25 places, is as true in the laſt as in any where, and may be examined in a few hours; ſo that any body, if he pleaſe, may be his own examiner and judge, if this way be not eaſy, certain, and ſpeedy.

Having made ſeveral logarithms for digit numbers, and mixt numbers, as for $1\frac{1}{4}$, $1\frac{1}{4}$, which are hereafter inſerted, I find the rule delivered by Michael Dairy is of admirable uſe and benefit in ſquaring the hyperbola, and making logarithms from it.

Some time ſince the death of the ſaid Michael Dairy, I ſhewed unto Mr. John Collins, who I knew had been a great familiar and friend to the ſaid Michael Dairy, the figurative work of my making the hyperbolical logarithms, according to the ſaid Dairy's rule; who ſeemed very well pleaſed with it, acknowledging it to be the ſpeedieſt way could poſſibly be of ſquaring the hyperbola, and making the logarithms from it; and, after a little pauſing on it, replied, "That Dairy muſt have had this rule out of the ſaid James Gregory's works." I made anſwer, "Not from his ſaid *Quadratura Circuli & Hyperbolæ*." He anſwered "No, from his *Exercitationes Geometricæ*, printed at London, 1668."—A book I had not ſeen or heard of till then. And as he the ſaid Mr. Collins had been always very frank and free to communicate any mathematical thing to me, ſo I held myſelf obliged to acquaint him firſt with this work. He ſeemed to admire that Michael Dairy ſhould keep ſuch a thing from him, who had been ſo great a familiar with him in theſe ſtudies. Not long after my diſcovery hereof to the ſaid Mr. Collins, he alſo departed this life; whoſe death, all that were mathematical, and knew him, lamented not a little: for he was not only excellent in mathematical arts and ſciences, but of a very good, affable, and frank nature to communicate any thing he knew to any lover and inquirer of thoſe things; and hath left behind him thoſe mathematical works which will continue his fame amongſt the lovers and ſtudents therein. He alſo, in his lifetime, promoted the publiſhing of other men's mathematical works, as the elaborate Algebra of the learned John Kerſy, who was my father's diſciple about $164\frac{5}{6}$; and alſo of the learned Baker's Algebra, and ſeveral others. He was a man of great correſpondence with mathematical perſons in foreign parts, and thereby could give information of any new or old mathematical book; and, till my acquaintance with him, I was ignorant of foreign authors; being but

but young when my father died, and not then having taken any pains in theſe ſtudies: ſo that, by the ſaid Collins's information and means, I have heard of, and ſeen, ſome mathematical authors of note and eſteem.

After the ſaid Mr. Collins had told me of James Gregory's ſaid *Exercitationes Geometricæ*, ſold by Moſes Pitts, in St. Paul's Church-yard, I bought there one of them; and do find, that Michael Dairy had deduced this rule from the ſaid book: wherein the ſaid James Gregory hath made the ſquaring of the hyperbola, an exerciſe geometrically demonſtrating the quadrature of the hyperbola, ſome time before publiſhed by the induſtrious and lucky Nicholas Mercator; who, by the happy diſcovery of ſome properties in the hyperbola, hath made all the ways of ſquaring the hyperbola flowing from the ſame, very eaſy, certain and delightful: and becauſe neither of them have exemplified their doctrine and rules with figurative work, ſo large as to 25 places. I have here, to illuſtrate their admirable works, inſerted divers figurative operations, whereby the reader and ſtudent may ſee, and have that ſatisfaction in fact and operation, which is ſo pleaſing and deſirable by every one.

I ſhall not here trouble the reader with any ſections of the cone, whereby he may ſee the riſe and geniture of an hyperbola from that body, but content myſelf to ſhew him from a ſquare and an infinite company of oblongs on a ſuperficies, each equal to that ſquare, how a curve is begotten which ſhall have the ſame properties and affections of an hyperbola inſcribed within a right-angled cone: and ſeeing a curve made after this manner following doth become ſuch an hyperbola, the doctrines and analogies delivered and diſcovered by thoſe two ingenious artiſts, Mercator and Gregory, may be applied to this curve as often as need and occaſion doth require.

And, not to detain the reader any longer from knowing how to make this curve, we proceed to deſcribe the ſame accordingly.

There is a ſquare ABCD, whoſe ſide or root is 10; let DB be prolonged *in infinitum*, and continually divided equally by the root, or DB, and thoſe equal diviſions numbered by 10, 20, 30, 40, 50, 60, 70, &c, *in infinitum*: upon theſe numbers let perpendiculars be erected, which call ordinates, and each of thoſe perpendiculars of that length, that perpendiculars let fall from the aforeſaid perpendiculars to the ſide or baſe CD (which call complement ordinates), the oblongs made of the ordinate perpendiculars, and complement ordinate perpendiculars, may be ever equal to the ſquare AD, which may eaſily be done thus; for it is $\frac{100}{20}$, $\frac{100}{30}$, $\frac{100}{40}$, $\frac{100}{50}$, &c, produces the length of the ordinate perpendiculars; for 100 divided by 20, maketh 5 for the length of the ordinate perpendicular 20E. And 100 divided by 30, giveth 3333333 &c for the ordinate perpendicular 30F; and 100 divided by 40, produceth 25 for the ordinate 40G, and ſo of the reſt. And, geometrically, it is as 20D is to BD, ſo is AB to AH equal to 20E, as before; for that the angle ACH is equal to C20D, and ſo of the reſt. And, for the length of the next ordinate, you ſay, as 30D to BD, ſo AB to AK, which is equal to 30F. And, for the ordinate 40G, ſay, as 40G to BD, ſo AB to AM, which will be equal to 40G, and ſo of all the reſt; whereby you have all the perpendiculars upon the prolonged ſide BD, both geometrically and arithmetically. The ſame proportion is to be obſerved for any intermediate parts.

Now,

Now, for all the perpendiculars which are let fall from the aforesaid perpendiculars or ordinates to the base CD, which call complement ordinates, the geometrical proportion for NE, equal D20 is as HA to AC, so CD to CL20 equal to NE; and for the complement ordinate OF equal D30, it is as KA to AC, so CD to D30 equal OF, and so of the rest. Now, for NE arithmetically, say, as 5 to 10, so 10 to 20 equal to NE, equal to D20; and for OF, say, as 33333333 to 10, so 10 to 30 equal OF, which is equal to CL30, and for PG equal to D40, say, as .25 is to 10, so 10 to 40 equal to PG, equal to D40; and so for all the rest of the complement ordinates standing upon the base CD, whereby it doth appear, that all the oblongs made of the ordinates, and complement ordinates, are each of them equal to the square AD, which is here 100; for the oblong ED being made of E20 and D20, is by the 13 of the 6 Euclide, equal to the square AD; for Q20 is a mean proportional between D20, and 20R, and Q20 is found to be equal AB, so is the oblong or parallelogram ED equal to the square AD, and the like demonstration serves for all the oblongs or parallelograms standing upon the base CD, by the tips or angular points of those parallelograms, or from the ends of all the ordinates standing upon 20, 30, 40, 50, 60, 70, *in infinitum*, draw the curve line from A towards E, so shall you describe the curve AEFGS, which curve you see is begotten without any consideration or respect to the section of a cone, and yet becomes the same in all respects, to have the same affections and properties of an hyperbola derived from the intersecting of a right-angled cone, as shall be shewed in the next chapter.

You may observe the complement ordinate NE, being equal to D20, is equal to twice radius. And if CD be made the radius of a circle, then is NE equal to D20, equal to the tangent of twice radius; for D20 becometh the tangent of twice radius. Also, it is manifest that the complement of the tangent equal to twice radius is also equal to half the radius; that is, the tangent complement of D20 is 20E equal to 5. And seeing the radius is ever a mean proportion between the tangent and the tangent complement, therefore each oblong is equal to the square AD.

CHAP. II.

IN the former chapter, we have shewed the begetting of a curve, without any regard to the section of any solid body; and now it remaineth to prove that this curve hath the same properties and affections that an hyperbola, deduced from the section of a right-angled cone.

I remember some time before the death of John Collins, he told me, it was a great work of the learned Vincent, or Magnan, to prove, that distances reckoned in the asymptote of an hyperbola, in a geometrical progression, and the spaces that the perpendiculars thereon erected, made in the hyperbola, were equal the one to the other. This property is now very well known; the hyperbola hath, and this curve hath the same property; which is discernible almost *intuitu*. In the hyperbola, they call the prolonged line DB *in infinitum*, from the point B, an asymptote. And here in this prolonged line from B, on 20, 40, 80, 160,

3 320,

320, 640, &c, let the ordinates touch the curve in EFGS &c; I ſay, that thoſe trapezias with the curve line (or hyperbolical ſpaces) are all equal the one to the other. In the right-lined trapezias thereon, it is manifeſt, they are all equal the one to the other, by ſeveral propoſitions of the 6th book of Euclid: for in the right-lined trapezia ZEAB, the ſide AB is twice EZ; and, by the former chapter, it was found, that GY is half EZ, by ſaying, as AB to EZ, ſo EZ to GY. And the right-lined trapezia ZEAB ſhall be therefore equal to 75. Now, foraſmuch as in the right-lined trapezia YGEZ, the baſe of that YZ is double to ZB, but the perpendiculars are in the ratio of AB to EZ; for, as before, it is as AB : EZ :: EZ :: GY; therefore the right-lined trapezia YGEZ equal to the right-lined trapezia ZEAB, and ſo will all right-lined trapezias, ſo baſed and perpendiculared, be equal the one to the other. The trapezia LYEZ is equal to the ſquare AD, becauſe ZY × ZE is equal to AB × AB, as in the foregoing chapter, the oblong GZ is half the parallelogram LZ, and the triangle GEI half the parallelogram LI. Now the parallelogram GZ + the triangle GEI (half the parallelogram LI) is equal to the right-lined trapezia ZEAB, for in numbers 20 × 25 = 50 + half 50 equal to 75.

Thus you ſee the right-lined trapezias, numbered upon the prolonged ſide in geometrical proportion, are equal the one to the other. It remaineth now to prove the mixed trapezias; that is, the trapezias ſtanding upon the ſame baſis, but joined aloft with this curve, are alſo equal the one to the other.

Firſt, let it be obſerved, that theſe curvilined trapezias (or hyperbolical ſpaces) are ever leſs than the right-lined trapezias, becauſe all the points in the curvilined trapezias fall within the right line that joins the right-lined trapezias; and is thus proved in the right-lined trapezia BZEA: let there be in the baſe ZB upon the point 5 erected a perpendicular to touch EA in T, then is T5 equal to AB leſs αT, which is half QE, that is, 10 leſs 2, 5 equal 7, 5 = TS. But, by the foregoing chapter, if a perpendicular be erected upon the ſaid point 5 to V (to touch the curve in V) ſo that the parallelogram VD ſhall be equal to AD, as in the former chapter; then will it be AD divided by DV = 5V, which is 100 divided by 15, produceth 6,666666 for the true length of 5V; whereas before 5T is 7,5. By the ſame means may all the intermediate points in this curve line EVA be found to fall within the right line AE, that is, between the line EA and ZB; and therefore the right-lined trapezia ZETAB greater than the curvilined trapezia (or hyperbolical ſpace) ZEVAB.

Now, foraſmuch as we have proved that the aforeſaid right-lined trapezias are ever equal the one to the other, it will now follow, that ſeeing the curve paſſing by all thoſe points which are extremities of the right-lined trapezia (as well as the curvilined ſpaces, being upon the ſame baſis always), and this curve being generated continually by one and the ſame ratio, as in the former chapter. That therefore the curvilined trapezias ſtanding upon geometrical proportional baſes, ſhall be alſo equal the one to the other; which is the affection and property of the hyperbola. And ſo the doctrines and precepts delivered by thoſe two famous Geometers, Mercator and Gregory, for the ſquaring of the hyperbola, be applied to this geometrical curvilined figure, and from it derived logarithms, which may be called hyperbolical logarithms.

The way and means to find the hyperbolical ſpaces in numbers ſhall be ſhewed in the following chapters.

CHAP. III.

IN this chapter we will consider that most admirable discovery (I suppose Mercator made) upon drawing the diagonal CB, which, by construction, cutteth all the perpendiculars standing upon the base CD at equal angles, and in such distances from the base CD as doth unravel the mystery of his infinite series, and make the quadrature of the hyperbola more easy and certain than any I ever saw or heard of.

The diagonal CB being drawn, doth give the first term of a geometrical progression, or infinite series, between 10 and 20, or 30, 40, 50, 60, 70, 80, 90, &c.

That is to say, would you know the first term of an infinite series (or numbers geometrical proportional continued) between 10 and 20, the sum of all which shall be just 20. Having from Z drawn the line ZC, to cut BA into H, which taken off and applied to CD, from C to N equal NB, because the angle BCN is equal to the angle CBN; I say, that NB is the first term of an infinite series between CD equal AC, and the perpendicular NE equal DZ; which may be done by squaring AC, and dividing it by the side or number given, the complement whereof to 10 is the first mean or term of that infinite series, so shall the first term of the infinite series between 10 and 20 be found 5. Thus in numbers, $10 \times 10 = 100$, $\frac{100}{20} = 5$, the complement whereof to 10 is 5, equal CN, equal NB for the first term of an infinite series between 10 and 20, whose sum is 20, as by the arithmetical work in the margin, where *a*, *b*, *c*, *d*, *e*, *f*, &c are a rank of geometrical progressional numbers, whose infinite sum would make but 20, and is demonstrated by the 7th and 8th of Euclid.

a.	10
b..	5
c..	2.5
d..	1.25
e....	625
f....	3125
h....	15625
i.....	78125
k.....	390625
l.....	1953125
m......	9765625
	19..99.0234375

And in numbers, thus: as 10 less 5 is to 10, what 10? the quotient will be found 20 for the whole sum of that infinite series between 10 and 20 whose first term is 5.

In like manner, would you know the first term of an infinite series between 10 and 30, divide the square of ac = 100 by 30, the quotient will be found 3333333, whose complement to 10 is 6666666; I say, that 6666666 is the first term of an infinite series between 10 and 30, as by the arithmetical operation in the margin; and briefly thus, as 10 :: 3333333 = 6666666 : 10 :: 10 : 30, so is 30 the whole sum of all those infinite progressional numbers between 10 and 30. In the figure you draw the line ΦC, which cutteth BA in K; I say, that BK transferred from C to O equal or is the first term of an infinite series between AC and OF equal DΦ. And PS be the first term equal 7,5, equal PC between AC and PG equal DY = 40 between 10 and 40; for, as before, 10 — 7,5 = 2,5 : 10 :: 10 = 40, so is 40 the whole sum of an infinite series between 10 and 40, whose first term is 7,5, and so of all the rest. On this great mystery depends much the following squaring of the hyperbola, and hath made it so intelligible and easy.

a.	10
b..	6.666666666
c..	4.444444444
d..	2.962962962
e..	1.975308641
f..	1.316872427
g....	877914951
h....	585276634
i....	390184422

Hence

Hence you may note, that if you would know the firſt mean of an infinite ſeries between any other number, and that number doubled, tripled, quadrupled, &c, you may from this root 10 deduce it; as, for example, let CD be 12, and I would know the firſt term of an infinite ſeries between 12 and 24, the double of 12, I ſay, as 10 is to 5 what 12 *facit* 6; for, ſuppoſing CD = 10 it was before found 5. Therefore between 12 and 24 you ſay 10 : 5 :: 12 : = 6 for the firſt mean of an infinite geometrical progreſſion between 12 and 24; and is thus, by the former analogy, proved by ſaying, as 12 — 6 = 6 : 12 :: 12 : = 24, ſo is 24 found to be the total ſum of an infinite ſeries between 12 and 24, as by the operation in the margin will appear.

Alſo, if it were required to find the firſt mean between 12 and three times that number, viz. 36, ſay, as 10 : 6666666 :: 12 : 8. And ſo by tabulating or working this example as you do the former, the total ſum of that infinite ſeries between 12 and 36 (the firſt term being found 8), will amount to 36; and, for proof, you ſay, 12 — 8 = 4 : 12 :: 12 : = 36, ſo is 36 the whole ſum of that infinite ſeries, or geometrical progreſſion, between 12 and 36, the firſt term being 8, as was deſired.

a. 12
b. . 6
c. . 3
d. . 1.5
e. . . . 75
f. . . . 375
g. . . . 1875
h. 9375
i. 46875
k. 234375
l. 1171875
m. 5859375

23,994140625

If it ſhall be required to know an infinite ſeries between 10 and any other number; as, to know the firſt term between 10 and 15, that is, between DC and D5, draw 5C, and it cutteth BA in E; I ſay, that BE is the firſt term of an infinite ſeries between 10 and 15, as by the operation in the margin, and as before taught, $\frac{100}{15}$ = 6666666; which ſubtract from 10, leaveth 3333333 for the firſt term. And by the former rule is proved thus: as 10 — 3333333 = 6666666 : 10 :: 10 : = 15. Thus may you find the firſt of any term of an infinite ſeries between 10 and any other number.

a. 10
b. . 3333333333
c. . 1111111111
d. . . 370370370
e. . . 123456793
f. . . . 41152264
g. . . . 13717421
h. 4572473
i. 1524157
k. 508052
l. 169350
m. 56450

14,999971774

And if it ſhould be deſired to know the firſt term of an infinite ſeries between any two other numbers; as, for example, I would know the firſt term of an infinite ſeries between 12 and 16: to do it geometrically, you muſt ſuppoſe BA = 12; and then counting 16 from D in the line DB, prolonged by that point, and C draw a line which will cut the line BA (now repreſenting 12) in a point, which taken from B, will be the length of that line geometrically; and arithmetically, it will be found by ſaying, as 16D to DC, ſo 16 = 4 to BK in the line BA, numbered from B to A when BA is 12. In numbers the proportion ſtands thus:

thus: 16 : 12 :: 4 : = 3, which 3 is the first term of an infinite series between 12 and 16, as was required, and is manifest by the operation in the margin; which to prove by the foregoing rules, you say, as 12 — 3 = 9 : 12 :: 12 : = 16; which is to say, as 9 to 12, what 12 *facit* 16 for the whole sum of that infinite series between 12 and 16, the first term being found, as before, to be 3.

a.	12
b..	3
c...	75
d...	1875
e....	46875
f....	1171875
g.....	29296875
h......	732421875
i......	18310546875
	15,99993896484375

Thus have you that great mystery unfolded, of finding geometrically and arithmetically the first term of a geometrical progression with the whole sum of that infinite progression, or series, between any two numbers; which is the main thing I conceive that famous Mercator was so lucky in discovery thereof, and doth unravel the mystery of squaring the hyperbola, as will be manifest in the next chapter following.

CHAP. IV.

I Observe from the said learned Gregory's *Exercitationes Geometricæ*, he giveth three quantities, or spaces, contiguous to the vertex A, which shall be all equal the one to the other, which is very true and perspicuous; and then shews how to find the areas of them severally, as in page 9, 10, 11, and 12, of said book.

And here we must consider them all three before we come to understand Dairy's rule, which is but a deduction from these, as will appear hereafter. And now I begin to consider the said three several quantities, or spaces, all contiguous to the vertex A, and of a different form, and yet equal the one to the other.

Let it therefore now be shewn those three several spaces differing in form, and yet equal the one to the other, contiguous to the vertex A, which shall represent the curvilined trapezia, or hyperbolical space for 2. The first curvilined trapezia, or hyperbolical space for 2, let be ZBAVE, which is intelligible *intuitu*: the second let be AVENC, which is equal to the former ZBAVE, by the 43 of the 1 of Euclid, because the parallelogram EZBH (BH being equal to ZE) is equal to the parallelogram HACN (HA being equal to HB). Now, forasmuch as the curvilined triangle AVEH is common to both the said curvilined trapezias, or hyperbolical spaces, it remaineth therefore, that these two curvilined trapezias, or hyperbolical spaces, ZEVAB and AVENC, are equal the one to the other.

And now to find out the third curvilined space contiguous to the vertex A, and yet equal to either of the other two, but differing in form, doth require a little further consideration, which from him is directed thus: And is manifest by the figure, divide BZ in two equal parts in 5; then, as before taught, will it be as 5D : DC :: 5B : BK = DX make CX equal to CΠ, or find DΠ, it is as D5 : DZ :: DC : DΠ upon Π erect a perpendicular to touch the return or con-

tinuation

tinuation of the curve on the other ſide of A, from A towards Θ in Σ; ſo is this curvilined figure, or hyperbolical ſpace, ΠΣAVX (differing in form) equal to either of the other two ZEVAB or CAVN. And for finding the area of this curvilined figure, or hyperbolical ſpace, ΠΣAVX, is derived Dairy's rule, which is but a deduction from the finding of the areas of the other two, as will hereafter appear.

Arithmetically, DX is found by ſaying, as 15 to 10; what 10? *facit* 66666666, and 10 leſs 66666666 reſt 33333333 for cΠ, ſo is xΠ equal to 66666666; or DΠ may be found thus: as D5 : DZ :: DC : DΠ, which in numbers is, as 15 : 20 :: 10 : = 1.33333333 for DΠ.

James Gregory, in the 4th propoſition, page 10 and 11, of his *Exercitationes Geometricæ*, doth contemplate firſt, the ſecond of theſe three curvilined trapezias, or hyperbolical ſpaces; that is to ſay, the curvilined trapezia, or hyperbolical ſpace, CAVEN; and in that fourth propoſition, after a long and learned demonſtration, doth prove the ſpace CAVEN to be equal to his ſuppoſed quantity ω; and then refers you to Cavalerius's method of indiviſibles; a book I have not yet ſeen. Which briefly, I conceive, may be thus eaſily demonſtrated.

It is manifeſt, that all the perpendiculars let fall from the ordinates (ſtanding upon the prolonged ſide DB) to the baſe CD, doth not only deſcribe the curve, but would alſo fill the whole hyperbolical ſpace, were number, and the curve definitive. And thoſe perpendiculars let fall from the ordinates (ſtanding upon BD prolonged, numbered 20, 30, 40, 50, 60, &c), doth divide the baſe DC in $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$ &c. And whereas the diagonal CB, by croſſing all thoſe perpendiculars, doth give the firſt term of an infinite ſeries between the root, or ſide, AC (= NH), and the length of each of thoſe perpendiculars ſtanding upon the baſe CD. Therefore, to know the hyperbolical ſpace CAVEN, divide the parallelogram CH, making it the firſt term of an infinite ſeries by the ratio of NB = NC to NH *in infinitum*, and each of thoſe quotes, or proportional numbers, by 1, 2, 3, 4, 5, 6, 7, 8, &c, alſo *in infinitum*. The quotes of all the laſt diviſions added, will give you the area of the hyperbolical ſpace CAVEN, and ſo for any other curvilined or hyperbolical ſpace ſtanding upon the baſe CD, as by the calculation following.

And before I handle any of the other two curvilined ſpaces, differing in form, and yet equal to this hyperbolical ſpace CAVEN, we will exemplify this demonſtration in operation; and the figurative work thereof ſhall be the work of the next chapter.

CHAP. V.

LET it be required to calculate the area of the curvilined trapezia, or hyperbolical ſpace, CAVEN to 15 places, and hereafter you will ſee it done to 25 places, according to Dairy's rule, in chapter VII. but with greater diſpatch.

Thus by the calculation do you find the area of the curvilined trapezia, or hyperbolical ſpace, CAVEN to the 15 place, to be 693147180559945 equal to the

the curvilined trapezia ZEVAB, and alſo equal to the curvilined trapezia, or hyperbolical ſpace, ΠΣAVX; the calculation of the areas of any part of theſe two latter ſhall be ſhewn hereafter, which will differ in operation, yet bring out the ſame number; and in calculating the laſt, we ſhall uſe Dairy's directions.

It having been before ſhewn, that the hyperbolical ſpace ZAVAB, equal to the curvilined trapezia, or hyperbolical ſpace, CAVEN, is equal to the curvilined trapezia, or hyperbolical ſpace YGFEZ; that therefore the ſaid ZEVAB is a ſpace, or quantity, to repreſent the logarithm of 2. So then the aforeſaid number 693147180559945 is an hyperbolical logarithm of 2; and having the logarithm of 2, you have alſo the logarithm of all the powers of 2.

And, by this calculation, you have not only gotten the logarithm of 2, but gained alſo the logarithm of 3: for if you add all the quotes marked with this aſteriſm (*), the addition of them ſhall be half the ſum of the hyperbolical logarithm of 3, agreeable to the 4th conſectary of the 4th propoſition, and firſt inference on the 5th propoſition of ſaid James Gregory's *Exercitationes Geometricæ*; from whence, 'tis plain, that Michael Dairy had his rule, as will appear more manifeſt after we have contemplated the two other curvilined trapezias, or hyperbolical ſpaces, ZDVAB and ΠΣAVX.

The addition of the quotes marked with (*), make 549306144334055, which doubled, is 1098612288668110 for the logarithm of 3; and now, having gotten the logarithm of 3, you have alſo the logarithm of all the powers of 3, and of all the compoſites of 2 and 3.

Again, if you ſhall from 50 ſubtract 125, and to that add 4166666666666, and from that ſubtract 15625, and ſo on throughout, you ſhall have the logarithm of the difference between 2 and 3, or the logarithm of 1 and $\frac{1}{2}$, or 1 and $\frac{5}{10}$, correſpondent to the inference on the 5th prop. of James Gregory; all which ſhall be fully exemplified hereafter.

The calculation of the logarithm of 2, according to the method before going, is the ground-work of all the calculations following; and I ſhall only give the calculation of one more ſpace to repreſent the logarithm of 3, after that method, though we have, you ſee, gotten already the logarithm thereof; but ſuppoſing we had not, and were to find that firſt according to the ſaid method.

From F you let fall a perpendicular, as FO; ſo is the curvilined trapezia, or hyperbolical ſpace, AEFOC equal to the curvilined trapezia, or hyperbolical ſpace, AEFΦB, for the oblong AEOC, as before ſhewn, will be equal to the parallelogram EFΦB, and the curvilined triangle AEFE common to both parallelograms. Therefore the curvilined trapezia AEFOC equal to the curvilined trapezia, or hyperbolical ſpace, AEFΦB, to calculate whoſe area, which will be the logarithm of 3, you proceed as followeth, which work is but partly done, to ſhew the way thereof, the logarithm of 3 being hereafter done to 25 places, but with far greater diſpatch than this method will permit,

666666666666666	:	I	666666666666666
444444444444444	:	II	222222222222223
296296296296296	:	III	.98765432098765
197530864197530	:	IV	.49382716049382
131687242798352	:	V	.26337448559671
.87791495198902	:	VI	.14631915866484
.58527663465934	:	VII	..8361094780848
.39018443310630	:	VIII	..4877305288829
.26012294873752	:	IX	..2890254985973
.17341529915834	:	X	..1734152991583
.11561019943888	:	XI	..1051001813081
..7707346629258	:	XII	...642278885771

I have but gone twelve ſteps in the calculation of the logarithm of 3 after ſaid method, which will, if it were added, but give the logarithm of 3 to 5 places. I have left it unfiniſhed for the exerciſe of thoſe who ſhall take delight herein, and finiſh it throughout, to the intent of making the logarithm of 3 to 15 places, according to this method. By adding the quotes of ſo much as is done, the firſt five figures will be 10986, correſpondent to the logarithm of 3: this method being ſomewhat ſlow, I ſhall not calculate the logarithms of any other numbers according to it. And, by theſe two examples, the reader may ſee enough to calculate any other curvilined trapezia, or hyperbolical ſpace, ſtanding only in or upon the baſe CD, equal to an hyperbolical ſpace; reckoned in the prolonged ſide DB.

And ſo we will contemplate, in the next chapter, the affections and properties of the hyperbolical ſpace, or curvilined trapezia, AΣΘΥC, equal to the curvilined trapezia, or hyperbolical ſpace, ABZEV.

CHAP. VII.

BEFORE we ſhew how to calculate any part of the curvilined trapezia, or hyperbolical ſpace, AΣΘΥC, equal to the curvilined trapezia, or hyperbolical ſpace, AVEZB, we will inſert tables to illuſtrate the 1, 2, and 3 propoſitions of James Gregory's *Exercitationes Geometricæ*.

THE

THE FIRST TABLE OF ILLUSTRATION.

I. II.

A	5
B	25
C	125
D	625
E	3125
F	15625
G	78128
H	390625
I	1953125
K	9765625
L	48828125

z = 9999999999

The rule to find z,

$5 - 2,5 = 2,5 : 5 :: 5 : = 10 = z.$

III.

A C E G I L z =

The rule to find z is

$5 - 125 = 375 : 5 :: 5 : =$
66666666 impares = z.

IV.

B D F H K M z =

The proportion to find z,

$25 - 625 = 1875 : 25 :: 25 : =$
33333333 pares = z.

V.

The rule is A − B, C − D, E − F, G − H, I − K, L − M

$25 - 625 = 1875 : 25 ::$
$25 : = 33333333$
Equalis paribus.

VI.

\+ | + | + | + | + | + | +
N M L K I H G F E D C B A

The proportion is

$5 + 25 = 75 : 5 :: 5 : = 33333333$
Equales exceſſui imparium ſupra omnes pares.

VII.

| + | + | + | + | + | + | +
N M L K I H G F E D C B A

The proportion is

$5 - 125 = 375 : 5 :: 5 : = 66666666$
Equalis paribus.

VIII.

\+ + + + + + + + + + + + +
N M L K I H G F E D C B A
\+ − + − + − + − + − + − +
N M L K I H G F E D C B A

2A + 2C + 2E + 2G + 2I +

THE

THE EXPLANATION OF THE FOREGOING TABLE.

This table confifteth of eight columns. The firft is a fuppofed literal rank of quantities continually proportional. The fecond is of numbers correfpondent to the firft in a ratio, as 2 is to 1; or 5 to 2,5: What 2,5? And fo fuppofed continued *in infinitum*; with the rule how to find out the whole fum of thofe numbers fo continually proportional.

The third column fheweth how to find the whole fum of the odd quantities or numbers.

The fourth teacheth how to know the whole fum of the even quantities or numbers.

The fifth telleth how to find the whole fum of the difference of the quantities of the firft column.

The fixth fuppofeth A − B + C − D + E, and teacheth how to find the folution thereof.

The feventh fuppofeth A + B − C + D − E, and giveth a rule to refolve the fame.

The eighth and laft fuppofeth the whole rank, firft affirmative, and the fecond evenly or alternately lefs; and giveth a folution thereof.

A further explanation of this table will be when we come to calculate the area of any part of the curvilined trapezia AΣΘYC.

THE SECOND TABLE.

Complement ordinates ftanding upon CD (more than radius) or perpendiculars let fall from	Ordinates or perpendiculars ftanding upon DB prolonged.	Arithmetical complements.	What proportion the fecond column hath to radius.
10	10	0	
20	5	5	$\frac{1}{2}$
30	333333333	666666666	$\frac{1}{3}$
40	2,5	7,5	$\frac{1}{4}$
50	2	8,	$\frac{1}{5}$
60	1,66666666	8,33333333	$\frac{1}{6}$
70	1,42877143	8,57142857	$\frac{1}{7}$
80	1,25	8,75	$\frac{1}{8}$
90	1,11111111	8,88888888	$\frac{1}{9}$
100	1.	9.	$\frac{1}{10}$

This table confifteth of four columns. The firft is equal fpaces, numbered in the fide DB prolonged, or the tangents greater than radius.

The fecond fheweth the length of the perpendiculars ftanding upon the fide DB prolonged; which are tangents lefs than radius; and by the tops pafs the curve or hyperbolical line.

The third column is the arithmetical complements.

The fourth column fheweth what proportion the fecond column hath to radius.

The rectangle, or parallelogram, of the firft and fecond column, is equal always to the fquare AD.

This table is of uſe to find points to deſcribe the curve or hyperbolical line, or to examine if the curve paſs through ſuch points as the table mentions.

The making of this table hath been formerly ſhewn, when it was taught how to deſcribe the curve.

We now come to ſhew, how to make a table to find the length of the baſes of the compound curvilined trapezias, or hyperbolical ſpaces.

We call that a compound curvilined trapezia, or hyperbolical ſpace, when AC is in the middle of that baſe.

So AC ſtanding upon the middle of Πx hath perpendiculars or ſides ΠΣ and XV, ſo is the curvilined trapezia ΠΣAVX, to be hereafter underſtood a compound curvilined trapezia, or hyperbolical ſpace, and will be ſhewn as followeth to be equal to the aforeſaid ſpaces CAVEN, and to AVEZB for the logarithm ſpace of 2.

And the compound curvilined trapezia Δ⊙AVEN will be equal to the curvilined trapezia AVFOC, and to AVFΦB for the logarithm ſpace of 3.

The compound curvilined trapezia, or hyperbolical ſpace ΠΣAVX, we may prove to be equal to AVEZB, thus: by the 43 of the firſt of Euclid, the parallelogram CK is equal to K5, and the curvilined triangle AVK common to both; ſo then is AVKXC equal to AV5B. And the parallelogram Π[illegible] equal to the ſquare VZ, and the curvilined triangle Σ[illegible]A equal to the curvilined triangle V[illegible]E; and ſo the compound curvilined trapezia ΠΣAVX equal to the curvilined trapezia AVEZB for the logarithm ſpace of 2. For, by the 4th table following, look what proportion the perpendiculars or ſides of the compound curvilined trapezias have one to the other, the like proportion have the ſides or perpendiculars of the other two curvilined trapezias.

So in this compound curvilined trapezia ΠΣ and XV, the ſides or perpendiculars are in a proportion as 2 is to 1 deſcending, or as 1 is to 2 aſcending; ſo likewiſe in the curvilined trapezia CAVEN (equal to the aforeſaid compound curvilined trapezia ΠΣAVX), the ſide or perpendicular NE is double to CA. And alſo in the curvilined trapezia AVEZB (equal, as before, to the compound curvilined trapezia ΠΣAVX), the ſide, or perpendicular, BA, twice ZE, as before taught.

Thus, by the ratio of the 2 tables following, you may make a compound curvilined trapezia equal to either of the other two curvilined trapezias, or hyperbolical ſpaces; and the calculating the area of the compound curvilined trapezias, will be found to be of far greater diſpatch than the former method; by which we ſhall make uſe of Dairy's rule, or rather the learned James Gregory's, from his firſt inference on his 5th propoſition.

We come now to inſert the third table, which is a table of ratios, to find the length of the baſes of the compound curvilined trapezias.

You may note, that in all the three different ſorts of curvilined trapezias, or hyperbolical ſpaces, equal the one to the other, if on the middle of their baſes you ſhall erect perpendiculars to touch the curve, the greater part or ſegments in each is equal to either greater ſegment of the other, and ſo is the leſſer part or ſegment of the one equal to the leſſer ſegment of either of the other.

THE

THE THIRD TABLE.

Being a Table of Ratios to find the Length of the Bases of the compound curvilined Trapezias, or hyperbolical Spaces.

I	II	III	IV	V	VI
5D	: DC	:: DC	: DX		
15	: 10	:: 10	: ,666666666	For	Length of the
5D	: ZD	:: DC	: DΠ	2	base
15	: 20	:: 10	: 1,333333333		666666666
ZD	: DC	:: DC	: DN		
20	: 10	:: 10	: 5	3	10
ZD	: ΦD	:: DC	: D△		
20	: 30	:: 10	: 15		
25	: 10	:: 10	: 4	4	12
25	: 40	:: 10	: 16		
30	: 10	:: 10	: .333333333	5	1333333333
30	: 50	:: 10	: 1,666666666		
35	: 10	:: 10	: 285714285	6	1428571430
35	: 60	:: 10	: 1,714285715		

This table consisteth of six columns. The first four shew the proportion or ratio to find the lengths of the bases; and the number in the sixth column is the length of the base for so many spaces as the fifth column signifies.

And, by the same reason, you may find the lengths of the bases for any other curvilined trapezia, or hyperbolical space.

Thus is 666666666, of the sixth column (the difference of the two first numbers in the fourth column), the length of the base, for the curvilined compound trapezia, or hyperbolical space, to represent the logarithm of 2.

And 10 the length of the base for 3, so is 12 for 4; and 1,333333333 for 5: and so is 1428571430 for the length of the base for the hyperbolical space for 6. And thus may you do for any other space or number.

The numbers in the fourth column for 2, 3, 4, 5, 6, &c, are in proportion as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, &c; and added are equal to twice radius, or 20 = D♉.

We proceed next to shew how to make a table of ratios, to find the lengths of both the perpendiculars, or sides, of the compound curvilined trapezias.

THE FOURTH TABLE.

Being a Table of Ratios to find the Length of both the Perpendiculars, or Sides, of the compound curvilined Trapezias, or hyperbolical Spaces.

I		II		III		IV	V
CD − CX = XD	:	DB	::	DB	:	D5 =	XV
10 − 333333333 = 666666666	:	10	::	10	:	15	2
CD + CΠ (= CX) = DΠ	:	DB	::	DB	:	DΣ	
10 + 333333333 = 1333333333	:	10	::	10	:	7,5	
CD − CN = ND	:	DB	::	DB	:	DZ =	NE
10 − 5 = 5	:	10	::	10	:	20	3
CD + CΔ (= CN) = DΠ	:	DB	::	DB	:	Δ⊙	
10 + 5 = 15	:	10	::	10	:	666666666	
CD − C♋ = ♋D	:	DB	::	DB	:	DR =	♋R
10 − 6 = 4	:	10	::	10	:	25	4
CD + C♉ (= D♋) = D♉	:	DB	::	DB	:	♈♉	
10 + 6 = 16	:	10	::	10	:	625	
CD − C♌ = ♌D	:	DB	::	DB	:	DΦ =	F♌
10 − 666666666 = 333333333	:	10	::	10	:	30	5
CD + C♍ (= C♌) = D♍	:	DB	::	DB	:	♍♎	
10 + 666666666 = 1666666666	:	10	::	10	:	6	

This table consisteth of five columns. The first contains the quantities and numbers of the first term in the proportion. The second column, the quantities and numbers of the second term in the proportion. The third column, the quantities and numbers of the third term in the proportion. The fourth column, the quantities and numbers of the fourth proportional number or term; wherein are numbers for the length of both the perpendiculars for 2, 3, 4, 5, &c. The fifth column, is the numerical order of the compound curvilined trapezias, or hyperbolical spaces, of 2, 3, 4, 5, &c.

35 583333333	6
40 571428571	7
45 5625	8
50 555555555	9
55 55	10

And, by the same ratio, you may find the lengths of both the perpendiculars for any other compound curvilined trapezias to represent the logarithm of any other number.

By the fourth column, you may perceive the perpendiculars, or sides, of the compound curvilined trapezias, or hyperbolical spaces, are in such proportion the one to the other, as the number they represent are to unity.

That is to say, in the compound curvilined trapezia ΠΣAVX to represent the logarithm space of 2, the perpendicular XV is to ΠΣ as 2 is to 1.

And in the compound curvilined trapezia Δ⊙AVEN to represent the logarithm space of 3, the perpendicular NE is in proportion to Δ⊙ as 3 is to 1.

And so the perpendiculars of the fourth logarithm space as 4 to 1; and of the fifth space as 5 to 1, &c; as by the fourth column of this fourth table appeareth.

And

And the perpendiculars of both the other ſorts of the curvilined trapezia, or hyperbolical ſpaces, are likewiſe in the very ſame proportion the one to the other; as you may note from what hath been ſaid before of them.

By theſe tables, and by what hath been ſaid formerly, theſe three curvilined trapezias have the ſame properties and affections as thoſe have in an hyperbola derived from the ſection of a right-angled cone.

We ſhall now, therefore, come to calculate ſome part of this latter hyperbolical ſpace, before we ſhew how to do it all at once; that is, of the hyperbolical ſpace △⊙AVEN to calculate the area of the ſpace △⊙AC, which is equal, as before ſhewn, to the ſpace of 5VAB. And when we have ſhewn to calculate this part, we ſhall, from this, and what hath been taught how to do the other part, come to derive Dairy's rule, or rather James Gregory's, which is compriſed in the firſt inference on his 5th propoſition.

CHAP. VII.

WE have, in the fifth chapter, calculated the area of the curvilined trapezia, or hyperbolical ſpace, CAVEN equal to AVEZB for the logarithm of 2.

In this curvilined trapezia CAVEN all the perpendiculars ſtanding upon the baſe CN, are each more than radius CA (or greater than the tangent of 45° 00′), being ſtill aſcending and affirmative; and therefore, by the firſt table, to be continually adding; as by the calculation thereof is alſo manifeſt.

We are now come to calculate the curvilined trapezia C△⊙A, part of the curvilined trapezia C♈⊖⊙A equal to AVEZB.

In this curvilined trapezia C△⊙A (equal to B5VA) all the perpendiculars ſtanding upon the baſe CA, are each leſs than the radius CA, being ſtill deſcending and negative; and, therefore, to be handled by the firſt table accordingly.

The baſe C△ is equal to CN of that ſpace, calculated, as before, in the 5th chapter.

If by the vertex A you draw a parallel to the diagonal CB as ZA♈, it is a tangent to the curve touching it in the point A, and AB doth cut all the perpendiculars contrarywiſe to CB. For CX = X♍ is not equal to ♈Π = Π♂, but Π♂ is equal to K♍ = KB, becauſe CΠ is equal to CX, and the angle Π♈♂ equal to the angle KB♍; ſo, therefore, by the 1 and 4 tables, all the perpendiculars ſtanding upon C♈ are leſs than radius. And ſeeing, by the ſixth column of the firſt table, and alſo by the fourth of the fourth table, we may find the length of △⊙. Therefore, to know the area of C△⊙A, making C⊖ the firſt term of an infinite ſeries in a continual proportion, as CA is to C△, that is, as 50 to 25: What 25? *Facit* 125, as in the infinite ſeries of numbers continually proportional for the calculation of the logarithms of 2, in chap. 5. You do therefore, as there ſaid, from 50 (of the ſecond part, under the title, The Quotes to be added)

added) ſubtract 125, and to that add 4166666666666666, and from that ſubtract 15625, and ſo on throughout, you ſhall have 405465108108165 for the area of the ſpace CΔΘA equal to BAV5. And thus may you find any other part of CϒΠA.

We ſhall ſhew how to do it for CΠΣA and CXVA, becauſe from them we ſhall derive Dairy's rule, or rather James Gregory's; for from them we have derived and calculated the logarithm for two to twenty-five places, as by the calculation following next after this will appear.

Now to calculate the area of the curvilined trapezia, or hyperbolical ſpaces, CΠΓA and CXVA, you make CX, or CH, the firſt term of any infinite ſeries, and the ſecond term in ſuch a ratio as ΠC is to CA for your proportionals of your infinite ſeries, and ſo proceed on, as in chap. V. and as here appeareth.

The Infinite Series of Numbers proportional.			*The Quotes to be added for* CXVA.
a	333333333333333.	I	A + 333333333333333 + A
aa	111111111111111.	II	B − .55555555555555 + B
aaa	.37037037037037.	III	C + .12345679012345 + C
a^4	.12345678012345.	IV	D − ..3086419753086 + D
a^5	..4115226337448.	V	E + ...823045267489 + E
a^6	..1371742112483.	VI	F − ...228623685414 + F
a^7	...457247370827.	VII	G + ...65321052927 + G
a^8	...152315790275.	VIII	H − ....19051973784 + H
a^9	....50805263425.	IX	I + ...5645029269 + I
a^{10}	....16935087808.	X	K − .....1693508781 + K
a^{11}	.....5645029269.	XI	L + ...513184479 + L
a^{12}	.....1881676423.	XII	M − ......156806369 + M

We have but gone through twelve ſteps of this calculation, to ſhew the manner thereof; but ſhould you proceed to go through it till it works off, as in chap. V. you may have both the ſegment CXVA, and CΠΓA: for, if you finiſh the calculation, and add up all the quotes, that ſum will be the area of CXVA, and be found 405465108108165, as in chap. V. and is equal to the greater ſegment WNHE☽ in the curvilined trapezia, or hyperbolical ſpace, CNHE☽A, and alſo WNHE☽ is equal to CΔΘA equal to AB5V.

And if from 333333333333333 you ſhall ſubtract the ſecond number 555555555555555, and to that add the third number 2345679012345, and then from that ſubtract the fourth, and ſo add and ſubtract according to the ſigns + and − throughout, you will have ..287682072451780 for the area of the leſſer ſegment of the compound curvilined trapezia WΠΓA☽, that is, the area of CΠΣA, equal to CWDA, equal to V5ZE.

And you have not only gotten, by this calculation, the area of each ſegment ſeparately, and ſo, conſequently, the area of the whole ſpace, by addition of theſe two, but you have alſo obtained the half of the whole area at once; for if you ſhall (correſpondent to the column of the firſt table) add the numbers with + affirmative, they will give you half the area of the compound curvilined

trapezia

trapezia XHPAV for the logarithm of 2, which you will ſee preſently exemplified, and done to 25 places: and this is the ſum of James Gregory's inference on his fifth propoſition of his *Exercitationes Geometricæ*; and ſo agreeable to the rule delivered to me, as before declared, by Michael Dairy. Having acquainted ſeveral perſons with Dairy's rule, in page 45, and ſhewn to them ſome figurative work thereupon, in order to make a logarithm, I was, notwithſtanding, ſome time, through inadvertency, almoſt diſcouraged of ever knowing how to cube *in infinitum* ſuch a number as there ſpoken of; neither did any of thoſe to whom I had communicated the ſame take any ſuch notice thereof (that I know of), ſo as to do it. And now I come to ſhew, how I overcame that difficulty of cubing a range of figures for 25 places, which he told me I muſt do *in infinitum*, before I could make the logarithm of ſo many places; and to remove the ſtumbling block, I do confeſs, took up ſome time; for Dairy had not then told me a word of ſuch an author as James Gregory, and I had not known his works but for John Collins, ſome years after Dairy's death. But, before I ever met with Gregory's book, I had obtained my deſire to cube *in infinitum* twenty-five figures; that is, twenty-five 3, by dividing by 9 continually, as in the calculation following, to find the logarithm of 2 all at once; which manner diſpatcheth the calculation much more ſpeedy than the method of calculation in the fifth chapter.

And now the reaſon of cubing twenty-five 3, by dividing only by 9, doth follow.

Foraſmuch as Dairy's rule, before declared, to make the logarithm of 2, doth bid you to 2, add 1, and from 2 ſubtract 1, ſo ſhall there be a reſult or fraction of $\frac{1}{3}$; and then divide 1, or 100,000,000,000,0000, by $\frac{1}{3}$, whoſe quotient is 333333333333333, which cube *in infinitum*, it had been as much as if he had ſaid cube $\frac{1}{3}$ fractionally, which is $\frac{1}{27}$, and divide 1000000000000000 by $\frac{1}{27}$, the quotient will be 37037037037037 for the cube of *a*, or $\frac{1}{3}$, as in the operation beforegoing. Now, foraſmuch as you would cube the number for $\frac{1}{3}$, viz. 333333333333333 (which is 1, or 100,000,000,000,0000, divided by 3), it is as if you ſhould ſay, as 27 to 1000000000000000: What 1? the quotient will be 37037037037037 for the cube of *a*, or 333333333333333, as before. Now, if you ſhall, as in the operation beforegoing, ſet down 333333333333333 (which is equal to $\frac{1}{3}$), you have no more to do but to divide by 9, for that $\frac{1}{9}$ of $\frac{1}{3}$ is equal to $\frac{1}{27}$; and, therefore, dividing 333333333333333 by 9, the quotient will be 37037037037037O, as before, for the cube of $\frac{1}{3}$, or *a*; and ſeeing $\frac{1}{3} \times \frac{1}{3}$ is equal to $\frac{1}{9}$, you have no more to do but to divide continually by 9, and they ſhall all be proportional numbers, by 7th of the 8th of Euclid, and conſequently correſpondent to the odd powers; for if the root be multiplied by the ſquare that begets the cube, and the cube again by the ſquare that begets the fifth power, and ſo on. So here, foraſmuch as dividing by 9 doth beget the third power; if you ſhall therefore continually divide by 9, you ſhall have the reſpective odd powers accordingly, as is alſo manifeſt by the laſt figurative calculation; and all for that a 1 doth neither multiply nor divide, and that $\frac{1}{9}$ of $\frac{1}{3}$ is equal to $\frac{1}{27}$; and if you ſhall divide $\frac{1}{27}$ of 1 by 9, the quotient will be 333333333333333, which is equal to *a* for the firſt number or root, as before.

Now, foraſmuch as to make a compound curvilined trapezia equal to an uncompounded; as, for inſtance, to make the compound curvilined trapezia WΠΓA Ɒ to be equal to the uncompounded CAVEN, equal to ABZE = AΘϒ for the logarithm of 2, and to find the length of the baſe, and both the perpendiculars, hath been diſcourſed, and may be ſeen, as in the third and fourth tables beforegoing. We come to handle and calculate the area of this compounded curvilined trapezia WΠΓ Ɒ A, for to make the half logarithm of 2 at once.

Seeing, by the ſixth column of the third table, the baſe WΠ is 6666666666666666, whoſe half is .. 333333333333333 for CW, or CΠ, equal to the firſt term in the former operation (and alſo the ſame as Dairy's reſult, or fraction, of $\frac{1}{3}$), and that I muſt divide in the ratio of AC to CΠ, or CW, *in infinitum*, as in the fifth chapter, and alſo as in this is ſhewn and taught, for to make the infinite ſeries of numbers proportional. It will appear, that if I do divide 33333333333333 by 9, it will give me the cube of the firſt term, and ſo dividing continually by 9, will produce the numbers appertaining to the odd powers, as by the large calculation to 25 places, next following. And ſeeing I am, by Dairy's rule, or rather James Gregory's, to divide each of the numbers of the infinite ſeries by the indices of the odd powers, it is manifeſt, that this rule of Dairy's is derivable from the 8th column of the firſt table;

$$\text{For } A + B + C + D + E + F + G + H + I$$
$$\text{And } A - B + C - D + E - F + G - H + I$$
$$\text{Doth make } {}^2A + {}^2C + {}^2E + {}^2G + {}^2I + {}^2L + {}^2N;$$

And therefore every other line of the quotes to be added in the former operation, doth make half the logarithm of 2.

In making the infinite ſeries, as in page 66, in order to make the half logarithm of 2 to 25 places, be very careful to ſet the figures in their due places, and to make that ſeries you are to divide continually by 9, which being done throughout, you may then prove your work by multiplication, in multiplying each line by 9; and if thoſe multiplications produce the foregoing numbers, you may conclude that part of the work to be well prepared. And ſeeing, by the direction over the figurative work in the ſecond column of page 66, you are to divide each of the numbers in the firſt by 1, 3, 5, 7, 9, &c, you muſt ſo order the quotes of the ſecond, that they may lie in the ſame line, or range with the reſpective dividends or numbers in the firſt. For the better preventing miſtakes, the letter figures do repreſent the diviſors proper to each line. And, would you make the logarithm of 2, according to the following method, for 7 or 8 places only, you may very well produce it in half an hour's time, as by that calculation is very perceptible. And ſome that have had thoſe two ſheets I formerly printed as a ſpecimen hereof, have told me, they have done the ſame; and were very ſolicitous I would, as ſoon as I could, publiſh the remainder; which, at length, as time and leiſure hath permitted, is done. And though I have not here inſerted many examples, yet, by what are herein done, you may perceive how to proceed for any other number propoſed. And, with the direction and reference at the end of this chapter, thoſe that are willing, and curious herein, may make a logarithm for any natural number deſired. I have not added hereto any table of logarithms at this time;

and

and what I may do hereafter, in order thereunto, I do not preſume to promiſe. I doubt not but ſome may both examine ſome table or other, or make by this method one *de novo*, and ſatisfy themſelves about the ſame; and ſome have told me, ſince my communicating this method unto them, that if the firſt makers of tables of theſe numbers had made them by ſuch eaſy ways, they did not doubt but their tables might have been ſomewhat more exact. Howſoever, it pleaſed God, who is the giver of every good and perfect gift, to raiſe and endue ſuch men with great ability and patience, to perform thoſe tables, with ſo much difficulty and labour as their methods did require, and for common uſes ſufficient. And with ſuch eagerneſs did that age embrace and purſue the invention of theſe numbers, that Ullach, a Dutchman, had exhibited a table of logarithms to 10 places for 100000, before the learned Henry Briggs's table, which he had in part done to 15 places, could be accompliſhed by him.—So exceeding glad were they of the invention. And the learned Henry Briggs, in his Epiſtle Dedicatory to our moſt gracious King's father, when Prince of Wales, ſaith, that amongſt the ancients there is not found any footſteps of theſe numbers. Of whoſe conſtruction and uſes the ſaid Henry Briggs hath written, in his *Arithmetica Logarithmica*, moſt learnedly and copiouſly. And now follows the figurative part of making the logarithm of 2 to 25 places.

 The

The Infinite Series, or Numbers continually proportional.

These numbers are continually divided by 9, in order to make the half logarithm of 2.

These numbers are quotes from those on the opposite side, by dividing them by 1, 3, 5, 7, 9, &c, and are $\frac{{}^2A}{2} + \frac{{}^2C}{2} + \frac{{}^2E}{2} + \frac{{}^2G}{2} + \frac{{}^2I}{2} + \frac{{}^2L}{2} + \frac{{}^2N}{2}$ + &c, correspondent to the last preceding calculation; which added, make half the area of the compound curvilined trapezia XΠΓAV for the half logarithm of 2 to 25 places.

a	3333333333333333333333333	I	3333333333333333333333333	A
a^3	370370370370370370370370	III	123456790123456790123457	C
a^5	41152263374485596707819	V	8230452674897119341564	E
a^7	4572473708276177411980	VII	653210529753739630283	G
a^9	508052634252908601331	IX	56450292694767622370	I
a^{11}	56450292694767622370	XI	5131844790433420216	L
a^{13}	6272254743863069152	XIII	482481134143313012	N
a^{15}	696917193762563239	XV	46461146250837549	P
a^{17}	77435243751395915	XVII	4555014338317407	R
a^{19}	8603915972377324	XIX	452837682756702	T
a^{21}	955990663597480	XXI	45523364933213	W
	106221184844164	XXIII	4618312384529	Y
	11802353871574	XXV	472094154863	
	1311372652397	XXVII	48569357496	
	145708072489	XXIX	5024416293	
	16189785832	XXXI	522251156	
	1798865092	XXXIII	54511063	
	199873899	XXXV	5710683	
	22208211	XXXVII	600222	
	2467579	XXXIX	63271	
Differentia 1	274175	XLI	6687	
Unitas 1	30464	XLIII	708^5	
Numerus Proposit. 2 $= \overset{a}{\frac{1}{3}} \times \frac{1}{3} = \overset{aaa}{\frac{1}{9}}$	3385	XLV	75	
	376	XLVII	8	
Summa - 3	42	XLIX		

You may perceive, that if 1 be added to 2, and subtracted from it, it leaveth a result of $\frac{1}{3}$; which multiplied into itself, maketh $\frac{1}{9}$; and therefore these numbers are continually divided by 9.

3465735902799726547086160^5 half the logarithm of 2,

693147180559945309417232I the logarithm of 2.

Thus

Thus have we calculated the logarithm for 2 to 25 places, after Dairy's rule, or rather James Gregory's, which method maketh far greater diſpatch than that in chap V; for this calculation, though to 25 places, is ſooner performed than that of 15 places, in chap. V; as, by comparing them, is very perſpicuous and manifeſt.

And now we have exemplified the rule Dairy declared; and I am apt to believe he had ſtudied well Gregory's ſaid *Exercitationes*, though he was not pleaſed to tell any more thereof, but that others ſhould take pains therein as well as he; and, truly, if John Collins had not acquainted me with Gregory's works, I had done the work, but not with that ſatisfaction I met with from James Gregory's books: and here you have it in a more familiar diſcourſe and dialect than that of James Gregory's, being altogether analitical; and if any letter or ſymbol be miſtaken in his, it is very great ſtudy and labour to find, and to ſet it to rights.

I find James Gregory had calculated the hyperbolical logarithm for 2, in his *Vera Circuli & Hyperbolæ Quadratura*, to 25 places, which agreeth with this calculation but to 17 places. I have not raiſed the logarithm for 2 to his doctrine in that book, but am ſatisfied this calculation for the logarithm of 2, in this chapter, is true, to an unit, in the 25th place, and may be, in two hours, very well examined by any one that will take the pains to do it, and they ſhall find it to be as herein calculated. And to this I have the concurrence of the moſt ingenious and laborious Mr. Abraham Sharp, who (from the occaſion of my publiſhing formerly two ſheets of the praxis hereof as a ſpecimen) hath ſhewn me his calculation of the logarithm of 2, and ſome others, to 40 places; the like, I ſuppoſe, not hitherto heard of or ſeen. Without all doubt, Gregory found, that Mercator's lucky invention of ſquaring the hyperbola, was of far more diſpatch than that of his *Vera Circuli & Hyperbolæ Quadratura*, or elſe he would not have wrote on Mercator: but ſo excellently hath Gregory illuſtrated Mercator, that a better way of ſquaring the hyperbola, I ſuppoſe, hath not, nor may be found.

We ſhall, for the future, keep to this laſt method of calculating thoſe examples hereafter following. And now we proceed to calculate a logarithm for $1\frac{1}{4}$; for having got the logarithm of 2, we have the logarithm of 4, and all the powers thereof; ſo then, if we make a logarithm of $1\frac{1}{4}$, and add to it the logarithm of 4, we ſhall have the logarithm of 5; becauſe that $4 \times 1\frac{1}{4}$, makes 5; that is, 4 multiplied by $1\frac{1}{4}$, makes 5: and this method I took to make the logarithm of 5; and having got the logarithm of 5, to the logarithm of 5 add the logarithm of 2, and you have the logarithm of 10. And, when I have ſhewn this, I ſhall produce Briggs's logarithm of 2, by one ſingle diviſion; for that all ſorts of logarithms are proportional the one to the other. We therefore now haſten to make the logarithm for $1\frac{1}{4}$.

The Calculation of the Logarithm for $1\frac{1}{4}$ to 25 Places.

The Infinite Series, or Numbers continually proportional.

Theſe numbers are continually divided by 81.

Theſe numbers are quotes from thoſe on the oppoſite ſide, they being divided by 1, 3, 5, 7, 9, &c; which added, make half the logarithms for $1\frac{1}{4}$ to 25 places.

a	1111111111111111111111111	I	1111111111111111111111111	A
a^3	1371742112482853223594O	III	457247370827617741198O	C
a^5	16935087808430286711 1	V	338701756168605734 22	E
a^7	2090751581287689718	VII	298678797326812817	G
	25811747917131972	IX	2867971990792441	I
	318663554532493	XI	28969414048408	L
	3934117957191	XIII	302624458245	N
	48569357496	XV	3237957166	P
	599621697	XVII	35271865	R
	7402737	XIX	389618	T
	91392	XXI	4352	W
	1128	XXIII	49	Y
	14		1	

This number is $\frac{1}{2}$ the logarithm for $1\frac{1}{4}$, 1115717756571048778831475

This number is the logarithm of $1\frac{1}{4}$, 2231435513142097557662951

Adding 1 to $1\frac{1}{4}$, it maketh $2\frac{1}{4}$, and ſubtracted from it, leaveth $\frac{1}{4}$, which reſult maketh a fraction of $\frac{1}{9}$; for $2\frac{1}{4}$ being reduced into fourths, make $\frac{9}{4}$, ſo the reſult of $\frac{1}{4}$ and $\frac{9}{4}$ (rejecting the denominators) is $\frac{1}{9}$, as above; which ſquared, maketh $\frac{1}{81}$: ſo are theſe numbers, therefore, continually divided by 81, to make the infinite ſeries. By decimals, it preſently ſheweth itſelf to be a fraction of $\frac{1}{9}$. Thus the difference, or numerator, is 25, the ſum, or denominator, 2,25; which fraction $\frac{25}{225}$ is equal to $\frac{1}{9}$.

Differentia -	25
Unitas	1
Numerus Propoſitus	1,25
Summa -	2,25

To divide the numbers on the other ſide (to make the infinite ſeries) by 81, is eaſy enough; for it is but dividing twice by 9, or taking the ninth part twice, and rejecting or cancelling the firſt; ſo is it very readily done; and the whole operation hereof may very well be performed in two hours' time: and thus have we got the logarithm for $1\frac{1}{4}$ to 25 places; and now ſhall proceed to make the logarithm of 10, which is, by adding together the logarithm of 2, 3 times, and that makes the logarithm of 8; and that added to the logarithm of $1\frac{1}{4}$, makes the logarithm of 10; and the logarithm of 2 ſubtracted from the logarithm of 10, leaveth the logarithm of 5, and is to the ſame effect as is before.

Logarithm

Logarithm of	2.	6931471805599453094172321
Logarithm of	8.	20794415416798359282516963
Logarithm of	$1\frac{1}{4}$.	2231435513142097557662951
Logarithm of	10.	23025850929940456840179914
Logarithm of	5.	16094379124341003746007593

We have now made and exhibited the logarithms of 2, 5, and 10, and from these you may make all their composites.

And now we proceed to make the logarithm of 3 to 25 places, which we shall shew two ways; first, all at once, from a compound curvilined trapezia, or hyperbolical space; secondly, by a composition of two logarithms, viz. of 2 and $1\frac{1}{2}$, for that $2 \times 1\frac{1}{2}$, maketh 3; and this latter we chiefly recommend. The compound curvilined trapezia, or hyperbolical space, NΔ $\odot$ AVE, we have, in the foregoing chapter, shewn to be equal to the uncompounded CAVFO, and also equal to AVFΦB; we shall calculate the logarithm for 3 according to the compounded space, and by the third and fourth tables you may know the lengths of the base and perpendiculars.

The base NΔ is 10, therefore CA = 5 = CX. Now, forasmuch as Dairy's rule of adding to, and subtracting 1 from 3, produces the half length of the base, agreeable to the third table, we shall shew how to calculate the half of the logarithm for 3, as we did for the half logarithm of 2.

Adding 1 to 3, it makes 4, and subtracting 1 from 3, leaveth 2; which maketh a result or fraction of $\frac{2}{4} = \frac{1}{2}$.

Now, dividing 1 and 25 cyphers by $\frac{1}{2}$, the quotient is 5, for the first term of your infinite series. Now, forasmuch as $\frac{1}{2} \times \frac{1}{2}$ maketh $\frac{1}{4}$, the second term must be, therefore, $\frac{1}{4}$ of the first, and so on; as was discoursed before, in making the half logarithm of 2.

Having made the infinite series as followeth, you divide each of those numbers (which, as before taught, are proportional) by 1, 3, 5, 7, 9, &c, and these added, make half the area of N$\Delta\Phi$AVE for the half logarithm of 3.

The

The Infinite Series, or Numbers continually proportional.

These numbers are continually divided by 4, in order to make the $\frac{1}{2}$ log. of 3 to 25 places.

These numbers are quotes from those on the opposite side, those being divided by 1, 3, 5, 7, 9, &c; which added, make $\frac{1}{2}$ the log. of 3 to 25 places.

a	5000000000000000000000000	I	5000000000000000000000000
aaa	125	III	416666666666666666666666
$aaaaa$	3125	V	625
a^7	78125	VII	11160714285714285714285
a^9	1953125	IX	217013888888888888888
a^{11}	48828125	XI	443892045454545454545
a^{13}	1220703125	XIII	93900240384615384615
a^{15}	30517578125	XV	2034505208333333333
a^{17}	762939453125	XVII	4487879136029411765
a^{19}	19073486328125	XIX	1003867701480263158
a^{21}	476837178203125	XXI	227065313430059523
	11920928955078125	XXIII	51830125891644022
	298023223876953125	XXV	11920928855078125
	74505805969238281	XXVII	2759474295156975
	18626451492309570	XXIX	642291430769295
	4656612873077392	XXXI	150213318486367
	1164153218269343	XXXIII	322247067220283
	291038304567336	XXXV	8315389130495
	72759576141834	XXXVII	1966475030861
	18189894035458	XXXIX	466407539294
	4547473508864	XLI	110913988021
	1136868377216	XLIII	26438799470
	284217094304	XLV	6315935429
	71054273576	XLVII	1511793055
	17763568394	XLIX	362521804
	4440892099	LI	87076316
	1110223025	LIII	20947604
	277555756	LV	5046469
	69333936	LVII	1216385
	17333485	LIX	293788
	4333371	LXI	71039
Differentia 2	1083343	LXIII	17196
—	270836	LXV	4167
Unitas 1	67709	LXVII	1010
Numerus Propositus 3 $=\overset{a}{\frac{1}{2}}\times\frac{1}{2}=\overset{aaa}{\frac{1}{4}}$	16927	LXIX	245
	4232	LXXI	59
—	1058	LXXIII	15
Summa - 4	264	LXXV	3

Half the log. of 3. 5493061443340548456976226

The log. of 3. 10986122886681096913952452

We

We ſhall now proceed to make the logarithm for 3 the ſecond way, which is from the logarithms of 2 and $\frac{1}{2}$.

The Infinite Series, or Numbers continually proportional.

Theſe numbers are continually multiplied by 4, in order to make the half of the logarithm of $1\frac{1}{2}$ to 25 places.

a	2000000000000000000000000
aaa	8
aaaaa	32
a^7	128
a^9	512
a^{11}	2048
	8192
	32768
	131072
	524288
	2097152
	8388608
	33554432
	1342177
	53687
	2147
	85

Differentia $\frac{1}{2}$
—
Unitas - 1
Numerus Propoſitus $1\frac{1}{2}$
—
Summa - $2\frac{1}{2}$

$\overset{a}{\frac{1}{5}} \times \overset{aaa}{\frac{1}{5}} = \frac{1}{25}$

To make this infinite ſeries, I ſhould divide by 25 continually; but if you multiply by 4, and transfer it anſwerably, it will be the ſame thing: becauſe $\frac{1}{5}$ of 1 is $\frac{2}{10}$, and that multiplied in itſelf, is $\frac{4}{100}$. Therefore, multiplying by 4 and dividing by 100, is the ſame thing as multiplying by 25. And thus this infinite ſeries is made very ſpeedily, in order to make the half of the logarithm for 1 and $\frac{1}{2}$.

Theſe numbers are quotes from thoſe on the oppoſite ſide, they being divided by 1, 3, 5, 7, 9, 11, &c; and added, make half the logarithm for $1\frac{1}{2}$ to 25 places.

I	2000000000000000000000000
III	26666666666666666666666
V	64
VII	18285714285714285714
IX	568888888888888888
XI	18618181818181818
XIII	630153846153846
XV	21845333333334
XVII	771011764706
XIX	27594105204
XXI	998643809
XXIII	36472209
XXV	1342177
XXVII	49711
XXIX	1851
XXXI	69
XXXIII	3

2027325540540821909890065
half the logarithm of $1\frac{1}{2}$,
4054651081081643819780131
the logarithm of $1\frac{1}{2}$,
6931471805599453094172321
the logarithm for 2,
10986122886681096913952452
the logarithm for 3.

Having now made the logarithm for $1\frac{1}{2}$, you add to it the logarithm of 2, and that makes the logarithm for 3, which will be found, as before, to be the ſame number.

And now we proceed to make the logarithm for 7, and then we ſhall have all to 11: in order thereunto, we make the logarithm for $1\frac{2}{5}$, or $1\frac{4}{10}$, and add that to the logarithm of 5, and it will produce the logarithm of 7, for that $1\frac{2}{5}$ multiplied by 5, maketh 7, or $1\frac{4}{10} \times 5 = 7$.

The

The Infinite Series, or Numbers continually proportional.

These numbers are continually divided by 36, in order to make half the logarithm of $1\frac{2}{5}$, or $1\frac{4}{10}$.

These numbers are quotes from those on the opposite side, those being divided by 1, 3, 5, 7, 9, 11, &c; and added, make half the logarithm of $1\frac{2}{5}$, or $1\frac{4}{10}$.

a	1666666666666666666666666	I	1666666666666666666666666
.	277777777777777777777777		
aaa	46296296296296296296296	III	15432098765432098765432
.	7716049382716049382716		
aaaaa	1286008230452674897119	V	257201646090534979424
.	214334705075445816186		
a^7	35722450845907636031	VII	5103207263701090863
.	5953741807651272672		
a^9	992290301275212112	IX	110254477819468014
.	165381716879202019		
a^{11}	27563619479867003	XI	2505783589078819
	4593936579977864		
	765656096662972	XIII	58896622820229
.	127609349443829		
	21268224907304	XV	1417881660487
	3544704151217		
	590784025203	XVII	34752001483
	98464004200		
	16410667366	XIX	863719335
	2735111228		
	455851871	XXI	21707237
	75975312		
Differentia 4	12662552	XXIII	550546
	2110425		
Unitas 1	351737	XXV	14069
Numerus Proposit. 1 4 ; *a* *aaa*	58623		
$=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$	9737	XXVII	360
	1623		
Summa - 2 4	270	XXIX	9

This series is made by dividing twice by 6, which is all one as if you divided at once by 36; and so every other number is the proper number of the series to be divided by 1, 3, 5, 7, 9, 11, &c, as in the opposite column, to make the half logarithm for $1\frac{4}{10}$.

1682361183106064652522972 half the logarithm of $1\frac{4}{10}$,
3364722366212129305045944 the logarithm of $1\frac{2}{5}$, or $1\frac{4}{10}$,
1609437912434100316888 95254 the logarithm of 5,
1945910149055313247394 1198 the logarithm of 7.

N. B. The logarithm of 5, in page 69, being put in the room of this in the second column, will produce 1945910149055313305105353 for the true logarithm of 7; those two last numbers being part mistaken.

Having

Having by this calculation made the half logarithm of $1\frac{4}{10}$, if we double it, and to that add the logarithm of 5, that addition will produce the logarithm of 7, as was required. And now we have all the logarithms to 11, and to make the logarithms from 10 to 100, it will not be much difficulty to proceed after the foregoing methods; as to make the logarithm of 11, you have for the first term *a*, the result or fraction $\frac{5}{6}$, and for *aa*, it will be $\frac{25}{36}$, which is very easy to work; and for the logarithm of 13, you make it of 12 multiplied by $1\frac{1}{12}$; and so it is for the first term *a*, the result or fraction $\frac{1}{25}$; and for the second *aa*, it is $\frac{1}{625}$, which 625 is $= \frac{5}{8}$; and so may you make many easy compendiums for the prime numbers between 10 and 100, and also, not with great difficulty, from 1000 to 10000; and when you have made some logarithms, you will perceive how the differences arise; and having for composites, logarithms in a readiness, greater and lesser than the prime or incomposite very near, it will be, by the difference, no great difficulty to make a logarithm for such a prime very readily and easily. And they that are curious herein, may have compendiums hereof in James Gregory's aforesaid *Vera Circuli & Hyperbolæ Quadratura*, to make logarithms for prime or incomposite numbers, to which I shall refer him. And here I shall content myself to have exemplified James Gregory's method in his *Exercitationes Geometricæ* to so many examples of logarithms as I have herein calculated to 25 places, and shall, in the next chapter, shew how to produce from these geometrical logarithms Briggs's logarithms.

CHAP. VIII.

HAVING, in the preceding chapter, made the logarithms for 2, 3, 4, 5, 6, 7, 8, 9, and 10, according to the geometrical figure, or hyperbola, I require the logarithm of 2, according to Briggs's table. Forasmuch as all logarithms are proportional, it is as the hyperbolical logarithm of 10, is to its logarithm of 2 :: so is Briggs's logarithm of 10 to this logarithm of 2. The operation followeth.

By this division it doth appear, that this quotient doth agree with Briggs's table of logarithms for his logarithm number of 2, whereby it is apparent he did produce the logarithm for 2 to 15 places very true; though I have been told it was eight persons' work for a year's time after his method, which was by large and many extractions of the square root; and if it was so to 15 places, it would have been very tedious, if not impossible, for them to have produced the logarithm of 2 to 25 places, as before herein is shewn, and done by us; and both the hyperbolical and Briggs's logarithm to 25 places, may very well be calculated and done, according to the foregoing method, in half a day's time; by which method herein beforegoing one may make a table of logarithms, in a short space, to what Pardie (a French author), in his Elements of Geometry, hath declared; for he saith, he knew more than 20 persons engaged for 20 years, with indefatigable assiduity, to calculate the logarithms. He doth not say to how many places; but the greatest radius that I have seen of any

French author is but 11 places, which, I ſuppoſe, muſt be the ſame as Vlacq's. And the logarithm for 2, 3, 4, or 5, &c to 11 places, according to the method in this book, may be very well done and performed in leſs than two hours' time.

This diviſor is half the logarithm of 10, according to the hyperbola.
This quotient is the logarithm of 2, according to Briggs's table.
This dividend is compounded of half the hyperbolical logarithm of 2, and Briggs's logarithm of 10.

Diviſor. Quotient.

115129254649702281) (301029995663981190

Dividend.

34657359027997264,00000000000000000

1185820330865797.

3453378436877419.

115079334388337338.

114630052036052851

110137228513207981

65208993284759281

76443659599081405

73561068092600364

45835153027789954

112963766328792697

93474371440605431

13709677208436262

21967517434660339

104445919096901109

8295905121690561

The reader may now ſee, that logarithms derived from this figure, or the hyperbola, are not only more perceptible and intelligible, but with far more certainty and expedition produced, than what was known in former times.

The diviſor in the foregoing work differs 2 units in the 18th place from the half logarithm of 10 before herein calculated; and the reaſon is, that I took Gregory's logarithm of 10, in his *Vera Circuli & Hyperbolæ Quadratura de bene eſſe*; and having calculated the half logarithm of 2, as before, I was very deſirous to ſee if we could produce Briggs's logarithm of 2 to 15 places, as by the diviſion is manifeſt; and this I did long before I met with Gregory's other book of his *Exercitationes Geometricæ*; for, ſince I got that book, I did calculate, *de novo*, the logarithm of 10 to 25 places, according to his doctrine in that book, and as before herein is done. And the calculation of the logarithm of 10, as before, doth agree with Gregory's former book but to 17 places. Howſoever, the diviſion beforegoing is ſufficient to produce Briggs's logarithm for 2 to 15 places; and if any ſhall be ſo curious as to produce Briggs's

Briggs's logarithm for 25 places, he may rely on the foregoing examples herein, and may, in 4 hours' time, examine the foregoing calculations thereof, and in as little time produce Briggs's logarithm for them to the like number of places.

Having this divifion ready done, long before the publifhing hereof, I have contented myfelf to infert it here, whereby the ftudious may foon perceive what to do further to gratify himfelf herein.

I do not add hereto any table of logarithms, that being not my defign at this time, but only to fhew how Briggs's, or any other logarithms, may be derived from the doctrine beforegoing; and alfo for the curious, at his will and pleafure, to examine, whether any logarithms formerly publifhed be truly made or not.

As for the various ufes of logarithms I add none here, but refer the reader to fuch authors (whereof there are plenty) who have, long before, written largely and learnedly, as the firft inventor, the famous Lord Napier, Henry Briggs, Edmund Gunter, Richard Norwood, Wingate, and divers others; as alfo my father, John Speidell; in which the reader may meet with many excellent ufes of the logarithms in all parts of the mathematicks. And I do find, my father printed feveral forts of logarithms, but at laft concluded, that the decimal, or Briggs's logarithms, were the beft fort for a ftandard logarithm; and did alfo print the fame feveral ways, fo ordered, whereby they might be applied to arithmetical queftions, and other operations for the folution thereof, with eafe and readinefs.

END OF SPEIDELL'S LOGARITHMOTECHNIA.

EXTRACTS

FROM THE

PHILOSOPHICAL TRANSACTIONS,

VOL. XIX.

I. *An easy Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line, or Sum of the Secants; with various Methods for computing the same to the utmost Exactness. By Dr. E. Halley.*

IT is now near one hundred years since our worthy countryman, Mr. Edward Wright, published his Correction of Errors in Navigation; a book well deserving the perusal of all such as design to use the sea. Therein he considers the course of a ship on the globe, steering obliquely to the meridian; and having shewn, that the departure from the meridian is, in all cases, less than the difference of longitude, in the ratio of radius to the secant of the latitude, he concludes, that the sum of the secants of each point of the quadrant being added successively, would exhibit a line divided into spaces, such as the intervals of the parallels of latitude ought to be in a true sea chart, whereon the meridians are made parallel lines, and the rhombs, or oblique courses, represented by right lines. This is commonly known by the name of the meridian line, which, though it generally be called Mercator's, was yet undoubtedly Mr. Wright's invention, as he has made it appear in his preface. And the table thereof is to be met with in most books treating of navigation, computed with sufficient exactness for the purpose.

It was first discovered by chance, and, as far as I can learn, first published by Mr. Henry Bond, as an addition to Norwood's Epitome of Navigation, about

about fifty years ſince, that the meridian line was analogous to a ſcale of logarithmick tangents of half the complements of the latitudes. The difficulty to prove the truth of this propoſition, ſeemed ſuch to Mr. Mercator, the author of Logarithmotechnia, that he propoſed to wager a good ſum of money, againſt whoſo would fairly undertake it, that he ſhould not demonſtrate either that it was true or falſe. And, about that time, Mr. John Collins, holding a correſpondence with all the eminent mathematicians of the age, did excite them to this inquiry.

The firſt that demonſtrated the ſaid analogy, was the excellent Mr. James Gregory, in his *Exercitationes Geometricæ*, publiſhed *anno* 1668; which he did not without a long train of conſequences, and complication of proportions, whereby the evidence of the demonſtration is in a great meaſure loſt, and the reader wearied before he attain it. Nor with leſs work and apparatus hath the celebrated Dr. Barrow, in his Geometrical Lectures (Lect. XI. app. 1.), proved, that the ſum of all the ſecants of any arch is analogous to the logarithm of the ratio of radius + ſine to rad. — ſine; or, which is all one, that the meridional parts anſwering to any degree of latitude, are as the logarithms of the rationes of the verſed ſines of the diſtances from both the poles. Since which, the incomparable Dr. Wallis (on occaſion of a paralogiſm committed by one Mr. Norris in this matter) has more fully and clearly handled this argument, as may be ſeen in number 176 of the Tranſactions. But neither Dr. Wallis nor Dr. Barrow, in their ſaid Treatiſes, have any where touched upon the aforeſaid relation of the meridian line to the logarithmick tangent; nor hath any one, that I know of, yet diſcovered the rule for computing independently the interval of the meridional parts anſwering to any two given latitudes.

Wherefore having attained, as I conceive, a very facile and natural demonſtration of the ſaid analogy, and having found out the rule for exhibiting the difference of meridional parts between any two parallels of latitude, without finding both the numbers whereof they are the difference, I hope I may be intitled to a ſhare in the improvements of this uſeful part of geometry. And firſt let us demonſtrate the following

PROPOSITION.

The meridian line is a ſcale of logarithmick tangents of the half complements of the latitudes.

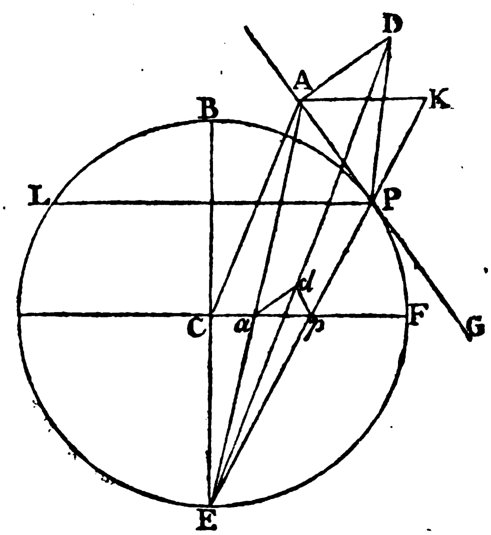
For this demonſtration it is requiſite to premiſe theſe four lemmata.

Lemma 1. In the ſtereographick projection of the ſphere upon the plane of the equinoctial, the diſtances from the center, which, in this caſe, is the pole, are laid down by the tangents of half thoſe diſtances; that is, of half the complements of the latitudes.—This is evident from Eucl. 3, 20.

Lemma 2. In the ſtereographick projection, the angles under which the circles interſect each other are, in all caſes, equal to the ſpherical angles they repreſent; which is, perhaps, as valuable a property of this projection as that of all the

the circles of the ſphere thereon appearing circles. But this not being vulgarly known, muſt not be aſſumed without a

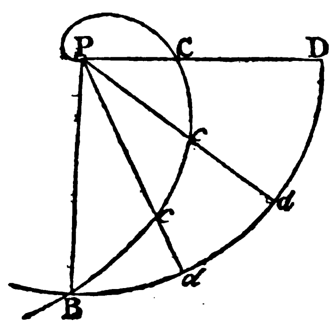
DEMONSTRATION.



Let EBPL be any great circle of the ſphere, E the eye placed in its circumference, C its centre, P any point thereof, and let FCO be ſuppoſed a plane erected at right angles to the circle EBPL, on which FCO we deſign the ſphere to be projected. Draw EP croſſing the plain FCO in *p*, and *p* ſhall be the point P projected. To the point P draw the tangent APG, and on any point thereof, as A, erect a perpendicular AD at right angles to the plane EBPL, and draw the lines PD, AC, DC, and the angle APD ſhall be equal to the ſpherical angle contained between the planes APC, DPC. Draw alſo AE, DE, interſecting the plain FCO in the points *a* and *d*, and join *ad*, *pd*; I ſay, the triangle *adp* is ſimilar to the triangle ADP, and the angle *apd* equal to the angle APD. Draw PL, AK, parallel to FO, and, by reaſon of the parallels, *ap* will be to *ad* as AK to AD, But (by Eucl. 3, 32) in the triangle AKP, the angle AKP = LPE, is alſo equal to APK = EPG; wherefore the ſides AK, AP, are equal, and it will be as *ap* to *ad*, ſo AP to AD; whence the angles DAP, *dap*, being right, the angle APD will be equal to the angle *apd*; that is, the ſpherical angle is equal to that on the projection, and that in all caſes; which was to be proved.

This lemma I lately received from Mr. Demoivre, though I ſince underſtand, from Dr. Hook, that he long ago produced the ſame thing before the ſociety. However, the demonſtration, and the reſt of the diſcourſe, is my own.

Lemma 3. On the globe the rhomb lines make equal angles with every meridian, and, by the foregoing lemma, they muſt likewiſe make equal angles with the meridians in the ſtereographick projection on the plane of the equator; they are therefore, in that projection, proportional ſpirals about the pole point.



Lemma 4. In the proportional ſpiral it is a known property, that the angles BPC, or the arches BD, are exponents of the rationes of BP to PC; for if the arch BD be divided into innumerable equal parts, right lines drawn from them to the centre P, ſhall divide the curve Bccc into an infinity of proportionals; and all the lines Pc ſhall be an infinity of proportionals between PB and PC, whoſe number is equal to all the points *dd* in the arch BD; whence, and by what I have delivered in the next enſuing diſcourſe, it follows, that as BD to B*d*, or as the angle

BPC to the angle BPc, so is the logarithm of the ratio of PB to PC, to the logarithm of the ratio of PB to Pc.

From these lemmata our proposition is very clearly demonstrated; for by the first, PB, Pc, PC, are the tangents of half the complements of the latitudes in the stereographick projection; and by the last of them the differences of longitude, or angles at the pole between them, are logarithms of the rationes of those tangents one to the other. But the nautical meridian line is no other than a table of the longitudes, answering to each minute of latitude on the rhomb line, making an angle of 45 degrees with the meridian; wherefore the meridian line is no other than a scale of logarithmick tangents of the half complements of the latitudes; *quod erat demonstrandum.*

Coroll. 1. Because that in every point of any rhomb line the difference of latitude is to the departure, as the radius to the tangent of the angle that rhomb makes with the meridian; and those equal departures are every where to the differences of longitude, as the radius to the secant of the latitude; it follows, that the differences of longitude are, on any rhomb, logarithms of the same tangents, but of a differing species, being proportioned to one another as are the tangents of the angles made with the meridian.

Coroll. 2. Hence any scale of the logarithm tangents (as those of the vulgar tables made after Briggs's form, or those made to Napier's, or any other form whatsoever) is a table of the differences of longitude to the several latitudes, upon some determinate rhomb or other; and therefore as the tangent of the angle of such rhomb to the tangent of any other rhomb, so the difference of the logarithms of any two tangents to the difference of longitude on the proposed rhomb, intercepted between the two latitudes, of whose half complements you took the logarithm tangents.

And since we have a very complete table of logarithm tangents of Briggs's form, published by Vlacq, *anno* 1633, in his *Canon Magnus Triangulorum Logarithmicus,* computed to ten decimal places of the logarithm, and to every ten seconds of the quadrant (which seems to be more than sufficient for the nicest calculator) I thought fit to inquire the oblique angle with which that rhomb line crosses the meridian, whereon the said Canon of Vlacq precisely answers to the differences of longitudes, putting unity for one minute thereof, as in the common meridian line. Now the momentary argument or fluxion of the tangent line at 45 degrees is exactly double to the fluxion of the arch of the circle, as may easily be proved; and the tangent of 45 being equal to radius, the fluxion also of the logarithm tangent will be double to that of the arch, if the logarithm be of Napier's form; but for Briggs's form, it will be as the same doubled arch multiplied into 0,43429 &c, or divided by 2,30258 &c. Yet this must be understood only of the addition of an indivisible arch, for it ceases to be true if the arch have any determinate magnitude.

Hence it appears, that if any one minute be supposed unity, the length of the arch of one minute being ,000290888208665721596154 &c, in parts of the radius, the proportion will be as unity to 2,908882 &c, so radius to the tangent of 71° 1′ 42″, whose logarithm is 10.46372611720718325204 &c; and under that angle is the meridian intersected by that rhomb line, on which the differences of Napier's logarithm tangents of the half complements of the latitudes are the true

true differences of longitude eſtimated in minutes and parts, taking the firſt four figures for integers. But for Vlacq's tables we muſt ſay,

As 2302585 &c to 2908882 &c, ſo radius to 1,2633114387424456921 2 &c, which is the tangent of 51° 38′ 9″, and its logarithm 10,10151042850772 0941162 &c; wherefore in the rhomb line which makes an angle of 51° 38′ 9″ with the meridian, Vlacq's logarithm tangents are the true differences of longitude. And this compared with our ſecond corollary may ſuffice for the uſe of the tables already computed.

But if a table of logarithm tangents be made by extraction of the root of the infiniteth power whoſe index is the length of the arch you put for unity (as for minutes the ,0002908882th &c power), which we will call a; ſuch a ſcale of tangents ſhall be the true meridian line, or ſum of all the ſecants, taken infinitely many. Here the reader is deſired to have recourſe to my little Treatiſe of Logarithms, in the enſuing diſcourſe, that I may not need to repeat it. By what is there delivered, it will follow, that putting t for the exceſs or defect of any tangent above or under the radius or tangent of 45, the logarithm of the ratio of radius to ſuch tangent will be

$\frac{1}{m}$ into $t - \frac{1}{2}tt + \frac{1}{3}ttt - \frac{1}{4}tttt + \frac{1}{5}t^5$ &c when the arch is greater than 45gr, or

$\frac{1}{m}$ into $t + \frac{1}{2}tt + \frac{1}{3}ttt + \frac{1}{4}tttt + \frac{1}{5}t^5$ &c when it is leſs than 45gr. And, by the ſame doctrine, putting T for the tangent of any arch, and t for the difference thereof from the tangent of another arch, the logarithm of their ratio will be

$\frac{1}{m}$ into $\frac{t}{T} + \frac{tt}{2TT} + \frac{t^3}{3TT} + \frac{t^4}{4T^4} + \frac{t^5}{5T^5}$ &c when T is the greater term, or

$\frac{1}{m}$ into $\frac{t}{T} - \frac{tt}{2TT} + \frac{t^3}{3T^3} - \frac{t^4}{4T^4} + \frac{t^5}{5T^5}$ &c when T is the leſſer term.

And if m be ſuppoſed ,0002908882 &c $= a$, its reciprocal $\frac{r}{a}$ will be 3437, 74677078493925 26 &c, which multiplied into the aforeſaid ſeries, ſhall give preciſely the difference of meridional parts between the two latitudes to whoſe half complements the aſſumed tangents belong. Nor is it material from whether pole you eſtimate the complements, whether the elevated or depreſſed, the tangents being to one another in the ſame ratio as their complements, but inverted.

In the ſame diſcourſe I alſo ſhewed, that the ſeries might be made to converge twice as ſwift, all the even powers being omitted; and putting τ for the ſum of the two tangents, the ſame logarithm would be $\frac{2}{m}$ or $\frac{2r}{a}$ into $\frac{t}{\tau} + \frac{t^3}{3\tau^3} + \frac{t^5}{5\tau^5} + \frac{t^7}{7\tau^7} + \frac{t^9}{9\tau^9}$ &c; but the ratio of τ to t, or of the ſum of two tangents to their difference, is the ſame as that of the ſine of the ſum of the arches to the ſine of their difference: wherefore, if S be put for the ſine complement of the middle latitude, and s for the ſine of half the difference of latitudes, the ſame ſeries will be $\frac{2r}{a}$ into $\frac{s}{S} + \frac{s^3}{3S^3} + \frac{s^5}{5S^5} + \frac{s^7}{7S^7} + \frac{s^9}{9S^9}$ &c; wherein, as the differences of latitudes are ſmaller, fewer ſteps will ſuffice. And if the equator be put for the middle latitude, and conſequently S = R, and s to the ſine of

4 the

the latitude, the meridional parts reckoned from the equator will be $\frac{s}{a} + \frac{s^3}{3rra} + \frac{s^5}{5r^4a} + \frac{s^7}{7r^6a}$ &c, which is coincident with Dr. Wallis's ſolution, in number 176 of the Philoſophical Tranſactions. And this ſame ſeries being half the logarithm of the ratio of $R + s$ to $R - s$, that is, of the verſed ſines of the diſtances from both poles, does agree with what Dr. Barrow had ſhewn in his XIth lecture.

The ſame ratio of τ to t may be expreſſed alſo by that of the ſum of the co-ſines of the two latitudes to the ſine of their difference, as likewiſe by that of the ſines of the ſum of the two latitudes to the difference of their co-ſines, or by that of the verſed ſine of the ſum of the co-latitudes to the difference of the ſines of the latitudes, or as the ſame difference of the ſines of the latitudes to the verſed ſine of the difference of the latitudes; all which are in the ſame ratio of the co-ſine of the middle latitude, to the ſine of half the difference of the latitudes. As it were eaſy to demonſtrate, if the reader were not ſuppoſed capable to do it himſelf, upon a bare inſpection of a ſcheme duly repreſenting theſe lines.

This variety of expreſſion of the ſame ratio I thought not fit to be omitted, becauſe by help of the rationality of the ſine of 30gr. in all caſes where the ſum or difference of the latitudes is 30gr. 60gr. 90gr. 120gr. or 150 degrees, ſome one of them will exhibit a ſimple ſeries, wherein great part of the labour will be ſaved. And beſides, I am willing to give the reader his choice, which of theſe equippolent methods to make uſe of; but for his exerciſe ſhall leave the proſecution of them, and the compendia ariſing therefrom, to his own induſtry; contenting myſelf to conſider only the former, which, for all uſes, ſeems the moſt convenient, whether we deſign to make the whole meridian line or any part thereof, viz. $\frac{2r}{a}$ into $\frac{s}{S} + \frac{s^3}{3S^3} + \frac{s^5}{5S^5} + \frac{s^7}{7S^7} + \frac{s^9}{9S^9}$ &c; wherein a is the length of any arch which you deſign ſhall be the integer or unity in your meridional parts (whether it be a minute, league, or degree, or any other), S the co-ſine of the middle latitude, and s the ſine of half the difference of latitudes; but the ſecants being the reciprocals of the co-ſines, $\frac{s}{S}$ will be equal to $\frac{ſs}{rr}$ putting ſ for the ſecant of the middle latitude; and $\frac{2r}{a}$ into $\frac{s}{S}$ will be $= \frac{2ſs}{ar}$. This multiplied by $\frac{ss}{3SS}$, that is, by $\frac{ſſss}{3rrrr}$ will give the ſecond ſtep; and that again by $\frac{3ſſss}{5rrrr}$, the third ſtep, and ſo forward till you have completed as many places as you deſire. But the ſquares of the ſines being in the ſame ratio with the verſed ſines of the double arches, we may, inſtead of $\frac{ss}{3SS}$, aſſume for our multiplicator $\frac{v}{3V}$, or the verſed ſine of the difference of the latitudes, divided by thrice the verſed ſine of the ſum of the co-latitudes, &c, which is the utmoſt compendium I can think of for this purpoſe, and the ſame ſeries will become $\frac{2sr}{aS}$ into $1 + \frac{v}{3V} + \frac{v^2}{5V^2} + \frac{v^3}{7V^3} + \frac{v^4}{9V^4}$ &c.

Hereby

Hereby we are enabled to eſtimate the default of the method of making the meridian line by the continued addition of the ſecants of æquidifferent arches, which, as the difference of thoſe arches are ſmaller, does ſtill nearer and nearer approach the truth. If we aſſume, as Mr. Wright did, the arch of one minute to be unity, and one minute to be the common difference of a rank of arches, it will be, in all caſes, as the arch of one minute to its chord :: ſo the ſecant of the middle latitude to the firſt ſtep of our ſeries. This, by reaſon of the near equality between a and $2s$, which are to one another in the ratio of unity to 1 — 0,0000000035256645 7713 &c, will not differ from the ſecant $ſ$ but in the ninth figure, being leſs than it in that proportion. The next ſtep being $+\frac{2ſ^3 s^3}{3ar^5}$ will be equal to the cube of the ſecant of the middle latitude multiplied into $\frac{2sss}{3arr} = 0{,}00000000705132908715$; which therefore, unleſs the ſecant exceed ten times radius, can never amount to 1 in the fifth place. Theſe two ſteps ſuffice to make the meridian line, or logarithm tangent to far more places than any tables of natural ſecants yet extant are computed to; but if the third ſtep be required, it will be found to be $+ſ^5$ into $\frac{2s^5}{5ar^4} = 0{,}000000000000000089498$: by all which it appears, that Mr. Wright's table does no where exceed the true meridian parts by fully half a minute; which ſmall difference ariſes by his having added continually the ſecants of 1′, 2′, 3′, &c, inſtead of $0\frac{1}{2}'$, $1\frac{1}{2}'$, $2\frac{1}{2}'$, $3\frac{1}{2}'$, &c; but, as it is, it is abundantly ſufficient for nautical uſes. That in Sir Jonas Moor's New Syſtem of the Mathematics is much nearer the truth, but the difference from Wright is ſcarce ſenſible till you exceed thoſe latitudes where navigation ceaſes to be practicable; the one exceeding the truth by about half a minute, the other being a very ſmall matter deficient therefrom.

For an example eaſy to be imitated by whoſoever pleaſes, I have added the true meridional parts to the firſt and laſt minutes of the quadrant; not ſo much that there is any occaſion for ſuch accuracy, as to ſhew that I have obtained, and laid down herein, the full doctrine of theſe ſpiral rhombs which are of ſo great concern in the art of navigation.

The firſt minute is 1.00000001410265862178.
The ſecond - 2,00000005641063806707.
The laſt, or 89° 59′, is 30374,9634311414228643.

And not 32348,5279, as Mr. Wright has it, by adding the ſecants of every whole minute; nor 30249,8, as Mr. Oughtred's rule makes it, by adding the ſecants of every other half minute; nor 30364,3, as Sir Jonas Moor had concluded it, by I know not what method, though in the reſt of his table he follows Oughtred.

And this may ſuffice to ſhew how to derive the true meridian line from the ſines, tangents, or ſecants, ſuppoſed ready made; but we are not deſtitute of a method for deducing the ſame independently from the arch itſelf. If the latitude from the equator be eſtimated by the length of its arch A, radius being unity, and the arch put for an integer be a, as before, the meridional parts anſwering to that latitude will be $\frac{1}{a}$ into $A + \frac{1}{6}A^3 + \frac{1}{24}A^5 + \frac{\frac{61}{84}}{60}A^7$ or $\frac{61}{5040}A^7$

1

$+ \frac{277}{72576} A^9$ or $\frac{1385}{362880} A^9$ &c; which converges much ſwifter than any of the former ſeries, and beſides has the advantage of A increaſing in arithmetical progreſſion, which would be of great eaſe, if any ſhould undertake, *de novo*, to make the logarithm tangents, or the meridian line, to many more places than now we have them. The logarithm tangent to the arch of $45 + \frac{1}{2}A$ being no other than the aforeſaid ſeries $A + \frac{1}{6}A^3 + \frac{1}{24}A^5$ &c, in Napier's form, or the ſame multiplied into 0,43429 &c, for Briggs's.

But becauſe all theſe ſeries toward the latter end of the quadrant do converge exceeding ſlowly, ſo as to render this method almoſt uſeleſs, or, at leaſt, very tedious, it will be convenient to apply ſome other arts, by aſſuming the ſecants of ſome intermediate latitudes; and you may for s, or the ſine of α, the arch of half the difference of latitudes, ſubſtitute $\alpha - \frac{1}{6}\alpha^3 + \frac{1}{120}\alpha^5 - \frac{1}{5040}\alpha^7 + \frac{1}{362880}\alpha^9$ &c, according to Mr. Newton's rule for giving the ſine from the arch; and if α be no more than a degree, a very few ſteps will ſuffice for all the accuracy that can be deſired.

And if α be commenſurable to a; that is, if it be a certain number of thoſe arches with which you make your integer, then will $\frac{\alpha}{a}$ be that number; which if we call n, the parts of the meridional line will be found to be

$$\frac{sn}{r} \text{ into } \left\{ \begin{array}{llll} 1 & + \frac{ss\alpha\alpha}{3r^4} & + \frac{s^4\alpha^4}{5r^8} & + \frac{s^6\alpha^6}{7r^{12}} \text{ \&c.} \\ & - \frac{\alpha\alpha}{6rr} & - \frac{ss\alpha^4}{6r^6} & - \frac{s^4\alpha^6}{6r^{10}} \text{ \&c.} \\ & & + \frac{1\alpha^4}{120r^4} & + \frac{13s^2\alpha^6}{360r^8} \text{ \&c.} \\ & & & - \frac{1\alpha^6}{5040r^6} \text{ \&c.} \end{array} \right.$$

In this the firſt two ſteps are generally ſufficient for nautical uſes, eſpecially when neither of the latitudes exceed 60 degrees, and the difference of latitudes doth not paſs 30 degrees.

But I am ſenſible I have already ſaid too much for the learned, though too little for the learner: to ſuch I can recommend no better treatiſe than Dr. Wallis's precedent Diſcourſe, wherein he has, with his uſual brevity and that perſpicuity peculiar to himſelf, handled this ſubject from the firſt principles, which here, for the moſt part, we ſuppoſe known.

I need not ſhew how, by regreſſive work, to find the latitudes from the meridional parts, the method being ſufficiently obvious: I ſhall only conclude with the propoſal of a problem which remains to make this doctrine complete, and that is this:

A ſhip ſails from a given latitude, and, having run a certain number of leagues, has altered her longitude by a given angle; it is required to find the courſe ſteered. The ſolution hereof would be very acceptable, if not to the public, at leaſt to the author of this tract; being likely to open ſome further light into the myſteries of geometry.

M 2 To

To conclude, I ſhall only add, that unity being radius; the co-ſine of the arch A, according to the ſame rules of Mr. Newton, will be $1 - \frac{1}{2}A^2 + \frac{1}{24}A^4 - \frac{1}{720}A^6 + \frac{1}{40320}A^8 - \frac{1}{3628800}A^{10}$ &c; from which, and the former ſeries exhibiting the ſine by the arch, by diviſion, it is eaſy to conclude, that the natural tangent of the arch A is $A + \frac{1}{3}A^3 + \frac{2}{15}A^5 + \frac{17}{315}A^7 + \frac{62}{2835}A^9$, &c, and the natural ſecant to the ſame arch $1 + \frac{1}{2}A^2 + \frac{5}{24}A^4 + \frac{61}{720}A^6 + \frac{277}{8064}A^8$ &c; and from the arithmetick of infinites, the number of theſe ſecants being the arch A, it follows, that the ſum total of all the infinite ſecants on that arch is $A + \frac{1}{6}A^3 + \frac{1}{24}A^5 + \frac{61}{5040}A^7 + \frac{277}{72576}A^9$ &c; the which, by what foregoes, is the logarithm tangent of Napier's form, for the arch of 45gr. $+ \frac{1}{2}A$, as before.

And collecting the infinite ſum of all the natural tangents on the ſaid arch A, there will ariſe $\frac{1}{2}AA + \frac{1}{12}A^4 + \frac{1}{45}A^6 + \frac{17}{2520}A^8 + \frac{31}{14175}A^{10}$ &c, which will be found to be the logarithm of the ſecant of the ſame arch A.

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2. *A moſt compendious and facile Method of conſtructing the Logarithms, exemplified and demonſtrated from the Nature of Numbers, without any Regard to the Hyperbola; with a ſpeedy Method for finding the Number from the Logarithm given. By Dr. E. Halley.*

THE invention of the logarithms is juſtly eſteemed one of the moſt uſeful diſcoveries in the art of numbers, and accordingly has had an univerſal reception and applauſe; and the great Geometricians of this age have not been wanting to cultivate this ſubject with all the accuracy and ſubtilty a matter of that conſequence doth require; and they have demonſtrated ſeveral very admirable properties of theſe artificial numbers which have rendered their conſtruction much more facile than by thoſe operoſe methods at firſt uſed by their truly noble inventor, the Lord Napier, and our worthy conntryman, Mr. Briggs.

But, notwithſtanding all their endeavours, I find very few of thoſe who make conſtant uſe of logarithms, to have attained an adequate notion of them, to know how to make or examine them, or to underſtand the extent of the uſe of them; contenting themſelves with the tables of them, as they find them, without daring to queſtion them, or caring to know how to rectify them, ſhould they be found amiſs; being, I ſuppoſe, under the apprehenſion of ſome great difficulty therein. For the ſake of ſuch, the following tract is principally intended; but not without hopes, however, to produce ſomething that may be acceptable to the moſt knowing in theſe matters.

But firſt it may be requiſite to premiſe a definition of logarithms, in order to render the enſuing Diſcourſe more clear, the rather, becauſe the old one, *Numerorum Proportionalium æqui Differentes comites*, ſeems too ſcanty to define them fully. They may more properly be ſaid to be *Numeri Rationum Exponentes*; wherein we conſider ratio as a *quantitas ſui generis*, beginning from the ratio of equality, or 1 to 1 = 0; being affirmative when the ratio is increaſing, as of unity to a greater number, but negative when decreaſing; and theſe rationes

we ſuppoſe to be meaſured by the number of ratiunculæ contained in each. Now theſe ratiunculæ are ſo to be underſtood as in a continued ſcale of proportionals infinite in number, between the two terms of the ratio, which infinite number of mean proportionals is to that infinite number of the like and equal ratiunculæ between any two terms, as the logarithm of one ratio is to the logarithm of the other. Thus if there be ſuppoſed between 1 and 10 an infinite ſcale of mean proportionals, whoſe number is 100000 &c, *in infinitum*, between 1 and 2 there ſhall be 30102 &c, of ſuch proportionals, and between 1 and 3 there will be 47712 &c, of them; which numbers therefore are the logarithms of the rationes of 1 to 10, 1 to 2, and 1 to 3; and not ſo properly to be called the logarithms of 10, 2, and 3.

But if, inſtead of ſuppoſing the logarithms compoſed of a number of equal ratiunculæ proportional to each ratio, we ſhall take the ratio of unity to any number, to conſiſt always of the ſame infinite number of ratiunculæ, their magnitude in this caſe will be as their number in the former; wherefore if between unity and any number propoſed there be taken any infinity of mean proportionals, the infinitely little augment or decrement of the firſt of thoſe means from unity, will be a ratiuncula; that is, the momentum or fluxion of the ratio of unity to the ſaid number. And ſeeing that in theſe continual proportionals all the ratiunculæ are equal, their ſum, or the whole ratio, will be as the ſaid momentum is directly; that is, the logarithm of each ratio will be as the fluxion thereof. Wherefore if the root of any infinite power be extracted out of any number, the differentiola of the ſaid root from unity ſhall be as the logarithm of that number. So that logarithms thus produced may be of as many forms as you pleaſe to aſſume infinite indices of the power whoſe root you ſeek; as if the index be ſuppoſed 100000 &c, infinitely, the roots ſhall be the logarithms invented by the Lord Napier; but if the ſaid index were 2302585 &c, Mr. Briggs's logarithms would immediately be produced. And if you pleaſe to ſtop at any number of figures, and not to continue them on, it will ſuffice to aſſume an index of a figure or two more than your intended logarithm is to have, as Mr. Briggs did; who, to have his logarithms true to 14 places, by continual extraction of the ſquare root, at laſt came to have the root of the 140737488355328th power; but how operoſe that extraction was will be eaſily judged by whoſo ſhall undertake to examine his Calculus.

Now, though the notion of an infinite power may ſeem very ſtrange, and, to thoſe that know the difficulty of the extraction of the roots of high powers, perhaps, impracticable; yet, by the help of that admirable invention of Mr. Newton, whereby he determines the unciæ or numbers prefixed to the members compoſing powers (on which chiefly depends the doctrine of ſeries), the infinity of the index contributes to render the expreſſion much more eaſy; for if the the infinite power to be reſolved be put (after Mr. Newton's method) $p + pq$, $\overline{p + pq}|^{\frac{1}{m}}$ or $\overline{1 + q}|^{\frac{1}{m}}$, inſtead of $1 + \frac{1}{m}q + \frac{1-m}{2mm}qq + \frac{1-3m+2mm}{6m^3}q^3 + \frac{1-6m+11mm-6m^3}{24m^4}q^4$ &c (which is the root when m is finite), becomes $1 + \frac{1}{m}q - \frac{1}{2m}qq + \frac{1}{3m}q^3 - \frac{1}{4m}q^4 + \frac{1}{5m}q^5$ &c, *mm* being *infinite* infinite; and conſequently,

quently whatever is divided thereby vanishing. Hence it follows, that $\frac{1}{m}$ multiplied into $q - \frac{1}{2}qq + \frac{1}{3}qqq - \frac{1}{4}q^4 + \frac{1}{5}q^5$ &c, is the augment of the first of our mean proportionals between unity and $1 + q$, and is therefore the logarithm of the ratio of 1 to $1 + q$; and whereas the infinite index m may be taken at pleasure, the several scales of logarithms to such indices will be as $\frac{1}{m}$, or reciprocally as the indices. And if the index be taken 10000 &c, as in the case of Napier's logarithms, they will be simply $q - \frac{1}{2}qq + \frac{1}{3}qqq - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c.

Again, if the logarithm of a decreasing ratio be sought, the infinite root of $1 - q$, or $\overline{1 - q}\,\rceil^{\frac{1}{m}}$, is $1 - \frac{1}{m}q - \frac{1}{2m}q^2 - \frac{1}{3m}q^3 - \frac{1}{4m}q^4 - \frac{1}{5m}q^5 - \frac{1}{6m}q^6$ &c; whence the decrement of the first of our infinite number of proportionals will be $\frac{1}{m}$ into $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$, &c; which therefore will be as the logarithm of the ratio of unity to $1 - q$. But if m be put 10000 &c, then the said logarithm will be $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$ &c.

Hence the terms of any ratio being a and b, q becomes $\frac{b-a}{a}$, or the difference divided by the lesser term when it is an increasing ratio; or $\frac{b-a}{b}$ when it is decreasing, or as b to a. Whence the logarithm of the same ratio may be doubly expressed; for putting x for the difference of the terms a and b, it will be either

$$\frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} \text{ \&c,}$$

$$\text{or } \frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} \text{ \&c.}$$

But if the ratio of a to b be supposed divided into two parts, viz. into the ratio of a to the arithmetical mean between the terms, and the ratio of the said arithmetical mean to the other term b, then will the sum of the logarithms of those two rationes be the logarithm of the ratio of a to b; and substituting $\frac{1}{2}z$ instead of $\frac{1}{2}a + \frac{1}{2}b$, the said arithmetical mean, the logarithms of those rationes will be, by the foregoing rule,

$$\frac{1}{m} \text{ in } \frac{x}{z} + \frac{xx}{2zz} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} \text{ \&c,}$$

$$\text{and } \frac{1}{m} \text{ in } \frac{x}{z} - \frac{xx}{2zz} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} \text{ \&c,}$$

$$\text{the sum whereof } \frac{1}{m} \text{ in } \frac{2x}{z} + * + \frac{2x^3}{3z^3} + * + \frac{2x^5}{5z^5} + * + \frac{2x^7}{7z^7} \text{ \&c,}$$

will be the logarithm of the ratio of a to b, whose difference is x and sum z. And this series converges twice as swift as the former, and therefore is more proper for the practice of making logarithms; which it performs with that expedition, that where x the difference is but the hundredth part of the sum, the

first

first ſtep $\frac{2x}{z}$ ſuffices to ſeven places of the logarithm, and the ſecond ſtep to twelve. But if Briggs's firſt twenty chiliads of logarithms be ſuppoſed made, as he has very carefully computed them to fourteen places, the firſt ſtep alone is capable to give the logarithm of any intermediate number true to all the places of thoſe tables.

After the ſame manner may the difference of the ſaid two logarithms be very fitly applied to find the logarithms of prime numbers, having the logarithms of the two next numbers above and below them: for the difference of the ratio of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b, is the ratio of ab to $\frac{1}{4}zz$, and the half of that ratio is that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical. And conſequently the logarithm thereof will be the half difference of the logarithms of thoſe rationes, viz. $\frac{1}{m}$ into $\frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8}$, &c; which is a theorem of good diſpatch to find the logarithm of $\frac{1}{2}z$. But the ſame is yet much more advantageouſly performed by a rule derived from the foregoing, and beyond which, in my opinion, nothing better can be hoped. For the ratio of ab to $\frac{1}{4}zz$, or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, has the difference of its terms $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$, or the ſquare of $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$, which, in the preſent caſe of finding the logarithms of prime numbers, is always unity, and calling the ſum of the terms $\frac{1}{4}zz + ab = yy$, the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$, or $\frac{1}{2}z$ will be found $\frac{1}{m}$ in $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}}$, &c, which converges very much faſter than any theorem hitherto publiſhed for this purpoſe.

Here note $\frac{1}{m}$ is all along applied to adapt theſe rules to all ſorts of logarithms. If m be 10000 &c, it may be neglected, and you will have Napier's logarithms, as we hinted before; but if you deſire Briggs's logarithms, which are now generally received, you muſt divide your ſeries by

2,302585092994045684017991454684364207601101488628772976033328,

or multiply it by the reciprocal thereof, viz.

0,434294481903251827651128918916605082294397005803666566114454.

But to ſave ſo operoſe a multiplication (which is more than all the reſt of the work), it is expedient to divide this multiplicator by the powers of z or y continually, according to the direction of the theorem, eſpecially where x is ſmall and integer, reſerving the proper quotes to be added together, when you have produced your logarithm to as many figures as you deſire; of which method I will give a ſpecimen.

If the curioſity of any gentleman that has leiſure would prompt him to undertake to do the logarithms of all prime numbers under 100000 to 25 or 30 figures, I dare aſſure him, that the facility of this method will invite him thereto; nor can any thing more eaſy be deſired. And, to encourage him, I here give the logarithms of the firſt prime numbers under 20 to 60 places, computed by the accurate pen of Mr. Abraham Sharp (from whoſe induſtry and capacity the world may in time expect great performances), as they were communicated to me by our common friend, Mr. Euclid Speidell.

N°.

N°,	Logarithm.
2	0,301029995663981195213738894724493026768189881462108541310427
3	0,477121254719662437295027903255115309200128864190695864829866
7	0,845098040014256830712216258592636193483572396323965406503835
11	1,041392685158225040750199971243024241706702190466453094596539
13	1,113943352306837769206541895026246254561189005053673288598083
17	1,230448921378273028540169894328337030007567378425046397380368
19	1,278753600952828961536333475756929317951129337394497598906819

The next prime number is 23, which I will take for an example of the foregoing doctrine; and, by the first rules, the logarithm of the ratio of 22 to 23 will be found to be either

$$\frac{1}{22} - \frac{1}{968} + \frac{1}{31944} - \frac{1}{937024} + \frac{1}{25768160} \text{ \&c,}$$
$$\text{or } \frac{1}{23} + \frac{1}{1058} + \frac{1}{36501} + \frac{1}{1119364} + \frac{1}{32181715} \text{ \&c;}$$

as likewise that of the ratio of 23 to 24 by a like process,

$$\frac{1}{23} - \frac{1}{1058} + \frac{1}{36501} - \frac{1}{1119364} + \frac{1}{32181715} \text{ \&c,}$$
$$\text{or } \frac{1}{24} + \frac{1}{1152} + \frac{1}{41472} + \frac{1}{1327104} + \frac{1}{39813120} \text{ \&c.}$$

And this is the result of the doctrine of Mercator, as improved by the learned Dr. Wallis. But by the second theorem, viz. $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5}$, &c, the same logarithms are obtained by fewer steps; to wit,

$$\frac{2}{45} + \frac{2}{273375} + \frac{2}{922640625} + \frac{2}{2615686171875} \text{ \&c,}$$
$$\text{and } \frac{2}{47} + \frac{2}{311469} + \frac{2}{1146725035} + \frac{2}{3546361843241} \text{ \&c;}$$

which was invented and demonstrated in the hyperbolick spaces analogous to the logarithms, by the excellent Mr. James Gregory, in his *Exercitationes Geometricæ*, and since further prosecuted by the aforesaid Mr. Speidell, in a late Treatise, in English, by him published on this subject. But the demonstratiou, as I conceive, was never till now perfected without the consideration of the hyperbola, which, in a matter purely arithmetical, as this is, cannot be so properly applied. But what follows, I think, I may more justly claim as my own, viz. that the logarithm of the ratio of the geometrical mean to the arithmetical between 22 and 24, or of $\sqrt{528}$ to 23, will be found to be either

$$\frac{1}{1058} + \frac{1}{1119364} + \frac{1}{888215334} + \frac{1}{626487882248} \text{ \&c,}$$
$$\text{or } \frac{1}{1057} + \frac{1}{3542796579} + \frac{1}{6596765584852 85} \text{ \&c.}$$

All these series being to be multiplied into 0,4342944819 &c, if you design to make the logarithm of Briggs. But, with great advantage in respect of the work,

work, the said 4342944819 &c, is divided by 1057, and the quotient thereof again divided by three times the square of 1057, and that quotient again by $\frac{5}{3}$ of that square, and that quotient by $\frac{7}{5}$ thereof, and so forth, till you have as many figures of your logarithm as you desire; as, for example, the logarithm of the geometrical mean between 22 and 24 is found by the logarithms of 2, 3, and 11, to be

			1.361,316,961,266,906,129,450,091,726,698,05
1057)	43429	&c	(;...,410,874,628,101,468,143,473,158,863,68
3 in 1117249)	41087	&c	(;...,...,...,122,585,215,441,818,294,600,74
$\frac{5}{3}$ in 1117249)	12258	&c	(;...,...,...,...,...,.65,832,351,843,761,75
$\frac{7}{5}$ in 1117249)	65832	&c	(;...,...,...,...,...,...,...,.42,088,297,65
$\frac{9}{7}$ in 1117249)	42088	&c	(;...,...,...,...,...,...,...,...,...,.29,30
	Summa		1.361,727,836,017,592,878,867,777,112,251,17

Which is the logarithm of 23 to 32 places, and obtained by five divisions with very small divisors; all which is much less work than simply multiplying the series into the said multiplicator 43429, &c.

Before I pass on to the converse of this problem, or to shew how to find the number appertaining to a logarithm assigned, it will be requisite to advertise the reader, that there is a small mistake in the aforesaid Mr. James Gregory's *Vera Quadratura Circuli & Hyperbolæ*, published at Padua, *anno* 1667; wherein he applies his quadrature of the hyperbola to the making the logarithms. In page 48, he gives the computation of the Lord Napier's logarithm of 10 to 25 places, and finds it 2,302,585,092,994,045,624,017,870, instead of 2,302,585,092,994,045,684,017,991, erring in the eighteenth figure; as I was assured upon my own examination of the number I here give you, and by comparison thereof with the same wrought by another hand, agreeing therewith to 57 of the 60 places. Being desirous to be satisfied how this difference arose, I took the no small trouble of examining Mr. Gregory's work, and at length found, that in the inscribed polygon of 512 sides, in the eighteenth figure, was a 0, instead of 9; which being rectified, and the subsequent work corrected therefrom, the result did agree to an unit with our number. And this I propose, not to cavil at an easy mistake in managing of so vast numbers, especially by a hand that has so well deserved of the mathematical sciences, but to shew the exact co-incidence of two so very differing methods to make logarithms, which might otherwise have been questioned.

From the logarithm given to find what ratio it expresses, is a problem that has not been so much considered as the former, but which is solved with the like ease, and demonstrated by a like process, from the same general theorem of Mr. Newton. For, as the logarithm of the ratio of 1 to $1 + q$ was proved to be $\overline{1 + q}|^{\frac{1}{m}} - 1$, and that of the ratio of 1 to $1 - q$ to be $1 - \overline{1 - q}|^{\frac{1}{m}}$; so the logarithm (which we will from henceforth call L) being given, $1 + L$ will be equal to $\overline{1 + q}|^{\frac{1}{m}}$ in the one case, and $1 - L$ will be equal to $\overline{1 - q}|^{\frac{1}{m}}$ in the other; consequently $\overline{1 + L}|^{m}$ will be equal to $1 + q$, and $\overline{1 - L}|^{m}$ to $1 - q$; that is, according

according to Mr. Newton's ſaid rule, $1 + m\text{L} + \frac{1}{2}m^2\text{L}^2 + \frac{1}{6}m^3\text{L}^3 + \frac{1}{24}m^4\text{L}^4 + \frac{1}{120}m^5\text{L}^5$ &c, will be $= 1 + q$, and $1 - m\text{L} + \frac{1}{2}m^2\text{L}^2 - \frac{1}{6}m^3\text{L}^3 + \frac{1}{24}m^4\text{L}^4 - \frac{1}{120}m^5\text{L}^5$ &c, will be equal to $1 - q$, m being any infinite index whatſoever, which is a full and general propoſition from the logarithm given to find the number, be the ſpecies of logarithm what it will. But if Napier's logarithm be given, the multiplication by m is ſaved (which multiplication is indeed no other than the reducing the other ſpecies to his), and the ſeries will be more ſimple, viz. $1 + \text{L} + \frac{1}{2}\text{LL} + \frac{1}{6}\text{L}^3 + \frac{1}{24}\text{L}^4 + \frac{1}{120}\text{L}^5$ &c, or $1 - \text{L} + \frac{1}{2}\text{LL} - \frac{1}{6}\text{L}^3 + \frac{1}{24}\text{L}^4 - \frac{1}{120}\text{L}^5$ &c. This ſeries, eſpecially in great numbers, converges ſo ſlowly, that it were to be wiſhed it could be contracted.

If one term of the ratio, whereof L is the logarithm, be given, the other term will be eaſily had by the ſame rule; for if L were Napier's logarithm of the ratio of a the leſſer to b the greater term, b would be the product of a into $1 + \text{L} + \frac{1}{2}\text{LL} + \frac{1}{3}\text{LLL}$ &c, $= a + a\text{L} + \frac{1}{2}a\text{LL} + \frac{1}{6}a\text{L}^3$ &c. But if b were given, a would be $= b - b\text{L} + \frac{1}{2}b\text{LL} - \frac{1}{6}b\text{L}^3$ &c. Whence, by the help of the chiliads, the number appertaining to any logarithm will be exactly had to the utmoſt extent of the tables. If you ſeek the neareſt next logarithm, whether greater or leſſer, and call its number a, if leſſer, or b, if greater, than the given L, and the difference thereof from the ſaid neareſt logarithm you call l; it will follow, that the number anſwering to the logarithm L will be either a into $1 + l + \frac{1}{2}ll + \frac{1}{6}lll + \frac{1}{24}l^4 + \frac{1}{120}l^5$ &c, or elſe b into $1 - l + \frac{1}{2}ll - \frac{1}{6}lll + \frac{1}{24}l^4 - \frac{1}{120}l^5$ &c; wherein, as l is leſs, the ſeries will converge the ſwifter. And if the firſt 20000 logarithms be given to fourteen places, there is rarely occaſion for the three firſt ſteps of this ſeries to find the number to as many places. But for Vlacq's great canon of 100000 logarithms, which is made but to ten places, there is ſcarce ever need for more than the firſt ſtep $a + al$ or $a + mal$, in one caſe, or elſe $b - bl$ or $b - mbl$, in the other, to have the number true to as many figures as thoſe logarithms conſiſt of.

If future induſtry ſhall ever produce logarithmick tables to many more places than now we have them, the aforeſaid theorems will be of more uſe to reduce the correſpondent natural numbers to all the places thereof. In order to make the firſt chiliad ſerve all uſes, I was deſirous to contract this ſeries, wherein all the powers of l are preſent, into one, wherein each alternate power might be wanting; but found it neither ſo ſimple nor uniform as the other. Yet the firſt ſtep thereof is, I conceive, moſt commodious for practice, and withal exact enough for numbers not exceeding fourteen places, ſuch as are Mr. Briggs's large table of logarithms; and therefore I recommend it to common uſe.—It is thus: $a + \frac{al}{1 - \frac{1}{2}l}$ or $b - \frac{bl}{1 + \frac{1}{2}l}$ will be the number anſwering to the logarithm given, differing from the truth by but one half of the third ſtep of the former ſeries. But that which renders it yet more eligible, is, that, with equal facility, it ſerves for Briggs's, or any other ſort of logarithm, with the only variation of writing $\frac{1}{m}$ inſtead of 1; that is, $a + \frac{al}{\frac{1}{m} - \frac{1}{2}l}$ and $b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l}$ or $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l}$

and

and $\dfrac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l}$; which are eaſily reſolved into analogies, viz. As 43429 &c $- \frac{1}{2}l$ to $43429 + \frac{1}{2}l$ :: ſo is a to the number ſought; or, as 43429 &c $+ \frac{1}{2}l$ to $43429 - \frac{1}{2}l$ :: ſo is b to the ſaid number ſought.

If more ſteps of this ſeries be deſired, it will be found as follows, $a + \dfrac{al}{1 - \frac{1}{2}l}$ $- \dfrac{\frac{1}{12}al^3}{1 - l} + \dfrac{\frac{1}{33}al^5}{1 - 2l}$ &c, as may eaſily be demonſtrated by working out the diviſions in each ſtep, and collecting the quotes, whoſe ſum will be found to agree with our former ſeries.

Thus, I hope, I have cleared up the doctrine of logarithms, and ſhewn their conſtruction and uſe, independent from the hyperbola, whoſe affections have hitherto been made uſe of for this purpoſe, though this be a matter purely arithmetical, nor properly demonſtrable from the principles of geometry. Nor have I been obliged to have recourſe to the method of indiviſibles, or the arithmetick of infinites, the whole being no other than an eaſy corollary to Mr. Newton's General Theorem for forming Roots and Powers.

END OF DR. HALLEY'S DISCOURSES.

NOTES

ON

SOME OF THE MORE DIFFICULT PASSAGES

OF THE

FOREGOING DISCOURSE

OF

DR. EDMUND HALLEY.

BY *FRANCIS MASERES*, ESQ.
CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

THE foregoing discourse of Dr. Halley on the construction of logarithms has always been considered, even by persons of great skill in the mathematicks, as a very obscure and difficult tract: I have therefore thought it might be useful to draw up the following explanatory notes upon many of the more difficult passages of it, by which, I hope, I have rendered them sufficiently intelligible. The reader will, however, observe, that I have made no notes on some of the most important parts of this discourse, to wit, those parts in which the author gives us the investigation of the two logarithmick serieses $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c, *ad infinitum*, and $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, *ad infinitum*, and the two anti-logarithmick serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ &c, *ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c, *ad infinitum*. The reason of my omitting to make notes on these important passages, is, that I never have been able perfectly to understand them. I have, however, given full investigations of the two former of these infinite serieses, in the "Remarks on the said Serieses" (which were first published by Mr. Mercator and Dr. Wallis), contained in the former volume of this Collection of Tracts, from page 235 to page

page 344; and I have given the like investigations of the two latter of these serieses, or the two anti-logarithmick serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ &c, *ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c, *ad infinitum*, in the subsequent discourse, intitled, "An Appendix to the foregoing Remarks." And it seems probable, that these investigations of these several serieses are either the same with those given, or intended to be given, by Dr. Halley, in the preceeding discourse, or very analogous to them, and founded on the same principles of arithmetick and abstract reasoning on the nature of ratios, without having recourse to the hyperbola, or any other geometrical figure. I therefore refer the reader to those two discourses, in the former volume of those Tracts, for the investigations of these four serieses; and I flatter myself, that, by the attentive perusal of those two discourses, together with the following notes on those passages of Dr. Halley's preceeding tract which I have been able to understand, and which seemed to stand in need of explanation, he will be able to understand, and perceive the truth of, all the conclusions contained in the foregoing tract of Dr. Halley, notwithstanding its obscurity.

NOTE I.

In page 86, line 16, &c.—*Hence the terms of any ratio being a and b, q becomes* $\frac{b-a}{a}$, *or the difference divided by the lesser term, when it is an increasing ratio, or* $\frac{b-a}{b}$, *when it is decreasing, or as b to a.*

In this passage, Dr. Halley supposes b to be greater than a, and he calls a ratio of minority, as that of a to b, *an increasing ratio*, because it proceeds from a lesser term to a greater; and he calls a ratio of majority, as that of b to a, *a decreasing ratio*, because it proceeds from a greater term to a lesser. This seems to be an odd kind of language: but definitions and the meaning of words are arbitrary, or what we please to make them; and this is evidently the sense in which Dr. Halley here uses the expressions of *an increasing* and *a decreasing* ratio.

The propositions therefore that are affirmed by Dr. Halley in this passage of his discourse, are these two, to wit: first, that if a be to b as 1 is to $1 + q$, q will be $= \frac{b-a}{a}$; and, secondly, that if b be to a as 1 is to $1 - q$, q will be $= \frac{b-a}{b}$. Now these propositions may be proved in the manner following.

In

In the first place, if a is to b as 1 is to $1+q$, we shall have, *invertendo*, $b : a :: 1+q : 1$, and, *dividendo*, $b-a : a :: 1+q-1 : 1$, or $b-a : a :: q : 1$; and consequently $q \times a = \overline{b-a} \times 1$, and $q = \frac{\overline{b-a} \times 1}{a} = \frac{b-a}{a}$.

Q. E. D.

And secondly, if b is to a as 1 is to $1-q$, we shall have, *dividendo*, $b-a : a :: 1-\overline{1-q} : 1-q$, or $b-a : a :: q : 1-q$. But, by the supposition, $a : b :: 1-q : 1$; therefore, *ex æquo*, we shall have $b-a : b :: q : 1$, and consequently $q \times b = \overline{b-a} \times 1$, and $q = \frac{\overline{b-a} \times 1}{b} = \frac{b-a}{b}$. Q. E. D.

NOTE II.

IN page 86, line 18, &c.—*Whence the logarithm of the same ratio may be doubly expressed: for, putting x for the difference of the terms a and b, it will be either* $\frac{1}{m} \times$ *the infinite series* $\frac{x}{b} + \frac{xx}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + $ *&c, or* $\frac{1}{m} \times$ *the infinite series* $\frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + $ *&c.*

For if 1 be to $1+q$ as a is to b, the logarithm of the ratio of a to b will be equal to the logarithm of the ratio of 1 to $1+q$, that is, to $\frac{1}{m} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + $ &c. But q is in this case $= \frac{b-a}{a} = \frac{x}{a}$. Therefore $\frac{1}{m} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + $ &c is $= \frac{1}{m} \times$ the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + $ &c. Consequently $\frac{1}{m} \times$ the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + $ &c, is the logarithm of the ratio of a to b. Q. E. D.

Secondly, if 1 is to $1-q$ as b is to a, the logarithm of the ratio of b to a will be equal to the logarithm of the ratio of 1 to $1-q$, that is, to $\frac{1}{m} \times$ the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + $ &c. But q is in this case $= \frac{b-a}{b} = \frac{x}{b}$. Therefore $\frac{1}{m} \times$ the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + $ &c is $= \frac{1}{m} \times$ the series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + $ &c. Consequently $\frac{1}{m} \times$ the series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + $ &c is the logarithm of the ratio of b to a. But the logarithm of the ratio of b to a is equal to the logarithm of the ratio of a to b; because those two ratios are equal,

equal, though contrary, to each other. Therefore $\frac{1}{m}$ × the infinite series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3x^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} +$ &c is equal to $\frac{1}{m}$ × the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} +$ &c, which is the logarithm of the ratio of a to b. Therefore the logarithm of the ratio of a to b is equal either to $\frac{1}{m}$ × the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} +$ &c, or to $\frac{1}{m}$ × the infinite series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} +$ &c, agreeably to Dr. Halley's assertion in the text.

Q. E. D.

NOTE III.

IN page 86, line 23, &c.—*But if the ratio of a to b be supposed to be divided into two parts, viz. into the ratio of a to the arithmetical mean between the terms, and the ratio of the said arithmetical mean to the other term b, then will the sum of the logarithms of those two rationes be the logarithm of the ratio of a to b; and, substituting* $\frac{1}{2}z$ *instead of* $\frac{1}{2}a + \frac{1}{2}b$*, the said arithmetical mean, the logarithms of those rationes will be, by the foregoing rule,*

$$\frac{1}{m} \text{ into the series } \frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \text{\&c},$$

$$\text{and } \frac{1}{m} \text{ into the series } \frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \text{\&c},$$

and the sum of these serieses, or

$$\frac{1}{m} \text{ into the series } \frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \text{\&c},$$

will be the logarithm of the ratio of a to b, whose difference is x, and sum z.

This passage requires some explanation. Dr. Halley affirms in it, that if a and b are two given quantities, of which b is the greater, and x be put for their difference $b - a$, and z for their sum $b + a$, the logarithm of the ratio of a to $\frac{z}{2}$, or to $\frac{b+a}{2}$, will be $= \frac{1}{m}$ × the infinite series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} +$ &c *ad infinitum*, and that the logarithm of the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to b will

I

b will be $= \frac{1}{m} \times$ the infinite ſeries $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} +$ &c *ad infinitum*. Now this may be ſhewn in the manner following.

Let the ratio of a to $\frac{z}{2}$, or to $\frac{b+a}{2}$ (which is evidently a ratio of minority, becauſe b is greater than a, and conſequently $\frac{b+a}{2}$ is greater than $\frac{a+a}{2}$, or than a) be equal to the ratio of $1 - q$ to 1; then will $2a$ be to z in the ſame proportion of $1 - q$ to 1. Therefore, *dividendo*, $z - 2a$ will be to z as $1 - \overline{1 - q}$, or q, is to 1; and conſequently q will be $= \frac{\overline{z-2a} \times 1}{z} = \frac{z-2a}{z} = \frac{b+a-2a}{z} = \frac{b-a}{z} = \frac{x}{z}$. Therefore $\frac{1}{m} \times$ the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, *ad infinitum*, will be $= \frac{1}{m} \times$ the ſeries $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6}$ + &c *ad infinitum*. But $\frac{1}{m} \times$ the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, *ad infinitum*, is the logarithm of the ratio of $1 - q$ to 1; therefore $\frac{1}{m} \times$ the ſeries $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} +$ &c, *ad infinitum*, will be the logarithm of the ſame ratio of $1 - q$ to 1, and conſequently of the ratio of a to $\frac{z}{2}$, or $\frac{b+a}{2}$, which is equal to it.

Q. E. D.

Secondly, let the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to b (which is evidently a ratio of minority, becauſe b is greater than a, and conſequently $2b$ is greater than $b + a$, and $\frac{2b}{2}$, or b, than $\frac{b+a}{2}$) be equal to the ratio of 1 to $1 + Q$.

Then will z be to $2b$ in the ſame proportion of 1 to $1 + Q$, and conſequently, *dividendo*, $2b - z$ will be to z as $1 + Q - 1$, or Q, is to 1. Therefore Q will be $= \frac{\overline{2b - z} \times 1}{z} = \frac{2b - z}{z} = \frac{2b - \overline{b+a}}{z} = \frac{2b - b - a}{z} = \frac{b-a}{z} = \frac{x}{z}$. Therefore $\frac{1}{m} \times$ the infinite ſeries $Q - \frac{Q^2}{2} + \frac{Q^3}{3} - \frac{Q^4}{4} + \frac{Q^5}{5} - \frac{Q^6}{6} +$ &c, *ad infinitum*, will be $= \frac{1}{m} \times$ the infinite ſeries $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6}$ + &c *ad infinitum*. But $\frac{1}{m} \times$ the ſeries $Q - \frac{Q^2}{2} + \frac{Q^3}{3} - \frac{Q^4}{4} + \frac{Q^5}{5} - \frac{Q^6}{6} +$ &c, *ad infinitum*, is the logarithm of the ratio of 1 to $1 + Q$; therefore $\frac{1}{m} \times$ the ſeries $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} +$ &c, *ad infinitum*, is alſo the logarithm of the ratio of 1 to $1 + Q$, and conſequently of the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to b, which is equal to it.

Q. E. D.

And hence it follows that the logarithm of the ratio of a to b (which is equal to the ſum of the two ratios of a to $\frac{b+a}{2}$ and of $\frac{b+a}{2}$ to b) will be equal to the ſum

ſum of $\frac{1}{m}\times$ the ſeries $\frac{x}{z}+\frac{x^2}{2z^2}+\frac{x^3}{3z^3}+\frac{x^4}{4z^4}+\frac{x^5}{5z^5}+\frac{x^6}{6z^6}+$ &c *ad infinitum*, and $\frac{1}{m}\times$ the ſeries $\frac{x}{z}-\frac{x^2}{2z^2}+\frac{x^3}{3z^3}-\frac{x^4}{4z^4}+\frac{x^5}{5z^5}-\frac{x^6}{6z^6}+$ &c *ad infinitum*, or to $\frac{1}{m}\times$ the ſeries $\frac{2x}{z}+\frac{2x^3}{3z^3}+\frac{2x^5}{5z^5}+\frac{2x^7}{7z^7}+\frac{2x^9}{9z^9}+\frac{2x^{11}}{11z^{11}}+$ &c *ad infinitum*; that is, the logarithm of the ratio of *a* to *b*, whereof *b* is the greater, and of which the difference $b-a$ is denoted by *x*, and the ſum $b+a$ is denoted by *z*, is equal to $\frac{1}{m}\times$ the infinite ſeries $\frac{2x}{z}+\frac{2x^3}{z^3}+\frac{2x^5}{z^5}+\frac{2x^7}{z^7}+\frac{2x^9}{z^9}+\frac{2x^{11}}{z^{11}}+$ &c, agreeably to Dr. Halley's laſt aſſertion in the paſſage here explained.

Q. E. D.

Dr. Halley obſerves that this ſeries $\frac{2x}{z}+\frac{2x^3}{3z^3}+\frac{2x^5}{5z^5}+$ &c converges twice as ſwift as either of the two former ſerieſes $\frac{x}{z}+\frac{x^2}{2z^2}+\frac{x^3}{3z^3}+\frac{x^4}{4z^4}+\frac{x^5}{5x^5}+\frac{x^6}{6z^6}+$ &c and $\frac{x}{z}-\frac{x^2}{2z^2}+\frac{x^3}{3z^3}-\frac{x^4}{4z^4}+\frac{x^5}{5z^5}-\frac{x^6}{6z^6}+$ &c; and ſays, that, on that account, it is more proper than thoſe other ſerieſes for the practice of making logarithms. But this circumſtance is not of any great importance, becauſe there is almoſt as much trouble in computing four terms, or any other given number of terms, of the ſeries $\frac{2x}{z}+\frac{2x^3}{3z^3}+\frac{2x^5}{5z^5}+\frac{2x^7}{7z^7}+$ &c, as in computing twice the ſame number of terms of the ſeries $\frac{x}{z}+\frac{x^2}{2z^2}+\frac{x^3}{3z^3}+\frac{x^4}{4z^4}+\frac{x^5}{5z^5}+\frac{x^6}{6z^6}+\frac{x^7}{7z^7}+\frac{x^8}{8z^8}+$ &c, or the ſeries $\frac{x}{z}-\frac{x^2}{2z^2}+\frac{x^3}{3z^3}-\frac{x^4}{4z^4}+\frac{x^5}{5z^5}-\frac{x^6}{6z^6}+\frac{x^7}{7z^7}-\frac{x^8}{8z^8}+$ &c. For in both caſes we muſt compute the ſame number of powers of the fraction $\frac{x}{z}$; in which the principal labour of the buſineſs conſiſts; for when once we have computed the firſt eight powers of $\frac{x}{z}$, to wit, $\frac{x}{z}$, $\frac{x^2}{z^2}$, $\frac{x^3}{z^3}$, $\frac{x^4}{z^4}$, $\frac{x^5}{z^5}$, $\frac{x^6}{z^6}$, $\frac{x^7}{z^7}$, and $\frac{x^8}{z^8}$, or any other number of thoſe powers, the dividing them by their indexes 2, 3, 4, 5, 6, 7, 8, &c, which are very ſmall and eaſy numbers, is a matter of very little difficulty; and the avoiding half of theſe operations of diviſion, by making uſe of the third ſeries $\frac{2x}{z}+\frac{2x^3}{3z^3}+\frac{2x^5}{5z^5}+\frac{2x^7}{7z^7}$ + &c, is but a ſmall ſaving of trouble.

NOTE IV.

IN page 87, line 8, &c.—*For the difference of the ratios of a to $\frac{1}{2}z$, and of $\frac{1}{2}z$ to b, is the ratio of ab to $\frac{1}{4}zz$; and the half of that ratio is that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical; and consequently the logarithm thereof will be the half difference of the logarithms of those rationes, viz.* $\frac{1}{m}$ *into the series* $\frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} +$ *&c* ad infinitum; *which is a theorem of good dispatch to find the logarithm of* $\frac{1}{2}z$.

The propositions contained in this passage may be proved in the manner following.

The ratio of a to $\frac{1}{2}z$ is equal to the ratio of $a \times b$ to $\frac{1}{2}z \times b$, or of ab to $\frac{1}{2}bz$; and the ratio of $\frac{1}{2}z$ to b is equal to the ratio of $\frac{1}{2}z \times \frac{1}{2}z$ to $b \times \frac{1}{2}z$; that is, to the ratio of $\frac{zz}{4}$ to $\frac{1}{2}bz$. Therefore the excess of the ratio of a to $\frac{1}{2}z$ above the ratio of $\frac{1}{2}z$ to b will be equal to the excess of the ratio of ab to $\frac{1}{2}bz$ above the ratio of $\frac{zz}{4}$ to $\frac{1}{2}bz$. But the excess of the ratio of ab to $\frac{1}{2}bz$ above the ratio of $\frac{zz}{4}$ to $\frac{1}{2}bz$ is evidently the ratio of ab to $\frac{zz}{4}$. Therefore the excess of the ratio of a to $\frac{1}{2}z$ above the ratio of $\frac{1}{2}z$ to b is equal to the ratio of ab to $\frac{zz}{4}$; which is the first proposition affirmed in the foregoing passage. Q. E. D.

Secondly, the ratio of $\sqrt{ab}$ to $\frac{1}{2}z$ is equal to half the ratio of ab to $\frac{zz}{4}$; that is, the ratio of the geometrical mean proportional between the two quantities a and b to the arithmetical mean between the same quantities is equal to half the ratio of ab to $\frac{zz}{4}$. But since the ratio of ab to $\frac{zz}{4}$ has been shewn to be equal to the difference of the ratios of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b, it follows that the ratio of $\sqrt{ab}$ to $\frac{z}{2}$ will be equal to half the difference of the said ratios; that is, the ratio of the geometrical mean proportional between the two quantities a and b to the arithmetical mean between the same quantities will be equal to half the difference between the ratios of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b. Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the said geometrical mean to the said arithmetical mean, will be equal to half the difference of the logarithms of the ratios of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b. But the logarithm of the ratio of a to $\frac{1}{2}z$ has been shewn, in Note III. to be $= \frac{1}{m} \times$ the infinite series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} +$ &c *ad infinitum*; and the logarithm of the ratio of $\frac{1}{2}z$ to b has been shewn, in the same note, to be $= \frac{1}{m} \times$ the infinite series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} +$ &c *ad infinitum*. Therefore the excess of the logarithm

logarithm of the ratio of a to $\frac{1}{2}z$ above the logarithm of the ratio of $\frac{1}{2}z$ to b will be $= \frac{1}{m} \times$ the series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} +$ &c $- \frac{1}{m} \times$ the series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} +$ &c $= \frac{1}{m} \times$ the series $\frac{x}{z} - \frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \frac{x^6}{6z^6} +$ &c $= \frac{1}{m} \times$ the series $\frac{2x^2}{2z^2} + \frac{2x^4}{4z^4} + \frac{2x^6}{6z^6} +$ &c; and half the said excess will be $= \frac{1}{m} \times$ the series $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} +$ &c. Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}z$, or to $\frac{b+a}{2}$, or of the geometrical mean proportional between the quantities a and b to the arithmetical mean between the same quantities, will be $= \frac{1}{m} \times$ the infinite series $\frac{x^2}{2z^2} + \frac{x^4}{4z^2} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \frac{x^{10}}{10z^{10}} + \frac{x^{12}}{12z^{12}} +$ &c *ad infinitum*.

Q. E. D.

Thirdly, by this series we may derive the logarithm of the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to 1, from the logarithms of the ratios of a to 1, and of b to 1. For when these two logarithms are known, we need only add them together, and we shall thereby obtain the logarithm of the ratio of ab to 1, the half of which will be the logarithm of the ratio of $\sqrt{ab}$, or the geometrical mean between a and b, to 1. And if we add to this logarithm the logarithm of the ratio of $\frac{z}{2}$ to $\sqrt{ab}$, or of the arithmetical mean between a and b to the said geometrical mean, which may be computed by means of the expression $\frac{1}{m} \times$ the series $\frac{x^4}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \frac{x^{10}}{10z^{10}} + \frac{x^{12}}{12z^{12}} +$ &c, *ad infinitum*, the sum will be the logarithm of the ratio of $\frac{z}{2}$ to 1, or of $\frac{b+a}{2}$ to 1, or of the said arithmetical mean between a and b to 1.

Q. E. D.

If the number b exceeds the number a by 2, we shall have $x\ (= b - a) = 2$, and $z\ (= b + a = a + 2 + a) = 2a + 2$, and consequently $\frac{x}{z} = \frac{2}{2a+2} = \frac{1}{a+1}$. Therefore the series $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \frac{x^{10}}{10z^{10}} + \frac{x^{12}}{12z^{12}} +$ &c, *ad infinitum*, will, in this case, be $= \frac{1}{2 \times \overline{a+1}|^2} + \frac{1}{4 \times \overline{a+1}|^4} + \frac{1}{6 \times \overline{a+1}|^6} + \frac{1}{8 \times \overline{a+1}|^8} + \frac{1}{10 \times \overline{a+1}|^{10}} + \frac{1}{12 \times \overline{a+1}|^{12}} +$ &c *ad infinitum*; and consequently the logarithm of the ratio of $\frac{z}{2}$, or $\frac{2a+2}{2}$, or $a + 1$, to $\sqrt{ab}$, or $\sqrt{a \times \overline{a+2}}$, or $\sqrt{aa + 2a}$, or of the arithmetical mean between a and b, or between a and $a + 2$, to the geometrical mean between them, will be $= \frac{1}{m} \times$ the infinite series $\frac{1}{2 \times \overline{a+1}|^2} + \frac{1}{4 \times \overline{a+1}|^4} + \frac{1}{6 \times \overline{a+1}|^6} + \frac{1}{8 \times \overline{a+1}|^8} + \frac{1}{10 \times \overline{a+1}|^{10}} + \frac{1}{12 \times \overline{a+1}|^{12}} +$ &c *ad infinitum*.

Thus, for example, if a was $= 22$, and $b = 24$, and we had already computed the logarithms of the ratios of all the prime numbers under 23 to 1, we might make ufe of the laft-mentioned feries to find the logarithm of the ratio of the prime number 23 to 1.

For fince we had already computed the logarithms of the ratios of the prime numbers 2 and 3 and 11 to 1, we might from thence derive the logarithms of the ratios of 22 and 24 to 1, by mere addition; becaufe the logarithm of the ratio of 22 to 1 is $= \text{L.} \frac{22}{11} + \text{L.} \frac{11}{1} = \text{L.} \frac{2}{1} + \text{L.} \frac{11}{1}$, and the logarithm of the ratio of 24 to 1 is $= \text{L.} \frac{24}{8} + \text{L.} \frac{8}{1} = \text{L.} \frac{3}{1} + \text{L.} \frac{8}{1} = \text{L.} \frac{3}{1} + \text{L.} \overline{\frac{2}{1}}^{3} = \text{L.} \frac{3}{1} + 3 \times \text{L.} \frac{2}{1}$. And having thus found the logarithms of the ratios of 22 and 24 to 1, we fhould have the logarithm of the ratio of 22×24, or 528, to 1, $= \text{L.} \frac{528}{24} + \text{L} \frac{24}{1} = \text{L.} \frac{22}{1} = \text{L.} \frac{24}{1}$; or the fum of the logarithms of the ratios of 22 to 1 and 24 to 1, would be that of the ratio of 528 to 1. Therefore half the fum of thofe logarithms would be the logarithm of the ratio of the fquare-root of 528, or of the geometrical mean proportional between 22 and 24, to 1. And, laftly, if to this logarithm of the ratio of $\sqrt{528}$ to 1 we fhould add the logarithm of the ratio of 23, or the arithmetical mean between 22 and 24, to $\sqrt{528}$, which may be computed by means of the expreffion $\frac{1}{m} \times$ the infinite feries $\frac{1}{2 \times \overline{a+1}|^{2}} + \frac{1}{4 \times \overline{a+1}|^{4}} + \frac{1}{6 \times \overline{a+1}|^{6}} + \frac{1}{8 \times \overline{a+1}|^{8}} + \frac{1}{10 \times \overline{a+1}|^{10}} + \frac{1}{12 \times \overline{a+1}|^{12}} +$ &c, *ad infinitum*, or $\frac{1}{m} \times$ the feries $\frac{1}{2 \times \overline{23}|^{2}} + \frac{1}{4 \times \overline{23}|^{4}} + \frac{1}{6 \times \overline{23}|^{6}} + \frac{1}{8 \times \overline{23}|^{8}} + \frac{1}{10 \times \overline{23}|^{10}} + \frac{1}{12 \times \overline{23}|^{12}} +$ &c, *ad infinitum*, or $\frac{1}{m} \times$ the feries $\frac{1}{2 \times 529} + \frac{1}{4 \times \overline{529}|^{2}} + \frac{1}{6 \times \overline{529}|^{3}} + \frac{1}{8 \times \overline{529}|^{4}} + \frac{1}{10 \times \overline{529}|^{5}} + \frac{1}{12 \times \overline{529}|^{6}} +$ &c, the fum will be the logarithm of the ratio of 23 to 1.

Q. E. I.

This method of computing the logarithm of the ratio of 23 to 1 is the fame in fubftance with that by which we computed the fame logarithm in the 9th Example of the foregoing Remarks on Mr. Mercator's and Dr. Wallis's Seriefes, Art. 32, 80, and 95, pages 264, 304, and 328.

NOTE

NOTE V.

IN page 87, line 13.—*But the same is yet much more advantageously performed by a rule derived from the foregoing, and beyond which, in my opinion, nothing better can be hoped; for the ratio of ab to* $\frac{1}{4}zz$, *or* $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, *has the difference of its terms* $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$, *or the square of* $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$; *which, in the present case of finding the logarithms of prime numbers, is always unity; and, calling the sum of the terms* $\frac{1}{4}zz + ab = yy$, *the logarithm of the ratio of* $\sqrt{ab}$ *to* $\frac{1}{2}a + \frac{1}{2}b$, *or* $\frac{1}{2}z$, *will be found* $= \frac{1}{m} \times$ *the series* $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} +$ *&c*, ad infinitum; *which converges much faster than any theorem hitherto published for this purpose.*

Since z is $= b + a$, zz will be $= bb + 2ba + aa$. Now bb, ba, and aa, are three quantities in continued geometrical proportion, bb being to ba as b is to a, and ba being to aa also as b is to a. Therefore, by El. 5, 25, the sum of the two extreme terms will be greater than twice the middle term; that is, $bb + aa$ will be greater than $2ab$. Therefore $bb + aa + 2ab$, or $bb + 2ab + aa$, will be greater than $2ab + 2ab$, or $4ab$. Therefore zz (which is $= bb + 2ab + aa$) will be greater than $4ab$, and consequently $\frac{zz}{4}$ will be greater than ab.

Now let ab be put $= \mathrm{A}$, and $\frac{zz}{4} = \mathrm{B}$; then will B be greater than A. Let $\mathrm{B} - \mathrm{A}$, or the difference of B and A, or $\frac{zz}{4}$ and ab, be called D, and $\mathrm{B} + \mathrm{A}$, or the sum of B and A, or $\frac{zz}{4}$ and ab, be called S.

It is shewn above, in Note III. that, if a and b be any two unequal quantities, of which b is the greater, and x be put equal to their difference $b - a$, and z to their sum $b + a$, the logarithm of the ratio of a to b will be equal to $\frac{1}{m} \times$ the infinite series $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \frac{2x^9}{9z^9} + \frac{2x^{11}}{11z^{11}} +$ &c. It follows therefore, that, in the present case, in which there are two unequal quantities A and B, of which B is the greater, and the difference of the said quantities is denoted by D, and their sum by S, the logarithm of the ratio of A to B will be $= \frac{1}{m} \times$ the infinite series $\frac{2\mathrm{D}}{\mathrm{S}} + \frac{2\mathrm{D}^3}{3\mathrm{S}^3} + \frac{2\mathrm{D}^5}{5\mathrm{S}^5} + \frac{2\mathrm{D}^7}{7\mathrm{S}^7} + \frac{2\mathrm{D}^9}{9\mathrm{S}^9} + \frac{2\mathrm{D}^{11}}{11\mathrm{S}^{11}} +$ &c. But the ratio of A to B is equal to the ratio of ab to $\frac{zz}{4}$, because A is $= ab$, and B is $= \frac{zz}{4}$. Therefore the logarithm of the ratio of ab to $\frac{zz}{4}$ will be $= \frac{1}{m} \times$ the infinite series $\frac{2\mathrm{D}}{\mathrm{S}} + \frac{2\mathrm{D}^3}{3\mathrm{S}^3} + \frac{2\mathrm{D}^5}{5\mathrm{S}^5} + \frac{2\mathrm{D}^7}{7\mathrm{S}^7} + \frac{2\mathrm{D}^9}{9\mathrm{S}^9} + \frac{2\mathrm{D}^{11}}{11\mathrm{S}^{11}} +$ &c. Therefore the logarithm of half the ratio of ab to $\frac{zz}{4}$, or of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, will be $= \frac{1}{m} \times$ the infinite series $\frac{\mathrm{D}}{\mathrm{S}} + \frac{\mathrm{D}^3}{3\mathrm{S}^3} + \frac{\mathrm{D}^5}{5\mathrm{S}^5} + \frac{\mathrm{D}^7}{7\mathrm{S}^7} + \frac{\mathrm{D}^9}{9\mathrm{S}^9} + \frac{\mathrm{D}^{11}}{11\mathrm{S}^{11}} +$ &c; that is, the logarithm

logarithm of the ratio of $\sqrt{ab}$, the geometrical mean between the numbers a and b, to $\frac{z}{2}$, or $\frac{b+a}{2}$, the arithmetical mean between the ſame numbers, is $=$ $\frac{1}{m} \times$ the ſeries $\frac{D}{s} + \frac{D^3}{3s^3} + \frac{D^5}{5s^5} + \frac{D^7}{7s^7} + \frac{D^9}{9s^9} + \frac{D^{11}}{11s^{11}} +$ &c, *ad infinitum.*

Now let the difference of the numbers a and b be 2, or let b be $= a + 2$.

Then will ab be $(= a \times \overline{a+2}) = aa + 2a$; and $\frac{zz}{4}$ will be $(= \frac{\overline{a+b}|^2}{4} = \frac{\overline{a+a+2}|^2}{4} = \frac{\overline{2a+2}|^2}{4} = \frac{4a^2+8a+4}{4}) = a^2 + 2a + 1$; and conſequently $\frac{zz}{4} - ab$ will be $(= a^2 + 2a + 1 - \overline{(a^2+2a)}) = 1$. But $\frac{zz}{4} - ab$ is $=$ D. Therefore, in this caſe, or when b is $= a + 2$, D will be $= 1$; and conſequently D^3, D^5, D^7, D^9, D^{11}, and all the following powers of D, will alſo be equal to 1. Therefore the logarithm of the ratio of $\sqrt{ab}$, or the geometrical mean between the numbers a and b, to $\frac{z}{2}$, or $\frac{b+a}{2}$, the arithmetical mean between the ſame numbers, will, in this caſe, or when b is $= a + 2$, become $= \frac{1}{m} \times$ the infinite ſeries $\frac{1}{s} + \frac{1}{3s^3} + \frac{1}{5s^5} + \frac{1}{7s^7} + \frac{1}{9s^9} + \frac{1}{11s^{11}} +$ &c *ad infinitum.* Therefore, if, inſtead of s, we ſubſtitute yy in the terms of this ſeries, or ſuppoſe yy to be equal to $\frac{zz}{4} + ab$, or the ſum of $\frac{zz}{4}$ and ab, we ſhall have $\frac{1}{m} \times$ the ſeries $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \frac{1}{11y^{22}} +$ &c, *ad infinitum*, for the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of the geometrical mean between the two numbers a and b, or a and $a + 2$, to the arithmetical mean $(\frac{b+a}{2}$, or $\frac{a+2+a}{2}$, or $\frac{2a+2}{2}$, or$)$ $a + 1$, between the ſame numbers.

Q. E. I.

An Example of the Computation of a Logarithm by Means of the foregoing Series.

Let it be required to find, by means of this laſt ſeries, the logarithm of the ratio of the geometrical mean between the numbers 22 and 24 to the arithmetical mean between the ſame numbers; that is, to the number 23.

Here we have $a = 22$, $b = 24$, ab $(= 22 \times 24) = 528$, and $\sqrt{ab} = \sqrt{528}$, and z $(= b + a = 24 + 22) = 46$, and conſequently $\frac{z}{2} = 23$, and $\frac{zz}{4}$ $(= \overline{23}|^2) = 529$, and yy $(= \frac{zz}{4} + ab = 529 + 528) = 1057$, and y^4 $(= \overline{1057}|^2 = 1{,}117{,}249$. Therefore the ſeries $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \frac{1}{11y^{22}} +$ &c will be $= \frac{1}{1057} + \frac{1}{3 \times \overline{1057}|^3} + \frac{1}{5 \times \overline{1057}|^5} + \frac{1}{7 \times \overline{1057}|^7} + \frac{1}{9 \times \overline{1057}|^9} + \frac{1}{11 \times \overline{1057}|^{11}} +$ &c, and $\frac{1}{m} \times$ the ſeries $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \frac{1}{11y^{22}} +$ &c

&c will be $= \frac{1}{m} \times$ the series $\frac{1}{1057} + \frac{1}{3 \times \overline{1057}|^3} + \frac{1}{5 \times \overline{1057}|^5} + \frac{1}{7 \times \overline{1057}|^7} + \frac{1}{9 \times \overline{1057}|^9} + \frac{1}{11 \times \overline{1057}|^{11}} +$ &c; that is, the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of $\sqrt{528}$ to 23, will be $= \frac{1}{m} \times$ the series $\frac{1}{1057} + \frac{1}{3 \times \overline{1057}|^3} + \frac{1}{5 \times \overline{1057}|^5} + \frac{1}{7 \times \overline{1057}|^7} + \frac{1}{9 \times \overline{1057}|^9} + \frac{1}{11 \times \overline{1057}|^{11}} +$ &c *ad infinitum.*

Now $\frac{1}{m} \times$ the series $\frac{1}{1057} + \frac{1}{3 \times \overline{1057}|^3} + \frac{1}{5 \times \overline{1057}|^5} + \frac{1}{7 \times \overline{1057}|^7} + \frac{1}{9 \times \overline{1057}|^9} + \frac{1}{11 \times \overline{1057}|^{11}} +$ &c is $=$ the series $\frac{1}{m \times 1057} + \frac{1}{3m \times \overline{1057}|^3} + \frac{1}{5m \times \overline{1057}|^5} + \frac{1}{7m \times \overline{1057}|^7} + \frac{1}{9m \times \overline{1057}|^9} + \frac{1}{11m \times \overline{1057}|^{11}} +$ &c; which, if we put C for the first term $\frac{1}{m \times 1057}$, and D for the second term $\frac{1}{3m \times \overline{1057}|^3}$, and E for the third term $\frac{1}{5m \times \overline{1057}|^5}$, and F, G, H, &c, for the fourth, fifth, sixth, and other following terms of the series, will be $=$ the series $\frac{1}{m \times 1057} + \frac{C}{3 \times \overline{1057}|^2} + \frac{3D}{5 \times \overline{1057}|^2} + \frac{5E}{7 \times \overline{1057}|^2} + \frac{7F}{9 \times \overline{1057}|^2} + \frac{9G}{11 \times \overline{1057}|^2} +$ &c $= \frac{1}{m \times 1057} + \frac{C}{3 \times 1,117,249} + \frac{3D}{5 \times 1,117,249} + \frac{5E}{7 \times 1,117,249} + \frac{7F}{9 \times 1,117,249} + \frac{9G}{11 \times 1,117,249}$ + &c. Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of $\sqrt{528}$ to 23, is $=$ the series $\frac{1}{m \times 1057} + \frac{C}{3 \times 1,117,249} + \frac{3D}{5 \times 1,117,249} + \frac{5E}{7 \times 1,117,249} + \frac{7F}{9 \times 1,117,249} + \frac{9G}{11 \times 1,117,249} +$ &c *ad infinitum.*

But in Briggs's system of logarithms $\frac{1}{m}$ is $=$

$\frac{1}{2.302,585,092,994,045,684,017,991,454,684,364,207, \&c} = 0.434,294,481,903,251,827,651,128,918,916,605,082$, &c. Therefore $\frac{1}{m \times 1057}$ is $(= \frac{1}{m} \times \frac{1}{1057}) = 0.434,294,481,903,251,827,651,128,918,916,605,082, \&c \times \frac{1}{1057} = \frac{0.434,294,481,903,251,827,651,128,918,916,605,082, \&c}{1057} = 0.000,410,874,628,101,468,143,473,158,863,68$; and the series $\frac{1}{m \times 1057} + \frac{C}{3 \times 1,117,249} + \frac{3D}{5 \times 1,117,249} + \frac{5E}{7 \times 1,117,249} + \frac{7F}{9 \times 1,117,249} + \frac{9G}{11 \times 1,117,249} +$ &c is $=$ the series $0.000,410,874,628,101,468,143,473,158,863,68 + \frac{C}{3 \times 1,117,249} + \frac{3D}{5 \times 1,117,249} + \frac{5E}{7 \times 1,117,249} + \frac{7F}{9 \times 1,117,249} + \frac{9G}{11 \times 1,117,249} +$ &c $=$

$$\begin{array}{rl} & 0;000,410,874,628,101,468,143,473,158,863,68 \\ + & 0;\ldots,\ldots,\ldots,122,585,215,441,818,294,600,74 \\ + & 0;\ldots,\ldots,\ldots,\ldots,065,832,351,843,761,75 \\ + & 0;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots,042,088,297,65 \\ + & 0;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots,\ldots,\ldots,029,30 \\ + & \&c \\ = & 0;000,410,874,750,686,749,417,685,385,553,12 \ \&c. \end{array}$$

Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of $\sqrt{528}$ to 23 is $=$ 0.000,410,874,750,686,749,417,685,385,553,12, &c. Q. E. I.

The logarithm of the ratio of 1 to 23 is equal to the ſum of the logarithms of the ratios of 1 to $\sqrt{528}$ and of $\sqrt{528}$ to 23. The logarithm of the ratio of 1 to $\sqrt{528}$ is equal to half the logarithm of the ratio of 1 to 528, or to half the ſum of the logarithms of the ratios of 1 to 22 and of 22 to 528, or of 1 to 22 and of 1 to 24, or to half the ſum of the logarithms of the ratios of 1 to 2, and of 2 to 22, and of 1 to 3, and of 3 to 24, or to half the ſum of the logarithms of the ratios of 1 to 2, and of 1 to 11, and of 1 to 3, and of 1 to 8, or to $\frac{1}{2}$ L. $\frac{1}{2}$ + $\frac{1}{2}$ L. $\frac{1}{11}$ + $\frac{1}{2}$ L. $\frac{1}{3}$ + $\frac{1}{2}$ L. $\frac{1}{8}$, and therefore may be derived from the logarithms of thoſe leſſer ratios by addition. And, if it be ſo derived, it will be found to be $=$ 1.361,316,961,266,906,129,450,091,726,698,05, &c. If therefore we add to this logarithm the logarithm of the ratio of $\sqrt{528}$ to 23, which has juſt now been found to be $=$ 0.000,410,874,750,686,749,417,685,385, 553,12, &c, the ſum, to wit, 1.361,727,836,017,592,878,867,777,112,251,17, &c, will be the logarithm of the ratio of 1 to 23, or of 23 to 1, in Briggs's ſyſtem, or, according to the common way of expreſſing ourſelves on this ſubject, the logarithm of the number 23.

NOTE VI.

IN page 90, line 11.—*If one term of the ratio, whereof* L *is the logarithm, be given, the other term will eaſily be had by the ſame rule: for, if* L *was Napier's logarithm of the ratio of a, the leſſer, to b, the greater, term, b would be the product of a into the ſeries* $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c*, ad infinitum, $=$ *the ſeries* $a + aL + \frac{aL^2}{2} + \frac{aL^3}{2.3} + \frac{aL^4}{2.3.4} + \frac{aL^5}{2.3.4.5} + \frac{aL^6}{2.3.4.5.6} +$ *&c*, ad infinitum. *But if b was given, a would be* $=$ *the ſeries* $b - bL + \frac{bL^2}{2} - \frac{bL^3}{2.3} + \frac{bL^4}{2.3.4} - \frac{bL^5}{2.3.4.5} + \frac{bL^6}{2.3.4.5.6} -$ *&c*, ad infinitum.

The

The first proposition asserted in this passage is, "that, if any number, or quantity, called a, be given, of the ratio of which to another quantity called b, which is greater than it, L is the logarithm in Napier's System, the said greater quantity b, may be derived from a and L, by computing the value of the series $a + a\mathrm{L} + \frac{a\mathrm{L}^2}{2} + \frac{a\mathrm{L}^3}{2.3} + \frac{a\mathrm{L}^4}{2.3.4} + \frac{a\mathrm{L}^5}{2.3.4.5} + \frac{a\mathrm{L}^6}{2.3.4.5.6} +$ &c, *ad infinitum*, to which it will be equal." Now this may be proved in the manner following.

Suppose k to be of such a magnitude, that $1 + k$ shall be to 1 as b is to a. Then will L (which is the logarithm of the ratio of b to a) be also the logarithm of the ratio of $1 + k$ to 1. Therefore $1 + k$ will be $=$ the series $1 + \mathrm{L} + \frac{\mathrm{L}^2}{2} + \frac{\mathrm{L}^3}{2.3} + \frac{\mathrm{L}^4}{2.3.4} + \frac{\mathrm{L}^5}{2.3.4.5} + \frac{\mathrm{L}^6}{2.3.4.5.6} +$ &c. *ad infinitum*, and consequently that series will be to 1 as b is to a. Therefore $b \times 1$, or b, will be $= a \times$ the said series $1 + \mathrm{L} + \frac{\mathrm{L}^2}{2} + \frac{\mathrm{L}^3}{2.3} + \frac{\mathrm{L}^4}{2.3.4} + \frac{\mathrm{L}^5}{2.3.4.5} + \frac{\mathrm{L}^6}{2.3.4.5.6} +$ &c, *ad infinitum*, $=$ $a + a\mathrm{L} + \frac{a\mathrm{L}^2}{2} + \frac{a\mathrm{L}^3}{2.3} + \frac{a\mathrm{L}^4}{2.3.4} + \frac{a\mathrm{L}^5}{2.3.4.5} + \frac{a\mathrm{L}^6}{2.3.4.5.6} +$ &c, *ad infinitum*.

Q. E. D.

The Second Proposition asserted in the foregoing passage is, "That, if any number, or quantity, called b, be given, of the ratio of which to a lesser quantity, called a, the logarithm, in Napier's system, is known and denoted by L, the said lesser quantity a may be derived from the said greater quantity b, and the logarithm L, by computing the value of the series $b - b\mathrm{L} + \frac{b\mathrm{L}^2}{2} - \frac{b\mathrm{L}^3}{2.3} + \frac{b\mathrm{L}^4}{2.3.4} - \frac{b\mathrm{L}^5}{2.3.4.5} + \frac{b\mathrm{L}^6}{2.3.4.5.6} -$ &c, *ad infinitum*, to which it will be equal." This may be proved in the manner following.

Suppose b to be of such a magnitude that 1 shall be to $1 - k$ in the same proportion as b to a. Then will L be the logarithm of the ratio of 1 to $1 - k$, as well as of the ratio of b to a. Therefore $1 - k$ will be $=$ the series $1 - \mathrm{L} + \frac{\mathrm{L}^2}{2} - \frac{\mathrm{L}^3}{2.3} + \frac{\mathrm{L}^4}{2.3.4} - \frac{\mathrm{L}^5}{2.3.4.5} + \frac{\mathrm{L}^6}{2.3.4.5.6} -$ &c. *ad infinitum*. Therefore 1 will be to this series in the same proportion as b is to a; and consequently $a \times 1$, or a, will be $= b \times$ the series $1 - \mathrm{L} + \frac{\mathrm{L}^2}{2} - \frac{\mathrm{L}^3}{2.3} + \frac{\mathrm{L}^4}{2.3.4} - \frac{\mathrm{L}^5}{2.3.4.5} + \frac{\mathrm{L}^6}{2.3.4.5.6} -$ &c, *ad infinitum* $=$ the series $b - b\mathrm{L} + \frac{b\mathrm{L}^2}{2} - \frac{b\mathrm{L}^3}{2.3} + \frac{b\mathrm{L}^4}{2.3.4} - \frac{b\mathrm{L}^5}{2.3.4.5} + \frac{b\mathrm{L}^6}{2.3.3.5.6}$ $-$ &c, *ad infinitum*.

Q. E. D.

NOTE VII.

IN page 90, line 15th.—*Whence, by the help of the Chiliads, the number appertaining to any logarithm will be exactly had to the utmost extent of the Tables. If you seek the nearest next logarithm, whether greater or lesser; and call its number* a, *if lesser, or* b, *if greater, than the given* L, *and the difference thereof from the said nearest logarithm you call* l; *it will follow that the number answering to the logarithm* L *will be either* a × *the series* $1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120} + \frac{l^6}{720} +$ &c, ad infinitum, *or else* b × *the series* $1 - l + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120} + \frac{l^6}{720} -$ &c, ad infinitum.

The first proposition contained in this passage is as follows; to wit, "That, if any logarithm in Napier's system of Logarithms be given, and the same be denoted by L, and we seek in a Table of Napier's Logarithms the logarithm which is nearest to L of all those which are less than L, and denote the said logarithm by the Greek letter Λ and the number corresponding to it (or of the ratio of which to unity it is the logarithm) by a, and the excess of L above Λ by the small letter l, the number corresponding to the given logarithm L, (or of the ratio of which to unity the said given logarithm L is the logarithm,) will be $= a \times$ the series $1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120} + \frac{l^6}{720} +$ &c. *ad infinitum*." This may be shewn in the manner following.

Let b be put for the number hitherto unknown, which corresponds to the given logarithm L, or of the ratio of which to unity L is the logarithm. We shall then be to prove that b is $= a \times$ the series $1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120} + \frac{l^6}{720} +$ &c *ad infinitum*, or = the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c *ad infinitum*.

Now, since L is the logarithm of the ratio of b to 1, and Λ is the logarithm of the ratio of a to 1, it follows that l, or L − Λ, will be the logarithm of the ratio which is equal to the excess of the ratio of b to 1 above the ratio of a to 1; that is, l will be the logarithm of the ratio of b to a. But it has been shewn above in Note 6th, that if l is the logarithm of the ratio of b to a, (the quantity b being greater than a,) b will be equal to the series $a + al + \frac{al^2}{2} + \frac{al^3}{2.3} + \frac{al^4}{2.3.4} + \frac{al^5}{2.3.4.5} + \frac{al^6}{2.3.4.5.6} +$ &c, *ad infinitum*, or the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c, *ad infinitum*. Therefore the ratio of b to 1, or the ratio corresponding to the given logarithm L, is that of the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c, *ad infinitum* to 1. Q. E. D.

The second proposition contained in the foregoing passage is as follows, to wit, " That if any logarithm in Napier's System of logarithms be given, and the same be denoted by the letter L, and you seek in a table of logarithms the logarithm

logarithm which is nearest to L of all those that are greater than L, and denote the said logarithm by the Greek letter Λ, and the number corresponding to it (or of the ratio of which to unity it is the logarithm,) by b, and the excess of Λ above L by the small letter l, the number corresponding to the given logarithm L, (or of the ratio of which to unity L is the logarithm,) will be equal to $b \times$ the series $1 - l + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120} + \frac{l^6}{720} -$ &c, *ad infinitum*, or to the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, *ad infinitum*, or the ratio corresponding to the given logarithm L, is that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, *ad infinitum* to 1." This may be shewn in the manner following.

Let a be put for the number, hitherto unknown, which corresponds to the given logarithm L, or of the ratio of which to unity L is the logarithm. We shall then be to prove that a is equal to $b \times$ the series $1 - l + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120} + \frac{l^6}{720} -$ &c, *ad infinitum*, or to the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, *ad infinitum*.

Now, since L is the logarithm of the ratio of a to 1, and Λ is the logarithm of the ratio of b to 1, it follows that l, or Λ − L, will be the logarithm of the ratio which is equal to the excess of the ratio of b to 1 above the ratio of a to 1, that is, of the ratio of b to a. But it has been shewn above in Note 6th that, if l is the logarithm of the ratio of b to a, (the quantity b being greater than a,) a will be equal to the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{2.3} + \frac{bl^4}{2.3.4} - \frac{bl^5}{2.3.4.5} + \frac{bl^6}{2.3.4.5.6} -$ &c, *ad infinitum*, or the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, *ad infinitum*. Therefore the ratio of a to 1, or the ratio corresponding to the given logarithm L, is that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c *ad infinitum* to 1. Q. E. D.

In the foregoing passage, and the remarks made to explain it, the logarithms L, Λ, and l, have been supposed to be Napier's Logarithms of the ratios to which they respectively belong. But the tables of logarithms that are most in use at present, give us only the logarithms of Briggs's system, in which the logarithm of any given ratio is less than Napier's logarithm of the same ratio in the proportion of 1 to 2.302,585,092,994,045,684, &c, or of 0.434,294,481, 903,251,827, &c to 1. If therefore we are required to find the ratio corresponding to any given logarithm of Briggs's system, we must proceed as follows.

Let the given logarithm of Briggs's system, whereof we are required to find the corresponding ratio, be called B, and Napier's logarithm of the same ratio be called L. Also let the next lesser logarithm to B that is set down in the Tables be denoted by the small Greek letter λ, and Napier's logarithm of the same ratio be denoted by the capital Greek letter Λ. And let the difference be-

tween the logarithms B and λ of Briggs's ſyſtem be called d, and the difference between the correſponding logarithms L and Λ in Napier's Syſtem be called l. Then will L be $= 2.302,585,092,994,045,684$, &c $\times$ B, and Λ will be $=$ $2.302,585,092,994,045,684$, &c $\times\, \lambda$, and L $-$ Λ will be $= 2.302,585$, &c $\times$ B $-$ $2.302,585$ &c $\times\, \lambda = 2.302,585$ &c $\times\, \overline{B - \lambda}$; that is, l will be $=$ $2.302,585$, &c $\times\, d$; or, (if, for brevity's ſake, we put $m = 2.302,585,092,994,045,684$, &c,) l will be $= m\,d$, and conſequently l^2 will be $= m^2 d^2$, and $l^3 = m^3 d^3$, and $l^4 = m^4 a^4$, and l^5, l^6, &c $= m^5 d^5$, $m^6 d^6$, &c.

Further, let a be put for the number found in the tables correſponding to the logarithm λ, which is the next leſs logarithm to the given logarithm B, that is ſet down in the tables. Then will a be the number correſponding likewiſe to the logarithm Λ in Napier's ſyſtem. But it has been ſhewn in the firſt part of this note, that, upon theſe ſuppoſitions, the ratio correſponding to L, or of which L is the logarithm in Napier's ſyſtem, is that of the ſeries $a + a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} + \frac{a\,l^5}{120} + \frac{a\,l^6}{720} +$ &c *ad infinitum* to 1, which is equal to that of the ſeries $a + a\,m\,d + \frac{a\,m^2 d^2}{2} + \frac{a\,m^3 d^3}{6} + \frac{a\,m^4 d^4}{24} + \frac{a\,m^5 d^5}{120} + \frac{a\,m^6 d^6}{720} +$ &c *ad infinitum* to 1. Therefore the ratio correſponding to the given logarithm B, in Briggs's Syſtem, is that of the ſeries $a + a\,m\,d + \frac{a\,m^2 d^2}{2} + \frac{a\,m^3 d^3}{6} + \frac{a\,m^4 d^4}{24} + \frac{a\,m^5 d^5}{120} + \frac{a\,m^6 d^6}{720} +$ &c to 1. Q. E. I.

Secondly, Let the given logarithm of Briggs's ſyſtem, whereof we are required to find the correſponding ratio, be denoted, as before, by the letter B, and the logarithm of the ſame ratio in Napier's ſyſtem be likewiſe, as before, denoted by L. But let the ſmall Greek letter λ be put for the neareſt logarithm of Briggs's ſyſtem, to the given logarithm B of all thoſe that are greater than B, that is ſet down in the table, inſtead of being put for the next leſſer logarithm, as in the former caſe. And let the capital Greek letter Λ be put for Napier's logarithm of the ſame ratio of which λ is the logarithm in Briggs's Syſtem. Alſo let d be put for $\lambda - B$, or the difference of the logarithms λ and B in Briggs's ſyſtem, and l be put for $\Lambda - L$, or the difference of the logarithms Λ and L in Napier's ſyſtem. Then will L be $= 2.302,585,092,994,045,684$, &c $\times$ B, and $\Lambda = 2.302,585,092,994,045,684$, &c $\times\, \lambda$, and conſequently $\Lambda - L$ will be $= 2.302,585$, &c $\times\, \lambda - 2.302,585$, &c $\times$ B $= 2.302,585$ &c $\times\, \overline{\lambda - B} =$ $2.302,585$, &c $\times\, d$, that is, l will be $= 2.302,585,092,994,045,684$, &c. $\times$ d; or, (if we put $m = 2.302,585,092,994,045,684$, &c,) l will be $= m\,d$; and conſequently l^2 will be $= m^2 d^2$, and $l^3 = m^3 d^3$, and $l^4 = m^4 d^4$, and l^5, l^6, &c $= m^5 d^5$, $m^6 d^6$, &c.

Further, let b be put for the number found in the tables correſponding to the logarithm λ, which is the next greater logarithm to the given logarithm B, which is ſet down in the tables. Then will b be the number correſponding likewiſe to the logarithm Λ in Napier's ſyſtem. But it has been ſhewn in the ſecond part of this note, that, upon theſe ſuppoſitions, the ratio correſponding to L, or of which L is the logarithm in Napier's ſyſtem, is that of the ſeries $b - b\,l + \frac{b\,l^2}{2} - \frac{b\,l^3}{6} + \frac{b\,l^4}{24} - \frac{b\,l^5}{120} + \frac{b\,l^6}{720} -$ &c, *ad infinitum* to 1. Therefore the ratio correſponding

sponding to B, or of which B is the logarithm in Briggs's system, (which is the same ratio of which L is the logarithm in Napier's system,) is that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, *ad infinitum* to 1, or (because l is $= md$) that of $b - bmd + \frac{bm^2d^2}{2} - \frac{bm^3d^3}{6} + \frac{bm^4d^4}{24} - \frac{bm^5d^5}{120} + \frac{bm^6d^6}{720}$ — &c, *ad infinitum* to 1. Q. E. I.

In making the foregoing deductions I have constantly used the letters L, Λ, and l, to denote the logarithms of their respective ratios in Napier's system, and have employed the letters B, λ, and d, to denote the logarithms of the same ratios respectively in Briggs's system. This I have done, in order to preserve the reasonings as clear and distinct as possible. But Dr. Halley, in page 90, line 24th, uses the letter l to denote the logarithms of the same ratio both in Napier's and Briggs's systems. His words are these: "But, for Vlacq's great "canon of 100,000 logarithms, which is made but to ten places, there is scarce "ever need for more than the first step $a + al$, or $a + mal$, in one case, or "else $b - bl$, or $b - mbl$, in the other, to have the number true to as many "figures as those logarithms consist of." In this passage the letter l signifies a logarithm of Napier's system in the expressions $a + al$ and $b - bl$; and it signifies a logarithm of Briggs's system, and consequently ml signifies the corresponding logarithm in Napier's system, in the expressions $a + mal$ and $b - mbl$. These last expressions $a + mal$ and $b - mbl$ correspond to the two expressions $a + amd$ and $b - bmd$, in the foregoing deductions, or to the two first terms of the two infinite serieses $a + amd + \frac{am^2d^2}{2} + \frac{am^3d^3}{6} + \frac{am^4d^4}{24} + \frac{am^5d^5}{120} + \frac{am^6d^6}{720} +$ &c, *ad infinitum*, and $b - bmd + \frac{bm^2d^2}{2} - \frac{bm^3d^3}{6} + \frac{bm^4d^4}{24} - \frac{bm^5d^5}{120} + \frac{bm^6d^6}{720} -$ &c, *ad infinitum*, l signifying the same quantity in Dr. Halley's notation as d signifies in ours.

NOTE VIII.

IN page 90, line 29.—*If future industry shall ever produce logarithmick tables to many more places than now we have them, the aforesaid theorems will be of more use to deduce the correspondent numbers to all the places thereof. In order to make the first Chiliad serve all uses, I was desirous to contract the series, wherein all the powers of* l *are present, into one wherein each alternate power might be wanting; but found it neither so simple nor so uniform as the other. Yet the first step thereof is, I conceive, most commodious for practice, and withal exact enough for numbers*

4 *not*

not exceeding fourteen places, ſuch as are thoſe of Mr. Briggs's large table of logarithms; and therefore I recommend it to common uſe. It is thus; $a + \frac{al}{1-\frac{1}{2}l}$, *or* $b - \frac{bl}{1+\frac{1}{2}l}$, *will be the number anſwering to the logarithm given, differing from the truth, but by one half of the third ſtep of the former ſeries.*

This paſſage contains two propoſitions, to wit, 1ſt, "That the two terms $a + \frac{al}{1-\frac{1}{2}l}$, or $a + \frac{1l}{1-\frac{1}{2}l}$, are very nearly equal to the whole ſeries $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c, *ad infinitum*;" and 2dly, "That the two terms $b - \frac{bl}{1+\frac{1}{2}l}$, or $b - \frac{bl}{1+\frac{l}{2}}$, are very nearly equal to the whole ſeries $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, *ad infinitum.*

The firſt of theſe propoſitions may be proved in the manner following.

The fraction $\frac{al}{1-\frac{l}{2}}$ is equal to the geometrical progreſſion $al + \frac{al^2}{2} + \frac{al^3}{4} + \frac{al^4}{8} + \frac{al^5}{16} + \frac{al^6}{32} +$ &c, *ad infinitum.* Therefore the two terms $a + \frac{al}{1-\frac{l}{2}}$ are equal to the ſeries $a + al + \frac{al^2}{2} + \frac{al^3}{4} + \frac{al^4}{8} + \frac{al^5}{16} + \frac{al^6}{32} +$ &c, *ad infinitum,* which agrees with the ſeries $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c, *ad infinitum* in the three firſt terms, and exceeds it only by the ſeries $(\frac{al^3}{4} - \frac{al^3}{6} + \frac{al^4}{8} - \frac{al^4}{24} + \frac{al^5}{16} - \frac{al^5}{120} + \frac{al^6}{32} - \frac{al^6}{720} +$ &c, *ad infinitum,* or $\frac{3al^3}{12} - \frac{2al^3}{12} + \frac{3al^4}{24} - \frac{al^4}{24} + \frac{30al^5}{480} - \frac{4al^5}{480} + \frac{45al^6}{32\times 45} - \frac{2al^6}{2\times 720} +$ &c, *ad infinitum,* or $\frac{al^3}{12} + \frac{2al^4}{24} + \frac{26al^5}{480} + \frac{45al^6}{1440} - \frac{2al^6}{1440} +$ &c, *ad infinitum,* or$)$ $\frac{al^3}{12} + \frac{al^4}{12} + \frac{13al^5}{240} + \frac{43al^6}{1440} +$ &c, *ad infinitum,* of which the firſt term $\frac{al^3}{12}$ is equal to only half of $\frac{al^3}{6}$, which is the fourth term of the ſeries $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c, *ad infinitum,* or the third quantity that is to be computed and added to the firſt term a, or (according to Dr. Halley's expreſſion) the third ſtep of the ſaid ſeries.

Q. E. D.

The ſecond propoſition contained in the aforeſaid paſſage may be proved in the manner following.

The fraction $\frac{bl}{1+\frac{l}{2}}$ is equal to the ſeries $bl - \frac{bl^2}{2} + \frac{bl^3}{4} - \frac{bl^4}{8} + \frac{bl^5}{16} - \frac{bl^6}{32} +$ &c, *ad infinitum.* Therefore the two terms $b - \frac{bl}{1+\frac{l}{2}}$ are equal to $b -$ the ſecond $l - \frac{bl^2}{2} + \frac{bl^3}{4} - \frac{bl^4}{8} + \frac{bl^5}{16} - \frac{bl^6}{32} +$ &c, *ad infinitum,* that is, to the ſeries

ries $b - bl + \frac{bl^2}{2} - \frac{bl^3}{4} + \frac{bl^4}{8} - \frac{bl^5}{16} + \frac{bl^6}{32} -$ &c, *ad infinitum*, which agrees with the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, *ad infinitum* in the three first terms, and falls short of it by the series $\frac{bl^3}{4} - \frac{bl^3}{6} - \frac{bl^4}{8} + \frac{bl^4}{24} + \frac{bl^5}{16} - \frac{bl^5}{120} - \frac{bl^6}{32} + \frac{bl^6}{720} +$ &c, *ad infinitum*, or the series $\frac{bl^3}{12} - \frac{bl^4}{12} + \frac{13bl^5}{240} - \frac{43bl^6}{1440} +$ &c, *ad infinitum*, of which the first term $\frac{bl^3}{12}$ is equal to only half of $\frac{bl^3}{6}$ which is the fourth term of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} -$ &c, or the third quantity that is to be computed and added to the first term a, which is what I understand Dr. Halley to mean by the expression of the *the third step* of the said series. Q. E. D.

NOTE IX.

IN page 90, line 3 from the bottom.—*But that which renders it more eligible, is, that with equal facility it serves for Briggs's, or any other sort of logarithms, with the only variation of writing* $\frac{1}{m}$ *instead of* 1, *that is*, $a + \frac{al}{\frac{1}{m} - \frac{1}{2}l}$ *and* $b \frac{-bl}{\frac{1}{m} +}$, *or* $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l}$ *and* $\frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l}$, *which are easily resolved into analogies, viz. as* 0.434,29 *&c* — $\frac{1}{2}$l *is to* 0.434,29 *&c* + $\frac{1}{2}$l, *so is* a *to the number sought, or as* 0.434,29 *&c* + $\frac{1}{2}$l, *is to* 0.434,29 *&c* — $\frac{1}{2}$l, *so is* b *to the number sought.*

In this passage, (if I understand it rightly,) Dr. Halley uses the letter *l* for the logarithm of a ratio in some other system than Napier's, and supposes that the logarithms of any given ratios in such other system are to the logarithms of the same ratios in Napier's system in the proportion of 1 to *m*. This notation seems to throw some obscurity upon the passage, to remove which I shall substitute *d* instead of *l*, as above in Note 7th, and keep *l* to denote the logarithm of the same ratio in Napier's system. And then the foregoing passage will be found to contain the two following propositions; to wit, 1st, "That, if B be any given "logarithm in Briggs's system, or in any other system of logarithms different "from Napier's, and L be the logarithm of the same ratio in Napier's system "whereof B is the logarithm in the other system, and the proportion of the lo-"garithms of any given ratios in Napier's system to the logarithms of the same "ratios in the other system be that of *m* to 1; and λ be the next lesser loga-"rithm to B that is found in the tables, and *a* be the number found in the ta-"bles

" bles corresponding to λ, or of the ratio of which to unity λ is the logarithm; " and Λ be Napier's logarithm of the same ratio of a to 1; and d be $= \mathrm{B} - \lambda$, " and l be $= \mathrm{L} - \Lambda$; the ratio corresponding to the given logarithm B will be " very nearly equal to that of $a + \frac{ad}{\frac{1}{m} - \frac{1}{2}d}$ or $a + \frac{ad}{\frac{1}{m} - \frac{d}{2}}$, to 1; and, secondly, " That, if B be any given logarithm in Briggs's system, or in any other system " of logarithms different from Napier's, and L be the logarithm of the same " ratio in Napier's system whereof B is the logarithm in the other system; and " the proportion of the logarithms of any given ratios in Napier's system, " to the logarithms of the same ratios in the said other system be that of m to 1; " and λ be the next greater logarithm than B that is found in the tables, and b " be the number found in the tables corresponding to λ, or of the ratio of " which to unity λ is the logarithm; and Λ be Napier's logarithm of the same " ratio of b to 1; and d be $= \lambda - \mathrm{B}$, and l be $= \Lambda - \mathrm{L}$; the ratio corre- " sponding to the given logarithm B, will be very nearly equal to that of $b - \frac{bd}{\frac{1}{m} + \frac{1}{2}d}$, or $b - \frac{bd}{\frac{1}{m} + \frac{d}{2}}$, to 1."

These two propositions may be proved in the manner following.

By what is shewn in Note 7th it appears that, if L be any given logarithm of Napier's system, and Λ the next less logarithm set down in a table of Napier's logarithms, and a is the number corresponding to the said tabular logarithm Λ, (or the number of the ratio of which to unity Λ is the logarithm,) and l be $= \mathrm{L} - \Lambda$, the ratio corresponding to the given logarithm L, will be that of the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c, *ad infinitum* to 1, or, (because the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c, *ad infinitum*, is nearly equal to $a + \frac{al}{1 - \frac{l}{2}}$,) nearly, that of $a + \frac{al}{1 - \frac{l}{2}}$ to 1. Now the ratio corresponding to the logarithm L in Napier's system, is the same as the ratio corresponding to B in the other system, whose logarithms are to those of the same ratios in Napier's system as 1 to m. And l, or $\mathrm{L} - \Lambda$, is $(= m\mathrm{B} - m\lambda = m \times \overline{\mathrm{B} - \lambda}) = md$; and consequently al is $= amd$; and $\frac{l}{2}$ is $= \frac{md}{2}$, and $1 - \frac{l}{2}$ is $(= 1 - \frac{md}{2}) = \frac{2 - md}{2}$, and $\frac{al}{1 - \frac{l}{2}}$ is $(= \frac{amd}{\frac{2 - md}{2}} = amd \times \frac{2}{2 - md} = \frac{2amd}{2 - md} = \frac{amd}{1 - \frac{md}{2}}) = \frac{ad}{\frac{1}{m} - \frac{d}{2}})$ and therefore $a + \frac{al}{1 - \frac{l}{2}}$ is $= a + \frac{ad}{\frac{1}{m} - \frac{d}{2}}$. Therefore the ratio corresponding to the given logarithm B in the system of logarithms in which the logarithms of any given ratios are to Napier's logarithms of the same ratios in the proportion of 1 to m, is, nearly, that of $+ \frac{ad}{\frac{1}{m} - \frac{d}{2}}$ to 1.

Q. E. D

Secondly, it appears likewise by what is shewn in Note 7th, that, if L be any given logarithm in Napier's system, and Λ be the next greater logarithm set down in a table of Napier's logarithms, and b is the number corresponding to the said tabular logarithm Λ, (or the number, of the ratio of which to unity Λ is

is the logarithm,) and l be $= \Lambda - L$, the ratio corresponding to the given logarithm L will be that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720}$ — &c, *ad infinitum* to 1, or (because the said series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720}$ — &c, *ad infinitum* is nearly equal to $b - \frac{bl}{1+\frac{l}{2}}$,) it will be, nearly, that of $b - \frac{bl}{1+\frac{l}{2}}$ to 1. Now the ratio corresponding to the given logarithm L in Napier's system is the same as the ratio corresponding to the given logarithm B in the other system, whose logarithms are to those of the same ratios in Napier's system in the proportion of 1 to m. And l, or $\Lambda - L$, is ($= m\lambda - mB = m \times \overline{\lambda - B}$) $= md$; and consequently bl is $= bmd$, and $\frac{l}{2}$ is $= \frac{md}{2}$, and $1 + \frac{l}{2}$ is ($= 1 + \frac{md}{2}$) $= \frac{2+md}{2}$, and $\frac{bl}{1+\frac{l}{2}}$ is ($= \frac{bmd}{\frac{2+md}{2}} = bmd \times \frac{2}{2+md} = \frac{2bmd}{2+md} = \frac{bmd}{1+\frac{md}{2}}$) $= \frac{bd}{\frac{1}{m}+\frac{d}{2}}$, and $b - \frac{bl}{1+\frac{l}{2}}$ is $= b - \frac{bd}{\frac{1}{m}+\frac{d}{2}}$. Therefore the ratio corresponding to the given logarithm B in the system of logarithms in which the logarithms of any given ratios are to those of the same ratios in Napier's system in the proportion of 1 to m, is, nearly, that of $b - \frac{bd}{\frac{1}{m}+\frac{d}{2}}$ to 1. Q. E. D.

Besides the two propositions which we have just demonstrated, the foregoing passage contains the two following propositions, to wit, 1st, "That $a + \frac{al}{\frac{1}{m} - \frac{1}{2}l}$ is equal to $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l}$," and 2dly, "That $b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l}$ is equal to $\frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l}$;" or, according to the notation used in the foregoing part of this note, (in which d answers to the logarithm denoted by l in Dr. Halley's expression,) "That $a + \frac{ad}{\frac{1}{m} - \frac{1}{2}d}$ is equal to $\frac{\frac{1}{m}a + \frac{1}{2}da}{\frac{1}{m} - \frac{1}{2}d}$," and "That $b - \frac{bd}{\frac{1}{m} + \frac{1}{2}d}$ is equal to $\frac{\frac{1}{m}b - \frac{1}{2}db}{\frac{1}{m} + \frac{1}{2}d}$." Now this may be easily shewn in the manner following.

The former quantity $a + \frac{a d}{\frac{1}{m} - \frac{1}{2} d}$ is $(= \frac{a \times \overline{\frac{1}{m} - \frac{1}{2} d}}{\frac{1}{m} - \frac{1}{2} d} + \frac{a d}{\frac{1}{m} - \frac{1}{2} d}$ $= \frac{\frac{1}{m} a - \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d} + \frac{a d}{\frac{1}{m} - \frac{1}{2} d} = \frac{\frac{1}{m} a - \frac{1}{2} a d + a d}{\frac{1}{m} - \frac{1}{2} d}) = \frac{\frac{1}{m} a + \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d}$; and the latter quantity $b - \frac{b d}{\frac{1}{m} + \frac{1}{2} d}$ is $(= \frac{b \times \overline{\frac{1}{m} + \frac{1}{2} d}}{\frac{1}{m} + \frac{1}{2} d} - \frac{b d}{\frac{1}{m} + \frac{1}{2} d} =$ $\frac{\frac{1}{m} b + \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d} - \frac{b d}{\frac{1}{m} + \frac{1}{2} d} = \frac{\frac{1}{m} b + \frac{1}{2} b d - b d}{\frac{1}{m} + \frac{1}{2} d}) = \frac{\frac{1}{m} b - \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d}$.

Q. E. D.

Lastly, the expression $\frac{\frac{1}{m} a + \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d}$, which is equal to b, affords us the following analogy for determining the value of b; to wit, as $\frac{1}{m} - \frac{1}{2} d$ is to $\frac{1}{m} + \frac{1}{2} d$, so is a to $(\frac{\frac{1}{m} a + \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d}$, or) b, or the number sought; or, if d denote a logarithm in Briggs's system, and consequently m be $=$ 2.302,585,092,994,045,684, &c, and $\frac{1}{m}$ be $\frac{1}{2.302,585,092,994,045,684, \&c.}$ or, 0.434,294,481,903,251,827, &c, we shall have 0.434,294, &c, $- \frac{1}{2} d$ to 0.434,294, &c, $+ \frac{1}{2} d$ as a to b, or the number sought.

And the expression $\frac{\frac{1}{m} b - \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d}$, which is equal to a, will afford us the following analogy for determining the value of a; to wit, as $\frac{1}{m} + \frac{1}{2} d$ is to $\frac{1}{m} - \frac{1}{2} d$, so is b to $(\frac{\frac{1}{m} b - \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d}$, or) a, or the number sought; or, if d be a logarithm in Briggs's system, and consequently $\frac{1}{m}$ be $=$ 0.434,294,481,903,251,827, &c, we shall have 0.434,294, &c, $+ \frac{1}{2} d$ to 0.434,294, &c, $- \frac{1}{2} d$ as b is to a, or the number sought.

NOTE

NOTE X.

IN page 91, line 4th.——*If more steps of this series be desired, it will be found to be as follows, to wit,* $a + \frac{al}{1-\frac{1}{2}l} - \frac{\frac{1}{12}al^3}{1-l} + \frac{\frac{1}{35}al^5}{1-2l}$ &c; *as may easily be demonstrated by working out the divisions in each step, and collecting the quotes, whose sum will be found to agree with our former series.*

Dr. Halley here affirms that the series $a + \frac{al}{1-\frac{1}{2}l} - \frac{\frac{1}{12}al^3}{1-l} + \frac{\frac{1}{35}al^5}{1-2l}$ &c, is equal (as far as these four terms go, or as far as to the term that involves the fifth power of l,) to the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c. This assertion is true as far as the fourth power of l, and nearly, but not accurately, so with respect to the term $\frac{al^5}{120}$, which involves the fifth power of l. This will appear by converting the three fractions $\frac{al}{1-\frac{1}{2}l}$, $\frac{\frac{1}{12}al^3}{1-l}$, and $\frac{\frac{1}{35}al^5}{1-2l}$ into serieses by dividing their numerators by their denominators, and then adding together the first and third quotients, or serieses so obtained, and subtracting the second quotient from their sum. For, if this be done, we shall have $\frac{al}{1-\frac{1}{2}l}$, or $\frac{al}{1-\frac{l}{2}}$, $= al + \frac{al^2}{2} + \frac{al^3}{4} + \frac{al^4}{8} + \frac{al^5}{16} + \frac{al^6}{32} +$ &c. and $\frac{\frac{1}{12}al^3}{1-l} = \frac{1}{12}al^3 + \frac{1}{12}al^4 + \frac{1}{12}al^5 + \frac{1}{12}al^6 +$ &c, and $\frac{\frac{1}{35}al^5}{1-2l} = \frac{1}{35}al^5 + \frac{2}{35}al^6 +$ &c. Therefore $\frac{al}{1-\frac{1}{2}l} - \frac{\frac{1}{12}al^3}{1-l} + \frac{\frac{1}{35}al^5}{1-2l}$ will be $=$

$$
\begin{aligned}
& al + \frac{al^2}{2} + \frac{al^3}{4} + \frac{al^4}{8} + \frac{al^5}{16} + \frac{al^6}{32} + \&c. \\
& \qquad - \tfrac{1}{12}al^3 - \tfrac{1}{12}al^4 - \tfrac{1}{12}al^5 - \tfrac{1}{12}al^6 - \&c. \\
& \qquad\qquad + \tfrac{1}{35}al^5 + \tfrac{2}{35}al^6 + \&c. \\
= {} & al + \frac{al^2}{2} + \frac{3al^3}{12} + \frac{3al^4}{24} + \frac{al^5}{16} + \frac{al^6}{32} + \&c. \\
& \qquad - \frac{al^3}{12} - \frac{2al^4}{24} - \frac{al^5}{12} - \frac{al^6}{12} - \&c. \\
& \qquad\qquad + \frac{al^5}{35} + \frac{2al^6}{35} + \&c. \\
= {} & al + \frac{al^2}{2} + \frac{2al^3}{12} + \frac{al^4}{24} + \frac{al^5}{16} + \frac{al^6}{32} + \&c. \\
& \qquad - \frac{al^5}{12} - \frac{al^6}{12} - \&c. \\
& \qquad + \frac{al^5}{35} + \frac{2al^6}{35} + \&c. \\
= {} & al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{3al^5}{48} + \frac{3al^6}{96} + \&c. \\
& \qquad - \frac{4al^5}{48} - \frac{8al^6}{96} - \&c. \\
& \qquad + \frac{al^5}{35} + \frac{2al^6}{35} + \&c.
\end{aligned}
$$

$$= a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} - \frac{a\,l^5}{48} - \frac{5\,a\,l^6}{96} + \&c.$$
$$+ \frac{a\,l^5}{35} + \frac{2\,a\,l^6}{35} + \&c.$$

$= a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} + \frac{13\,a\,l^5}{1680} + \frac{17\,a\,l^6}{3360} + \&c$; and consequently the series $a + \frac{a\,l}{1 - \frac{l}{2}} - \frac{\frac{1}{12}a\,l^3}{1-l} + \frac{\frac{1}{35}a\,l^5}{1-2\,l}$ will be $= a + a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} + \frac{13\,a\,l^5}{1680} + \frac{17\,a\,l^6}{3360}$, which agrees with the original series $a + a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} + \frac{a\,l^5}{120} + \frac{a\,l^6}{720} + \&c$ *ad infinitum* in the five first terms, but differs from it in the sixth and seventh terms $\frac{13\,a\,l^5}{1680}$ and $\frac{17\,a\,l^6}{3360}$, of which the former is a little less than $\frac{a\,l^5}{120}$, and the latter is considerably greater than $\frac{a\,l^6}{720}$. For $\frac{a\,l^5}{120}$ is $= \frac{14\,a\,l^5}{14 \times 120} = \frac{14\,a\,l^5}{1680}$, which is greater than $\frac{13\,a\,l^5}{1680}$ by the difference $\frac{a\,l^5}{1680}$; and $\frac{a\,l^6}{720}$ is $= \frac{14\,a\,l^6}{14 \times 720} = \frac{14\,a\,l^6}{10,080}$, which is less than $\frac{51\,a\,l^6}{10,080}$, or $\frac{3 \times 17\,a\,l^6}{3 \times 3360}$, or $\frac{17\,a\,l^6}{3360}$, by $\frac{37\,a\,l^6}{10,080}$, or, nearly, $\frac{a\,l^6}{272}$, which is more than double of $\frac{a\,l^6}{720}$. Q. E. D.

In order to make the series $a + \frac{a\,l}{1-\frac{1}{2}l} - \frac{\frac{1}{12}a\,l^3}{1-l} + \frac{\frac{1}{35}a\,l^5}{1-2\,l} - \&c$, which is asserted by Dr. Halley to be equal to the original series $a + a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} + \frac{a\,l^5}{120} + \frac{a\,l^6}{720} + \&c$ *ad infinitum*, agree with the said original series in its sixth term $\frac{a\,l^5}{120}$, we must increase the fourth term $\frac{\frac{1}{35}a\,l^5}{1-2\,l}$ from $\frac{\frac{1}{35}a\,l^5}{1-2\,l}$, or $\frac{\frac{7}{245}a\,l^5}{1-2\,l}$, to $\frac{\frac{7}{240}a\,l^5}{1-2\,l}$. For then we shall have $\frac{a\,l}{1-\frac{1}{2}l} - \frac{\frac{1}{12}a\,l^3}{1-l} + \frac{\frac{7}{240}a\,l^5}{1-2\,l} =$

$$a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{4} + \frac{a\,l^4}{8} + \frac{a\,l^5}{16} + \frac{a\,l^6}{32} + \&c,$$
$$- \frac{a\,l^3}{12} - \frac{a\,l^4}{12} - \frac{a\,l^5}{12} - \frac{a\,l^6}{12} - \&c,$$
$$+ \tfrac{7}{240}a\,l^5 + \tfrac{14}{240}a\,l^6 + \&c.$$

$$= a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} + \frac{3\,a\,l^5}{48} + \frac{3\,a\,l^6}{96} + \&c,$$
$$- \frac{4\,a\,l^5}{48} - \frac{8\,a\,l^6}{96} - \&c,$$
$$+ \frac{7\,a\,l^5}{240} + \frac{7\,a\,l^6}{120} + \&c.$$

$$= a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} - \frac{a\,l^5}{48} - \frac{5\,a\,l^6}{96} - \&c,$$
$$+ \frac{7\,a\,l^5}{240} + \frac{7\,a\,l^6}{120} + \&c.$$

$$= a\,l + \frac{a\,l^2}{2} + \frac{a\,l^3}{6} + \frac{a\,l^4}{24} - \frac{5\,a\,l^5}{240} - \frac{50\,a\,l^6}{960} - \&c.$$
$$+ \frac{7\,a\,l^5}{240} + \frac{56\,a\,l^6}{960} + \&c.$$

$= al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{2al^5}{240} + \frac{6al^6}{960} + \&c = al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{160} + \&c$, and consequently $a + \frac{al}{1 - \frac{1}{2}l} - \frac{\frac{1}{12}al^3}{1-l} + \frac{\frac{7}{240}al^5}{1-2l} = a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{160} + \&c$, which agrees with the original ſeries $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$ *ad infinitum* in the ſix firſt terms, or as far as the ſixth term $\frac{al^5}{120}$ incluſively. Q. E. D.

Dr. Halley ſeems to think that there is much leſs trouble in computing the four terms $a + \frac{al}{1 - \frac{1}{2}l} - \frac{\frac{1}{12}al^3}{1-l} + \frac{\frac{7}{240}al^5}{1-2l}$ than in computing the firſt ſix terms of the original ſeries $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \&c$, to which ſix terms the foregoing four terms are equal. But I much doubt whether this opinion is well founded; becauſe the diviſors $1 - \frac{1}{2}l$, $1 - l$, and $1 - 2l$, of the ſecond, third, and fourth terms of the ſaid new ſeries will be much longer numbers, and conſequently more troubleſome to uſe as diviſors, than the numbers 2, 6, 24, and 120, which are the diviſors of the third, fourth, fifth, and ſixth terms of the ſaid original ſeries: and in both ſerieſes it will be neceſſary to raiſe l to the fifth power, in order to obtain the terms $\frac{\frac{7}{240}al^5}{1-2l}$ and $\frac{al^5}{120}$, in finding which power of l a conſiderable part of the difficulty of the computation will conſiſt. I ſhould therefore, in a computation of this kind, incline to make uſe of the original ſeries $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} +$ &c *ad infinitum*, rather than of Dr. Halley's ſaid new ſeries.

NOTE XI.

IN page 91, line 8th.—*Thus, I hope, I have cleared up the doctrine of logarithms, and ſhewn their conſtruction and uſe independent of the hyperbola, whoſe affections have hitherto been made uſe of for this purpoſe, though this be a matter purely arithmetical, nor properly demonſtrable from the principles of geometry. Nor have I been obliged to have recourſe to the method of Indiviſibles, or the arithmetick of Infinites; the whole being no other than an eaſy corollary to Mr. Newton's general theorem for forming roots and powers.*

ARTICLE I.—Dr. Halley here ſeems to aſſert, that he was the firſt perſon who had ſhewn the conſtruction of logarithms without the aſſiſtance, or intervention,

vention, of the hyperbola. But this aſſertion is not true in its full extent. For both Lord Napier, the firſt inventor of logarithms, and Mr. Henry Briggs, who invented, or firſt computed, thoſe of the ſyſtem now in uſe, which is called by his name, had conſtructed logarithms, and explained their manner of conſtructing them to the world, without any mention of the hyperbola. And ſo did the learned and ſagacious John Kepler afterwards in his two tracts, firſt publiſhed in the years 1624 and 1625, and now republiſhed at the beginning of this collection. And ſo likewiſe did Mr. Nicholas Mercator in the former and larger part of his tract, intitled, *Logarithmotechnia*, which was firſt publiſhed in the year 1668, and has now been republiſhed in this collection. Dr. Halley muſt therefore be ſuppoſed to mean only that the two logarithmick ſerieſes $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c, *ad infinitum*, and $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, *ad infinitum*, (which had been publiſhed by Mr. Mercator and Dr. Wallis for the purpoſe of conſtructing, or computing, logarithms more conveniently and expeditiouſly than by the methods uſed by Napier and Briggs, and the other old computers of logarithms,) had been derived by Mercator and Wallis, and other writers, (as, for inſtance, James Gregory,) in the tracts they had publiſhed on the ſubject, ſolely from the hyperbola. And this, I ſuppoſe, was true.

II. Dr. Halley ſays further, "*That the doctrine of logarithms, being a matter merely arithmetical, is not properly demonſtrable from the principles of geometry.*" But this is by no means a juſt concluſion; ſince numbers may as well be repreſented by geometrical lines, or areas, (ſuch as the aſymptotick ſpaces, or areas, of an hyperbola,) as by any other quantities whatſoever, and the inferences obtained by means of ſuch a repreſentation of numbers to the eyes, or ſenſes, are *at leaſt* as clear and ſatisfactory as thoſe obtained by conſidering them merely in the abſtract. Indeed, when we ſpeak of *abſtract numbers*, we can mean only *commenſurable quantities in general*, or *commenſurable quantities of any kind*, without confining our ideas to any one particular ſort of quantities. Thus, for example, when we ſay, "that 3 times the number 7 is equal to the number 21," the only meaning of the propoſition is, "that, if we take 3 parcels of men, or of horſes, or of houſes, or of cannon-balls, or of lines a mile long, or of any thing elſe, and each parcel conſiſts of 7 individuals, or units, of the things under conſideration, that is, 7 men, or 7 horſes, or 7 houſes, or 7 cannon-balls, or 7 miles, and we add the ſaid 3 parcels together, the ſum thereby ariſing will be 21 of the ſaid individuals, or units, to wit, men, horſes, houſes, cannon-balls, or miles; and that this will be equally true, whether we ſpeak of one of theſe ſorts of things, or of any other of them." Now certainly our ideas upon this ſubject will not be the more clear, and our concluſions the more certain, for our not confining ourſelves to the conſideration of only one of theſe ſubjects, but extending our thoughts to all of them at once, as we do when we conſider them in the abſtract: but they will, on the contrary, be rather leſs ſo, it being a plainer and eaſier propoſition to underſtand, "That 3 times 7 miles make 21 miles," than "That 3 times 7 things of any kind make 21 things of the ſame kind." When therefore it is ſhewn,

4 that

that the asymptotick spaces in an hyperbola that are intercepted between ordinates that are in the same ratio to each other are equal, and consequently that the asymptotick areas that are intercepted between ordinates whose ratios to each other are unequal, are proportional to, or measures of, the ratios of such ordinates, we have a clearer idea of these measures of the said ratios, or, in other words, of these logarithms of the said ratios, than when we endeavour to form a conception of the measures, or logarithms, of the ratios of mere abstract numbers to each other. And the conclusions that are drawn concerning these asymptotick spaces, or visible logarithms, are likely to be more clear and satisfactory than those that might be drawn concerning abstract logarithms, and at the same time must evidently be full as certain. It is not therefore true, as Dr. Halley has asserted, "That the doctrine of logarithms, being a matter purely arithmetical, is not properly demonstrable from the principles of geometry;" but it is true, on the contrary, "that the said doctrine may be very clearly and satisfactorily demonstrated in that manner."

III. In addition to what has been here advanced concerning Dr. Halley's assertion, we may observe, that Euclid himself, the most clear and accurate writer on the mathematicks that ever lived, has, in the fifth book of his Elements, (which is of the most abstract nature possible, and treats, not of lines and angles, and the areas of right-lined figures, as the former books do, but of quantities in general, and their proportions,) thought fit to represent abstract quantities by right lines, in order to assist his readers to conceive and understand what he says of them; thereby calling in the imaginations of his readers in aid, and to the ease and relief, of their understandings. And with respect to the subject now under consideration, to wit, the nature and construction of logarithms, I have always found right lines to convey to my mind the clearest conception both of logarithms, or the measures of the ratios of quantities to each other, and of the numbers, or quantities, themselves, of whose ratios to each other they are measures; and therefore I am of opinion that this doctrine cannot be better explained than by the assistance of the logarithmick curve, in which the ordinates increase on one side, and decrease on the other, *ad infinitum*, and consequently bear all possible ratios to each other, and therefore are capable of representing all sorts of numbers, however great or small, and the abscisses of the axis, or asymptote, belonging to any two different pairs of ordinates, are proportional to, or measures of, the ratios corresponding to the said ordinates. And further, I consider this curve as being somewhat preferable even to the hyperbola for the purpose of illustrating the doctrine of logarithms, because right lines, (such as the abscisses of the axis, or asymptote, of this curve,) are still more easily conceived in the imagination than the mixtilinear asymptotick areas of an hyperbola.

IV. However, though the illustrations and demonstrations of the properties of logarithms that have been given by Mr. Mercator and Dr. Wallis, and Mr. James Gregory, by means of the hyperbola, appear to me (for the reasons that have been mentioned,) to be very legitimate and satisfactory, yet I agree with Dr. Halley in thinking, that so useful and important a doctrine as this is, ought likewise to be explained, if possible, without the help of the hyperbola, and

upon

upon the abſtract and pure principles of arithmetick, to the end that it may be ſeen and underſtood in every poſſible light and mode of conſidering it: and therefore I have, in a preceding tract of Vol. I, intitled, "*Remarks on the two foregoing infinite Serieſes of Mr. Mercator and Dr. Wallis,*" endeavoured to deduce thoſe two serieſes from principles of pure arithmetick and the abſtract nature of proportion, and in a manner that I conjecture to be ſimilar to that in which Dr. Halley has deduced them in the foregoing diſcourſe which is the ſubject of theſe notes, though I have not been able, after the moſt earneſt endeavours, perfectly to comprehend the Doctor's own inveſtigation of them: and I have likewiſe, in the next following diſcourſe, in Vol. I, intitled, "*An Appendix to the foregoing Remarks,*" endeavoured to give a ſimilar inveſtigation of the two anti-logarithmick ſerieſes contained in the foregoing diſcourſe of Dr. Halley, that is grounded likewiſe on the principles of pure arithmetick, and which I conjecture to be, if not the very ſame with, at leaſt very ſimilar to, the inveſtigation of thoſe two ſerieſes given by Dr. Halley in the foregoing diſcourſe, which, however, from the extreme brevity with which it is expreſſed, I have not been able to underſtand. The other parts of the foregoing diſcourſe of Dr. Halley, (that is, all but the inveſtigations of Mr. Mercator's and Dr. Wallis's two logarithmick ſerieſes and the two anti-logarithmick ſerieſes derived from them,) I think I have been able to underſtand, (though not without great labour and difficulty,) and I hope I have explained them ſufficiently to the reader in the courſe of the ten foregoing notes; for the extraordinary length of which, the uncommon degree of obſcurity which runs through the text, muſt be my apology.

V. Dr. Halley tells us in the latter part of the paſſage cited in the beginning of this note, "*That in the foregoing diſcourſe he has not been obliged to have recourſe to the method of indiviſibles, or to the arithmetick of infinites;*" for which he ſeems much to applaud himſelf. But ſurely this applauſe is not deſerved. For what can be a recourſe to the arithmetick of infinites, if the following paſſages are not ſo? Firſt, in page 85, the Doctor's words are as follows: "*And theſe rationes we ſuppoſe to be meaſured by the number of ratiunculæ contained in each. Now theſe ratiunculæ are ſo to be underſtood as in a continued ſcale of proportionals, infinite in number, between the two terms of the ratio; which infinite number of mean proportionals is to that infinite number of the like and equal ratiunculæ between any other two terms as the logarithm of the one ratio is to the logarithm of the other. Thus, if there be ſuppoſed between* 1 *and* 10 *an infinite ſcale of mean proportionals, whoſe number is* 100,000, *&c. in infinitum; between* 1 *and* 2 *there ſhall be* 30,102, *&c. of ſuch proportionals; and between* 1 *and* 3 *there will be* 47,712, *&c. of them; which numbers therefore are the logarithms of the rationes of* 1 *to* 10, 1 *to* 2, *and* 1 *to* 3; *and not ſo properly to be called the logarithms of* 10, 2, *and* 3.

"*But, if, inſtead of ſuppoſing the logarithms compoſed of a number of equal ratiunculæ proportional to each ratio, we ſhall take the ratio of unity to any number to conſiſt always of the ſame infinite number of ratiunculæ, their magnitudes in this caſe will be as their numbers in the former. Wherefore, if between unity and any number propoſed, there be taken any infinity of mean proportionals, the infinitely little augment, or decrement, of the firſt of thoſe means from unity*

"*will*

" will be a ratiuncula, that is, the momentum, or fluxion, of the ratio of unity to " the said number. And, seeing that in these continual proportionals all the ratiun- " culæ are equal, their sum, or the whole ratio, will be as the said momentum is " directly; that is, the logarithm of each ratio will be as the fluxion thereof. " Wherefore, if the root of any infinite power be extracted out of any number, the " differentiola of the said root from unity shall be as the logarithm of that number. " So that logarithms thus produced may be of as many forms as you please to assume " infinite indices of the power whose root you seek; as, if the index be supposed " 100000 &c, infinitely, the roots shall be the logarithms invented by the Lord " Napier; but, if the said index were 2302585 &c, Mr. Briggs's logarithms " would immediately be produced. And, if you please to stop at any number of " figures, and not to continue them on, it will suffice to assume an index of a figure " or two more than your intended logarithm is to have, as Mr. Briggs did; who, to " have his logarithms true to 14 places, by continual extraction of the square root, " at last came to have the root of the 140,737,488,355,328th power; but how " operose that extraction was will be easily judged by whoso shall undertake to " examine his Calculus.

" Now, though the notion of an infinite power may seem very strange, and, to " those that know the difficulty of the extraction of the roots of high powers, " perhaps, impracticable: yet, by the help of that admirable invention of Mr. " Newton, whereby he determines the unciæ, or numbers prefixed to the members " composing powers (on which the doctrine of series chiefly depends), the infinity of " the index contributes to render the expression much more easy; for if the infinite " power to be resolved be put (after Mr. Newton's method) $p + pq$, $\overline{p + pq}|^{\frac{1}{m}}$ or " $\overline{1 + q}|^{\frac{1}{m}}$, instead of $1 + \frac{1}{m}q + \frac{1 - m}{2mm}qq + \frac{1 - 3m + 2mm}{6m^3}q^3 + \frac{1 - 6m + 11mm - 6m^3}{24m^4}$ " q^4 &c (which is the root when m is finite), becomes $1 + \frac{1}{m}q - \frac{1}{2m}qq + \frac{1}{3m}$ " $q^3 - \frac{1}{4m}q^4 + \frac{1}{5m}q^5$ &c, mm being infinitè infinite; and consequently whatsoever " is divided thereby vanishing. Hence it follows, that $\frac{1}{m}$ multiplied into $q - \frac{1}{2}qq$ " $+ \frac{1}{3}q^3 - \frac{1}{4}q^4 + \frac{1}{5}q^5$ &c, is the augment of the first of our mean proportionals " between unity and $1 + q$, and is therefore the logarithm of the ratio of 1 to " $1 + q$. And, whereas the infinite index m may be taken at pleasure, the several " scales of logarithms to such indices will be as $\frac{1}{m}$, or reciprocally as the indices: " And, if the index be taken 10000 &c, as in the case of Napier's logarithms, they " will be simply $q - \frac{1}{2}qq + \frac{1}{3}q^3 - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c.

" Again, if the logarithm of a decreasing ratio be sought, the infinite root of " $1 - q$, or $\overline{1 - q}|^{\frac{1}{m}}$, is $1 - \frac{1}{m}q - \frac{1}{2m}q^2 - \frac{1}{3m}q^3 - \frac{1}{4m}q^4 - \frac{1}{5m}q^5 - \frac{1}{6m}q^6$ " &c; whence the decrement of the first of our infinite number of proportionals will " be $\frac{1}{m}$ into $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$, &c; which therefore will be " as the logarithm of the ratio of unity to $1 - q$. But, if m be put 10000 &c, " then the said logarithm will be $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$ &c."

In this passage we have frequent mention made of *infinite numbers*, *infinite indices of powers*, *different infinite numbers bearing proportions to each other*, *fluxions*, *momenta*, *and differentiolæ*, and a quantity, *mm*, that is *infinitè* infinite. Surely these expressions partake both of *the method of indivisibles* and of *the arithmetick of infinites*, notwithstanding the author's boast of the contrary.

And, secondly, the following passage makes mention of infinite indexes of powers, and supposes that there may be as many different indexes of this kind as we please; which certainly is a branch of the arithmetick of infinites.

" *From the logarithm given to find what ratio it expresses, is a problem that has " not been so much considered as the former, but which is solved with the like ease, " and demonstrated by a like process, from the same general theorem of Mr. New- " ton. For, as the logarithm of the ratio of* 1 *to* 1 + q *was proved to be* " $\overline{1+q}|^{\frac{1}{m}} - 1$, *and that of the ratio of* 1 *to* 1 − q *to be* $1 - \overline{1-q}|^{\frac{1}{m}}$; *so the lo- " garithm (which we will from henceforth call* L*) being given,* 1 + L *will be " equal to* $\overline{1+q}|^{\frac{1}{m}}$ *in the one case, and* 1 − L *will be equal to* $\overline{1-q}|^{\frac{1}{m}}$ *in the other; " consequently* $\overline{1+L}|^{m}$ *will be equal to* 1 + q, *and* $\overline{1-L}|^{m}$ *to* 1 − q; *that is, " according to Mr. Newton's said rule,* $1 + mL + \frac{1}{2}m^2L^2 + \frac{1}{6}m^3L^3 + \frac{1}{24}m^4$ " $L^4 + \frac{1}{120}m^5L^5$ &c, *will be* = 1 + q, *and* $1 - mL + \frac{1}{2}m^2L^2 - \frac{1}{6}m^3L^3 +$ " $\frac{1}{24}m^4L^4 - \frac{1}{120}m^5L^5$ &c, *will be equal to* 1 − q, m *being any infinite index " whatsoever: which is a full and general proposition, from the logarithm given to " find the number, be the species of logarithm what it will. But, if Napier's lo- " garithm be given, the multiplication by* m *is saved (which multiplication is " indeed no other than reducing the other species to his), and the series will be " more simple, viz.* $1 + L + \frac{1}{2}LL + \frac{1}{6}L^3 + \frac{1}{24}L^4 + \frac{1}{120}L^5$ &c, or $1 - L +$ " $\frac{1}{2}LL - \frac{1}{6}L^3 + \frac{1}{24}L^4 - \frac{1}{120}L^5$ &c."

After reading these two passages of this discourse of Dr. Halley, (which are those which, with the greatest attention I could bestow upon them, I have not been able perfectly to understand) I presume, it will be readily allowed that the Doctor is not intitled to the merit he seems to claim, of having explained the doctrine of logarithms without having had recourse to the method of indivisibles, or to the arithmetick of infinites.

VI. It is, however, possible by another method of applying Sir Isaac Newton's excellent binomial theorem, to compute a table of logarithms either of Briggs's system, or any other that may be thought fit, in the manner which Dr. Halley recommends, or without the intervention of the hyperbola or logarithmick curve, or any other geometrical figure, and likewise without having recourse to the method of indivisibles or to the arithmetick of infinites, and by the help of the principles of pure and finite arithmetick only, and the contemplation of the nature and properties of ratios, which are the quantities of which logarithms are the measures. The method of doing this I shall now proceed to explain in the following discourse; which, as it has taken its rise from the reflections and observations of Dr. Halley contained in the foregoing tract on logarithms, I shall intitle an Appendix to the said tract.

AN

AN

APPENDIX

TO THE

FOREGOING TRACT

OF

DR. EDMUND HALLEY

UPON

LOGARITHMS;

BEING

A direct Method of computing the Logarithms of Ratios either in Briggs's System, or any other that may be proposed, by the help of Sir Isaac Newton's Binomial Theorem, without the intervention of the Hyperbola, or the Logarithmick Curve, or any other Geometrical Figure, and likewise without having recourse to the method of Indivisibles or the Arithmetick of Infinites.

By *FRANCIS MASERES*, Esq.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

ARTICLE I.

THE logarithms of ratios in Briggs's System are the numbers that express the magnitudes of the said ratios, or their proportions to each other, upon a supposition that an unit is taken for the representative, or logarithm, of the ratio of 10 to 1. And the logarithms of ratios in any other system are the numbers that represent the magnitudes of the said ratios, or their proportions to each other, upon a supposition that an unit is taken for the representative, or logarithm, of some other ratio than that of 10 to 1. If therefore we can find the proportion of any other ratio to the ratio of 10 to 1, we shall thereby be

enabled

enabled to discover Briggs's logarithm of such other ratio. For, if the number found for the representative of the ratio of 10 to 1 is 1, (which is Briggs's logarithm of that ratio,) the number found for the representative of the other ratio will be Briggs's logarithm of the said other ratio: and, if the number found for the representative of the ratio of 10 to 1 is not 1, but some other number, which we may call x, either greater or less than 1, we need only diminish or increase the number found for the representative of the said other ratio in the proportion of x to 1, and the number thereby obtained will be Briggs's logarithm of the said other ratio. Thus, for example, if we should seek the proportion between the ratio of 11 to 10, and the ratio of 10 to 1, and should find it to be that of 1 to 24.158,857,928,096,6, (as we shall find it to be in the course of the following pages,) of which two numbers the lesser, to wit, 1, is the representative of the smaller ratio of 11 to 10, and the greater number 24.158, 857,928,096,6, is the representative of the greater ratio of 10 to 1, we need only reduce the smaller number 1, (which is the representative of the ratio of 11 to 10,) in the proportion of 24.158,857,928,096,6, (the first representative of the ratio of 10 to 1,) to 1, (which is its second representative, or its representative in Briggs's system of logarithms,) and the number thereby obtained, to wit, $\frac{1}{24.158,857,928,096,6}$, or 0.041,392,685,158,228,29 &c, will be the new representative of the ratio of 11 to 10, or the logarithm of the said ratio in Briggs's system. And the like observation may be extended to the logarithms of any other system, in which 1 is not the logarithm of the ratio of 10 to 1, but of some other ratio. Our business therefore on the present occasion is to find a method of investigating the proportion between any given ratio (as, for example, the ratio of 11 to 10,) and the ratio of 10 to 1, which is the fundamental, or standard, ratio in Briggs's system of logarithms, or that to which all other ratios are, in that system, compared and referred. Or, if any other system of logarithms were to be computed, it would be necessary to find a method of investigating the proportion between any given ratio and the ratio of which 1 was the logarithm in that system. But, as no system of logarithms besides that of Briggs is now in use, it is needless to think of any other on this occasion.

2. Now in order to find the proportion between the standard ratio of 10 to 1 and any other ratio by the method which we are now going to explain, it will be most convenient (though not absolutely necessary) to suppose the said other ratio to be less than the ratio of 2 to 1, and indeed, for the most part, to be a great deal less than that ratio, and not to exceed the ratio of 11 to 10 or that of 10 to 9. But this will be sufficient for the purpose of computing the logarithms of all sorts of ratios, how great soever, because the greatest ratios are compounded of, and may be divided into, a number of other lesser ratios, of which some shall be exact multiples of the ratio of 10 to 1, and others shall be small ratios of the magnitudes that have been just described. Thus, for example, the ratio of 1788 to 1 is compounded of the ratio of 1788 to 1780, that of 1780 to 178, that of 178 to 89, that of 89 to 80, that of 80 to 8, and that of 8 to 1; of which the first is a very small ratio, the second is equal to the ratio of 10 to 1, the third is equal to the ratio of 2 to 1, the fourth is a small

ſmall ratio, very little greater than that of 88 to 80, or 11 to 10, the fifth is equal to that of 10 to 1, and the ſixth is equal to three times the ratio of 2 to 1. Therefore the logarithm of the ratio of 1788 to 1 is equal to the ſum of the logarithms of the ratios of 1788 to 1780, or of 447 to 445, and of 10 to 1, and of 2 to 1, and of 89 to 80, and of 10 to 1, and of three times the ratio of 2 to 1; or Log. $\frac{1788}{1}$ will be $=$ Log. $\frac{447}{445}$ $+$ Log. $\frac{10}{1}$ $+$ Log. $\frac{2}{1}$ $+$ Log. $\frac{89}{80}$ $+$ Log. $\frac{10}{1}$ $+$ 3 Log. $\frac{2}{1}$ $=$ Log. $\frac{447}{445}$ $+$ Log. $\frac{89}{80}$ $+$ 2 Log. $\frac{10}{1}$ $+$ 4 Log. $\frac{2}{1}$ $=$ Log. $\frac{447}{445}$ $+$ Log. $\frac{89}{80}$ $+ 2 \times 1 +$ 4 Log. $\frac{2}{1}$ $=$ Log. $\frac{447}{445}$ $+$ Log. $\frac{89}{80}$ $+ 2 +$ 4 Log. $\frac{2}{1}$. And conſequently, when the logarithm of the ratio of 2 to 1, and thoſe of the ſmall ratios of 447 to 445, and of 89 to 80, have been computed, that of the ratio of 1788 to 1 may be derived from them by mere addition. And, in like manner, the logarithm of any other great ratio may be obtained by adding together the logarithms of the ſmaller ratios of which it is compounded. It is not therefore neceſſary, in the preſent inveſtigation, to take into our conſideration any other ratios but ſuch ſmall ones as have been mentioned, to wit, thoſe of 10 to 9, and 11 to 10, or others of a ſtill ſmaller magnitude.

3. Now, as the reaſonings by which we may find the proportion of any one ſmall ratio, (ſuch as thoſe of 10 to 9 and of 11 to 10,) to the ſtandard ratio of 10 to 1, are exactly the ſame as thoſe by which we may find the proportion of any other ſuch ſmall ratio to the ſame ratio of 10 to 1;—and, as particular examples are ſimpler and eaſier to underſtand than general problems, I ſhall confine myſelf, in the courſe of the following pages, to the inveſtigation of the proportion which a ſingle particular ſmall ratio, namely, the ratio of 11 to 10, bears to the ſaid ſtandard ratio. This therefore will be the ſubject of the following Problem.

PROBLEM.

4. To find in numbers the proportion between the ratio of 11 to 10 and the greater ratio of 10 to 1.

SOLUTION.

The ratio of 11 to 10 is equal to the ratio of $\frac{11}{10}$ to $\frac{10}{10}$, or of $\frac{10+1}{10}$ to $\frac{10}{10}$, or of $1 + \frac{1}{10}$ to 1. We are therefore to find in numbers the proportion between the ratio of $1 + \frac{1}{10}$ to 1 and the ratio of 10 to 1.

Let this proportion be that of 1 to the unknown quantity x.

Then, ſince the ratio of 10 to 1 is to the ratio of $1 + \frac{1}{10}$ to 1 as x is to 1, and, from the nature of the powers of quantities, (which are only continual proportionals to unity and the quantities themſelves, of which they are ſaid to be

be powers,) the ratio of $\overline{1+\frac{1}{10}}\big|^{x}$ to 1 is to the ratio of $1+\frac{1}{10}$ to 1 in the same proportion of x to 1, it follows that the ratio of $\overline{1+\frac{1}{10}}\big|^{x}$ to 1 will be equal to the ratio of 10 to 1; and consequently $\overline{1+\frac{1}{10}}\big|^{x}$ will be $=10$. We must therefore endeavour to find the value of the index x in this equation $\overline{1+\frac{1}{10}}\big|^{x}$ $=10$.

5. Now, by Sir Isaac Newton's binomial theorem, $\overline{1+\frac{1}{10}}\big|^{x}$ is $=$ the series $1 + x \times \frac{1}{10} + x \times \frac{x-1}{2} \times \frac{1}{100} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{1}{1000} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{1}{10,000} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{x-4}{5} \times \frac{1}{100,000} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{x-4}{5} \times \frac{x-5}{6} \times \frac{1}{1000,000}$ + &c $= 1 + \frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000}$ + &c. Therefore this last series $1 + \frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000}$ + &c will be $=10$; and consequently (subtracting 1 from both sides) the series $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000}$ + &c will be $=9$. This equation we must now endeavour to resolve.

6. Now the true value of x in this equation, so far as it can be expressed by eighteen decimal figures, is 24.158,857,928,096,805,5; as may be collected from Mr. Abraham Sharp's computation of Briggs's logarithm of 11 to 61 places of figures. For the first twenty-two figures of that logarithm are 1.041,392,685, 158,225,040,750; that is, the first twenty-two figures of the logarithm of the ratio of 11 to 1 in Briggs's system are 1.041,392,685,158,225,040,750; and consequently the logarithm of the ratio of 11 to 10 will be equal to the excess of 1.041,392,685,158,225,040,750 above the logarithm of the ratio of 10 to 1, that is, above 1, or will be $=$ 0.041,392,685,158,225,040,750. Therefore the proportion of the ratio of 11 to 10 to the ratio of 10 to 1, is that of 0.041,392,685,158,225,040,750 to 1, and consequently is equal to that of 1 to $\frac{1.000,000,000,000,000,000,000,000}{0.041,392,685,158,225,040,750}$, or 24.158,857,928,096,805,5 &c. This therefore is the true value of x in this problem; which I have here set down before-hand, to the end that we may see, in the course of the following investigation of it, to what degree of exactness every new step in our gradual approaches towards it, will exhibit it.

7. To find the value of x in this equation $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000}$

$+$ &c $= 9$, we must take a few of the first terms of the series which forms the left-hand side of the equation, and suppose them to be equal to the whole series, and consequently to the absolute term 9, and then resolve the finite equations resulting from such suppositions. And it is evident that, in this way of proceeding, the more terms of the series we retain, the nearer will the value of x thereby obtained approach to its true value; but that it must always be somewhat greater than the said true value, on account of the subsequent terms of the series which have not been retained. The values of x arising from the retention of the first term only, the two first terms, the three first terms, and the four first terms, will be as follows.

8. If we suppose the first term $\frac{x}{10}$ alone to be equal to the whole series, and consequently to 9, we shall have $x = 10 \times 9 = 90$. This value of x is more than triple of its true value, which is 24.158,857, &c.

9. If we suppose the two first terms $\frac{x}{10} + \frac{xx-x}{200}$ to be equal to the whole series, and consequently to 9, we shall have $\frac{20x}{200} + \frac{xx-x}{200} \left(= \frac{x}{10} + \frac{xx-x}{200}\right) = 9$, or $\frac{19x+xx}{200} = 9$, and consequently $19x + xx \ (= 200 \times 9) = 1800$. Therefore $\frac{361}{4} + 19x + xx$ will be $\left(= \frac{361}{4} + 1800 = \frac{361}{4} + \frac{7200}{4}\right) = \frac{7561}{4}$; and consequently $\frac{19}{2} + x$ will be $\left(= \frac{\sqrt{7561}}{2}\right) = \frac{86.95}{2}$, and x will be $\left(= \frac{86.95}{2} - \frac{19}{2} = \frac{67.95}{2}\right) = 33.97$. This second value of x is much nearer to its true value than the former was, but is still a good deal too large.

10. Let us then, in the third place, suppose the three terms $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000}$ to be equal to the whole series, and consequently to the absolute term 9, and investigate the value of x resulting from this supposition.

Now $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000}$ are $= \frac{600x}{6000} + \frac{30xx-30x}{6000} + \frac{x^3-3xx+2x}{6000} = \frac{572x+27xx+x^3}{6000}$. Therefore $\frac{572x+27xx+x^3}{6000}$ will, upon this supposition, be $= 9$, and consequently $572x + 27xx + x^3$ will be $(= 6000 \times 9) = 54{,}000$. We must therefore endeavour to resolve this cubick equation.

Now, if we suppose x to be equal to 30, and substitute 30 instead of x in the compound quantity $572x + 27xx + x^3$, (which forms the left-hand side of the equation $572x + 27xx + x^3 = 54{,}000$,) we shall have $xx = 900$, and $x^3 = 27{,}000$, and consequently $27xx \ (= 27 \times 900) = 24{,}300$, and $572x \ (= 572 \times 30) = 17160$, and $572x + 27xx + x^3 \ (= 17160 + 24{,}300 + 27{,}000) = 68{,}460$; which is greater than the absolute term, 54,000, of the cubick equation $572x + 27xx + x^3 = 54{,}000$. Therefore 30 must be greater than the true value of x in that equation.

Let us therefore suppose the said true value to be $30 - z$; and let $30 - z$ be substituted instead of x in the terms of the said equation, but with an omission of all the quantities that would involve either zz or z^3. And we shall then have

$$xx\ (= \overline{30 - z}|^2 = \overline{30}|^2 - 2 \times 30 \times z + \&c)$$
$$= 900 - 60 \times z + \&c,$$
$$\text{and } x^3\ (= \overline{30 - z}|^3 = \overline{30}|^3 - 3 \times \overline{30}|^2 \times z + \&c$$
$$= \overline{30}|^3 - 3 \times 900 \times z + \&c)$$
$$= 27000 - 2700 \times z + \&c,$$
$$\text{and consequently } 27xx\ (= 27 \times \overline{900 - 60 \times z + \&c}$$
$$= 27 \times 900 - 27 \times 60 \times z + \&c)$$
$$= 24{,}300 - 1620 \times z + \&c,$$
$$\text{and } 572x\ (= 572 \times \overline{30 - z} =$$
$$572 \times 30 - 572 \times z)$$
$$= 17160 - 572 \times z,$$
$$\text{and } 572x + 27xx + x^3 =$$
$$17{,}160 - 572 \times z$$
$$+ 24{,}300 - 1620 \times z + \&c$$
$$+ 27{,}000 - 2700 \times z + \&c =$$

$68{,}460 - 4892 \times z + \&c$. Therefore this last quantity $68{,}460 - 4892 \times z + \&c$, will be $= 54{,}000$; and consequently, (adding $4892 \times z$ to both sides) we shall have $68{,}460 = 54{,}000 + 4892 \times z$, and (subtracting $54{,}000$ from both sides,) $4892 \times z = 14{,}460$, and, lastly, (dividing both sides by 4892) $z = 2.95$. Therefore x, or $30 - z$, will be $(= 30 - 2.95) = 27.05$; that is, the root of the cubick equation $572x + 27xx + x^3 = 54{,}000$ will be $= 27.05$, or (neglecting the decimal fraction .05) $= 27$. Therefore 27 is the third approximation to the value of x in the original equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \&c = 9$; and it is considerably nearer to the truth than the foregoing, or second, approximation, 33.95, which was derived from only the two first terms of the series.

11. And if, in the fourth place, we suppose the four first terms of the series, to wit, $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240{,}000}$, to be equal to the whole series, and consequently to the absolute term 9, we shall have another approximation to the value of x, which will come within less than a 25th part of its true value. For we shall then have $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240{,}000} = \frac{572x + 27xx + x^3}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240{,}000} = \frac{40 \times 572x + 40 \times 27xx + 40x^3}{240{,}000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240{,}000} = \frac{22880x + 1080xx + 40x^3}{240{,}000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240{,}000} = \frac{22{,}874x + 1091xx + 34x^3 + x^4}{240{,}000}$; and consequently $\frac{22{,}874x + 1091xx + 34x^3 + x^4}{240{,}000}$ will be $= 9$, and $22{,}874x + 1091xx + 34x^3 + x^4$ will be $(= 240{,}000 \times 9) = 2{,}160{,}000$. We must therefore now endeavour to resolve the biquadratick equation $22{,}874x + 1091xx + 34x^3 + x^4 = 2{,}160{,}000$.

12. Now the root of this equation must be less than 27. For, if we suppose x to be equal to 27, we shall have $xx = 729$, and $x^3 = 19{,}683$, and $x^4 = 531{,}441$,

531,441, and consequently $22874x$ $(= 22874 \times 27) = 617{,}598$, and $1091xx$ $(= 1091 \times 729) = 795{,}539$, and $34x^3$ $(= 34 \times 19683) = 669{,}222$, and $22{,}874x + 1091xx + 34x^3 + x^4$ $(= 617{,}598 + 795{,}539 + 669{,}222 + 531{,}441) = 2{,}613{,}800$; which is greater than 2,160,000, or the absolute term of the equation $22874x + 1091xx + 34x^3 + x^4 = 2{,}160{,}000$; and consequently 27 is greater than the true value of x in that equation.

13. Let us therefore suppose the true value of x in this equation to be $27 - z$. And let $27 - z$ be substituted in its terms instead of x, but with an omission of all the quantities that shall involve either zz, z^3, or z^4. And we shall then have

$$xx\ (= \overline{27 - z}|^2 = \overline{27}|^2 - 2 \times 27 \times z + \&c) = \overline{729 - 54 \times z + \&c},$$

$$\text{and } x^3\ (= \overline{27 - z}|^3 = \overline{27}|^3 - 3 \times \overline{27}|^2 \times z + \&c = \overline{27}|^3 - 3 \times 729 \times z + \&c = \overline{27}|^3 - 2187 \times z + \&c) = \overline{19683 - 2187 \times z + \&c},$$

$$\text{and } x^4\ (= \overline{27 - z}|^4 = \overline{27}|^4 - 4 \times \overline{27}|^3 \times z + \&c = \overline{27}|^4 - 4 \times 19683 \times z + \&c = \overline{27}|^4 - 78{,}732 \times z + \&c) = 531{,}441 - \underline{78{,}732} \times z + \&c,$$

$$\text{and } 22874x\ (= 22874 \times \overline{27 - z} = 22{,}874 \times 27 - 22{,}874 \times z) = 617{,}598 - 22{,}874z,$$

$$\text{and } 1091xx\ (= 1091 \times \overline{729 - 54 \times z + \&c} = 1091 \times 729 - 1091 \times 54 \times z + \&c) = 795{,}539 - 58{,}914 \times z + \&c,$$

$$\text{and } 34x^3\ (= 34 \times \overline{19683 - 2187 \times z + \&c} = 34 \times 19683 - 34 \times 2187 \times z + \&c) = 669{,}222 - 74{,}358 \times z + \&c,$$

and consequently $22874x + 1091xx + 34x^3 + x^4$

$$\left.\begin{array}{l} = 617{,}598 - 22{,}874 \times z \\ + 795{,}539 - 58{,}914 \times z + \&c \\ + 669{,}222 - 74{,}358 \times z + \&c \\ + 531{,}441 - 78{,}732 \times z + \&c \end{array}\right\} = 2{,}613{,}800 - 234{,}878 \times z + \&c.$$

Therefore this last quantity $2{,}613{,}800 - 234{,}878 \times z + \&c$, will be equal to the absolute term 2,160,000; and consequently (adding $234{,}878 \times z$ to both sides,) 2,613,800 will be $= 2{,}160{,}000 + 234{,}878 \times z$, and (subtracting 2,160,000 from both sides,) $234{,}878 \times z$ will be $= 453{,}800$, and consequently z will be $= \frac{453{,}800}{234{,}878} = 1.93$. Therefore x, or $27 - z$, will be $(= 27 - 1.93) = 25.07$, that is, the root of the biquadratick equation $22{,}874x + 1091xx + 34x^3 + x^4 = 2{,}160{,}000$ is $= 25.07$; which is therefore a fourth approximation to the true value of x in the original equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240{,}000} + \&c = 9$.

14. As this fourth approximation, 25.07, to the value of x in the original equation agrees with the third approximation to it, 27.05, in the firſt, or higheſt, figure 2, we might conclude (if we did not already know the true value of x,) that the firſt figure of the true value of x in the ſaid original equation muſt be 2, though, perhaps, the ſecond figure of it may be leſs than 5. And, further, as the difference between the third and fourth approximations, 27.05 and 25.07, is leſs than 2, whereas the difference between the ſecond and third approximations, 33.97 and 27.05, is more than 6, we have reaſon to ſuppoſe, (without taking the pains to try it,) that, if we were to take in another term of the ſeries $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3 - 3xx + 2x}{6000} +$ &c, and ſuppoſe the five firſt terms of it, to wit, $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000}$, to be equal to the whole ſeries, and conſequently to 9, and were to reſolve the equation of the fifth order thence reſulting, the difference between the value of x that would be thereby obtained, and the laſt, or fourth value of it, 25.07, would be leſs than 1, and conſequently that the ſaid fifth approximation to the value of x would be greater than 24. We might therefore conclude, without reſolving any more equations for the purpoſe, that the true value of x in the ſaid original equation $\frac{x}{10} + \frac{xx-x}{200} +$ &c $= 9$, or of the index x in the equation $\overline{1 + \frac{1}{10}}|^{x} = 10$, is greater than 24, but leſs than 25; and this concluſion we ſhall now proceed to verify by raiſing the ſaid binomial quantity $1 + \frac{1}{10}$ to the 24th and 25th powers.

15. Now $1 + \frac{1}{10}$ is $= 1.1$; of which the ſquare is 1.21, and the fourth power is $(= \overline{1.21}|^2) = 1.4641$, and the eighth power is $(= \overline{1.4641}|^2) = 2.143,588,81$, and the ſixteenth power is $(= \overline{2.143,588,81}|^2) = 4.594,972,986,357,216,1$. Therefore $\overline{1.1}|^{24}$ is $(= \overline{1.1}|^{16} \times \overline{1.1}|^{8} = 4.594,972,986,357,216,1 \times 2.143,588,81) = 9.849,732,675,807,611,094,711,841$, and $\overline{1.1}|^{25}$ is $(= \overline{1.1}|^{24} \times 1.1 = 9.849,732,675,807,611,094,711,841 \times 1.1) = 10.834,705,943,388,372,204,183,025,1$. It appears therefore that $\overline{1.1}|^{x}$, or $\overline{1 + \frac{1}{10}}|^{x}$ (which is ſuppoſed to be $= 10$,) is greater than $\overline{1.1}|^{24}$, or $\overline{1 + \frac{1}{10}}|^{24}$, but leſs than $\overline{1.1}|^{25}$, or $\overline{1 + \frac{1}{10}}|^{25}$, and nearer to the former than to the latter. Therefore the index x is greater than 24, but leſs than 25, as we had concluded it to be in conſequence of the foregoing inveſtigations. We therefore now know for certain that the two firſt, or higheſt, figures of the true value of the index x in the equation $\overline{1 + \frac{1}{10}}|^{x} = 10$ are 24.

16. We might now proceed to inveſtigate the value of x to a greater degree of exactneſs by taking in five, or ſix, or ſeven, or more, terms of the ſeries $\frac{x}{10} + \frac{xx-x}{200}$

$+ \frac{x^3 - 3xx + 2x}{6000} +$ &c, and supposing them to be equal to the whole series, and consequently to 9, and resolving the equations resulting from such suppositions. But the computations necessary to these resolutions would be found excessively laborious. We shall therefore lay aside the equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000}$ $+$ &c $= 9$, and shall proceed to obtain, in its stead, another equation, of which the root, or unknown quantity, shall be a much smaller quantity than x, and shall even be less than 1, and in which the powers of the unknown quantity shall consequently form a decreasing progression, and therefore a few of the first terms of the series will approach much nearer to an equality with the whole series than in the former equation, and consequently our approaches to the true value of the said unknown quantity by retaining first one term, then two terms, then three terms, and, lastly, four terms, of the series, and supposing them to be equal to the whole series, and resolving the equations resulting from these suppositions, will be much swifter than our approaches to the value of x in the foregoing operations. This may be done in the manner following.

17. Since x is greater than 24, let us suppose it to be $= 24 + z$.

Then will $\overline{1 + \frac{1}{10}}|^{24 + z}$ be $= \overline{1 + \frac{1}{10}}|^{x}$, and consequently $= 10$.

But $\overline{1 + \frac{1}{10}}|^{24 + z}$ is $= \overline{1 + \frac{1}{10}}|^{24} \times \overline{1 + \frac{1}{10}}|^{z}$.

Therefore $\overline{1 + \frac{1}{10}}|^{24} \times \overline{1 + \frac{1}{10}}|^{z}$ will be $= 10$.

But we have seen that $\overline{1 + \frac{1}{10}}|^{24}$ is $= 9.849,732,675,807,611,094,711,841$.

Therefore $9.849,732,675,807,611,094,711,841 \times \overline{1 + \frac{1}{10}}|^{z}$ is $= 10$; and consequently $\overline{1 + \frac{1}{10}}|^{z}$ will be $\left(= \frac{10.000,000,000,000,000,000,000,000,000,000}{9.849,732,675.807,611,094,711,841}\right) =$ $1.015,255,979,947,706,347,941$, &c. We must therefore endeavour to find the value of the index z in this new equation $\overline{1 + \frac{1}{10}}|^{z} = 1.015,255,979,947,$ $706,347,941$, &c.

18. Now, by the binomial theorem, $\overline{1 + \frac{1}{10}}|^{z}$ is $=$ the series $1 + z \times \frac{1}{10}$ $+ z \times \frac{z - 1}{2} \times \frac{1}{100} + z \times \frac{z - 1}{2} \times \frac{z - 2}{3} \times \frac{1}{1000} + z \times \frac{z - 1}{2} \times \frac{z - 2}{3} \times$ $\frac{z - 3}{4} \times \frac{1}{10000} +$ &c $= 1 + \frac{z}{10} + \frac{zz - z}{200} + \frac{z^3 - 3zz + 2z}{6000} + \frac{z^4 - 6z^3 + 11zz - 6z}{240,000}$ $+$ &c, or (to express the terms more correctly, because z is, in this case, less than 1, and consequently zz is less than z, and z^4 than z^3,) $1 + \frac{z}{10} -$ $\frac{\overline{z - zz}}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{\overline{6z - 11zz + 6z^3 - z^4}}{240,000} +$ &c. Therefore this series $1 +$ $\frac{z}{10} - \frac{\overline{z - zz}}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{\overline{6z - 11zz + 6z^3 - z^4}}{240,000} +$ &c will be $= 1.015,255,$ $979,947,706,347,941$, &c; and consequently (subtracting 1 from both sides,)

 the

the ſeries $\frac{z}{10} - \overline{\frac{z - zz}{200}} + \frac{2z - 3zz + z^3}{6000} - \overline{\frac{6z - 11zz + 6z^3 - z^4}{240,000}} +$ &c, will be $=$ 0.015,255,979,947,706,347,941, &c. This equation we muſt now endeavour to reſolve.

19. Now here, as in the former equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} +$ &c $= 9$, we may make continual approaches to the true value of the root by taking in more and more terms of the ſeries, and ſuppoſing them to be equal to the whole ſeries, and conſequently to the abſolute term 0.015,255,979,947,706,347,941, &c, and then reſolving the equations that will reſult from theſe ſuppoſitions. This may be done in the manner following.

20. If we ſuppoſe the firſt term, $\frac{z}{10}$, of the ſeries to be alone equal to the whole ſeries, and conſequently to 0.015,255,979,9, &c, we ſhall have $z = 10 \times 0.015,255,979,9$, &c $= 0.152,559,799$, &c; of which the two firſt figures .15 are exact. For, if z be $= 0.152,559,799$, &c, we ſhall have x ($= 24 + z$) $= 24.152,559,799$, &c, which agrees with the more exact value of x, to wit, 24.158,857,928,096,805,5, in the four higheſt figures. This is no inconſiderable approximation to the true value of z, and is obtained with hardly any trouble.

21. This firſt approximation 0.152, &c, to the value of z in the equation $\frac{z}{10} - \overline{\frac{z - zz}{200}} + \frac{2z - 3zz + z^3}{6000} - \overline{\frac{6z - 11zz + 6z^3 - z^4}{240,000}} +$ &c $= 0.015,255,979,9$, &c, is leſs than its true value, becauſe the firſt term of the ſeries, $\frac{z}{10}$, is not leſs than the whole ſeries, (as was the caſe with $\frac{x}{10}$ the firſt term of the former ſeries $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} +$ &c,) but is greater than the whole ſeries, on account of the diminution of its value by the ſubtraction of the following term $\frac{z - zz}{200}$. And for a like reaſon the value of z derived from the two firſt terms of the ſeries will be greater than the truth, and the value of it derived from the three firſt terms will be leſs than the truth, and the value of it derived from the four firſt terms will be greater than the truth, and ſo on alternately, becauſe any odd number of the terms will be greater than the whole ſeries, and any even number of them will be leſs: all which is the conſequence of z's being leſs than unity.

22. In the ſecond place, let us ſuppoſe the two firſt terms of the ſeries, to wit, $\frac{z}{10} - \overline{\frac{z - zz}{200}}$, to be equal to the whole ſeries, and conſequently to 0.015, 255,979,9 &c. We ſhall then have $\frac{20z}{200} - \overline{\frac{z - zz}{200}}$ ($= \frac{z}{10} - \overline{\frac{z - zz}{200}}$) $= 0.015,255,979,9$ &c, or $\frac{19z + zz}{200} = 0.015,255,979,9$ &c, and conſequently $19z + zz$ ($= 200 \times 0.015,255,979,9$ &c) $= 3.051,195,800$, &c, and $\frac{361}{4} + 19z + zz$

$+ zz \left(= \frac{361}{4} + 3.051,195,800 = \frac{361}{4} + \frac{4 \times 3.051,195,800 \ \&c}{4} = \frac{361}{4} + \frac{12.204,783,200}{4}\right) = \frac{373.204,783,200}{4}$, and consequently $\frac{19}{2} + z \left(= \frac{\sqrt{373.204,783,200, \ \&c}}{2}\right) = \frac{19.3185}{2}$, and $z \left(= \frac{19.3185}{2} - \frac{19}{2} = \frac{0.3185}{2}\right) = 0.1592$. Therefore 0.1592 is the second approximation to the value of z in the equation $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000}$ + &c $=$ 0.015,255,979,947,706,347, 941, &c, or in the original equation $\left(1 + \frac{1}{10}\right)^z = 1.015,255,979,947,706,347,941$, &c.

Q. E. I.

23. In the third place let us suppose the three first terms of the foregoing series, to wit, the three terms $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000}$ to be equal to the whole series, and consequently to the absolute term 0.015,255,979,947,706, 347,941, &c.

We shall then have $\frac{19z + zz}{200} + \frac{2z - 3zz + z^3}{6000}$, or $\frac{30 \times 19z + 30zz}{6000} + \frac{2z - 3zz + z^3}{6000}$, or $\frac{570z + 30zz}{6000} + \frac{2z - 3zz + z^3}{6000}$, or $\frac{572z + 27zz + z^3}{6000}$, $=$ 0.015,255,979,947,706, 347,941, &c, and consequently $572z + 27zz + z^3$ (= 6000 × 0.015,255, 979,947,706,347,941, &c) = 91.535,879,686,238,087,646, &c. We must therefore now resolve this cubick equation, $572z + 27zz + z^3 = 91.535,879,686,238,087,646$, &c.

Now, if we substitute 0.1592, or the value of z derived from the two first terms of the series, (and which we know to be not very different from the true value of z in this cubick equation,) instead of z in the compound quantity $572z + 27zz + z^3$ (which forms the left-hand side of this equation) we shall have

$$zz \ (= \overline{0.1592}|^2) = 0.025,344,64,$$
$$\text{and } z^3 \ (= \overline{0.1592}|^3) = 0.004,034,866,688,$$
$$\text{and } 27zz \ (= 27 \times 0.025,344,64) = 0.684,305,28,$$
$$\text{and } 572z \ (= 572 \times 0.1592) = 91.0624,$$
$$\text{and consequently } 572z + 27zz + z^3 =$$
$$\begin{aligned} &91.062,4 \\ + \ &0.684,305,28 \\ + \ &0.004,034,866,688 = \\ &91.750,740,146,688; \end{aligned}$$

which is greater than 91.535,879,686,238,087,646, &c, or the absolute term of the cubick equation $572z + 27zz + z^3 = 91.535,879,686,238,087,646$, &c. Therefore 0.1592 must be greater than the true value of z in that equation.

Let us therefore suppose the value of z in that equation to be $0.1592 - v$.

Then we shall have

$$\begin{aligned} zz \ (&= \overline{0.1592 - v}|^2 = \overline{0.1592}|^2 - 2 \times 0.1592 \times v + \&c \\ &= \overline{0.1592}|^2 - 0.3184 \times v + \&c) \\ &= 0.025,344,64 - 0.3184 \times v + \&c, \end{aligned}$$

and

$$
\begin{aligned}
\text{and } z^3\ (&= \overline{0.1592 - v}|^3 = \overline{0.1592}|^3 - 3 \times \overline{0.1592}|^2 \times v + \&c \\
&= \overline{0.1592}|^3 - 3 \times 0.025{,}344{,}64 \times v + \&c \\
&= \overline{0.1592}|^3 - 0.076{,}033{,}92 \times v + \&c) \\
&= 0.004{,}034{,}866{,}688 - 0.076{,}033{,}92 \times v + \&c,
\end{aligned}
$$

$$
\begin{aligned}
\text{and consequently } 572z\ (&= 572 \times \overline{0.1592 - v} \\
&= 572 \times 0.1592 - 572 \times v) \\
&= 91.0624 - 572 \times v,
\end{aligned}
$$

$$
\begin{aligned}
\text{and } 27zz\ (&= 27 \times \overline{0.025{,}344{,}64 - 0.3184 \times v + \&c} \\
&= 27 \times 0.025{,}344{,}64 - 27 \times 0.3184 \times v + \&c) \\
&= 0.684{,}305{,}28 - 8.5968 \times v + \&c,
\end{aligned}
$$

$$
\begin{aligned}
&\text{and } 572z + 27zz + z^3 = \\
&\quad 91.062{,}4 \qquad\qquad - 572 \times v \\
&+ 0.684{,}305{,}28 \qquad - 8.5968 \times v + \&c \\
&+ 0.004{,}034{,}866{,}688 - 0.076{,}033{,}92 \times v + \&c
\end{aligned}
$$

$= 91.750{,}740{,}146{,}688 - 580.672{,}833{,}92 \times v$ + &c. Therefore this last quantity $91.750{,}740{,}146{,}688 - 580.672{,}833{,}92 \times v$ + &c will be equal to the absolute term 91.535,879,686,238,087,646, &c; and consequently (adding $580.672{,}833{,}92 \times v$ to both sides,) 91.750,740,146,688 will be = 91.535,879, 686,238,087,646, &c + $580.672{,}833{,}92 \times v$, and (subtracting 91.535,879, 686,238,087,646, &c from both sides,) $580.672{,}833{,}92 \times v$ will be = 0.214, 860,460,449,912,354, and consequently z will be $\left(= \frac{0.214{,}860{,}460{,}449{,}912{,}354}{580.672{,}833{,}92}\right)$ = 0.000,370. Therefore z, or $0.1592 - v$, will be (= 0.1592 − 0.000, 370) = 0.15883. Therefore 0.15883 is a third approximation to the true value of z in the equation $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240{,}000}$ + &c = 0.015,255,979,947,706,347,941, &c, or in the original equation $\overline{1 + \frac{1}{10}}|^x$ = 1.015,255,979,947,706,347,941, &c. And it is a pretty near approximation to it, the four first figures of it, 0.1588, being exact. For, if we suppose z to be = 0.15883, we shall have x, or $24 + z$, = 24.15883, of which the first six figures 24.1588 are true, the more accurate value of x being (as has been already observed,) 24.158,857,928,096,805,5.

24. In order to obtain a still nearer approximation to the value of z in the equation $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240{,}000}$ + &c = 0.015, 255,979,947,706,347,941, &c, let us suppose the four first terms of the said series to be equal to the whole series, and consequently to the absolute term 0.015,255,979,947,706,347,941, &c, and resolve the biquadratick equation that will result from that supposition.

The four terms $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240{,}000}$ are equal to $\frac{572z + 27zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240{,}000} = \frac{40 \times 572z + 40 \times 27zz + 40z^3}{240{,}000} - \frac{6z - 11zz + 6z^3 - z^4}{240{,}000} = \frac{22880z + 1080zz + 40z^3}{240{,}000} - \frac{6z - 11zz + 6z^3 - z^4}{240{,}000} =$

$\frac{22874z + 1091zz + 34z^3 + z^4}{240,000}$. Therefore this last quantity $\frac{22874z + 1091zz + 34z^3 + z^4}{240,000}$ is $=$ 0.015,255,979,947,706,347,941, &c; and consequently $22874z + 1091zz + 34z^3 + z^4$ is ($=$ 240,000 $\times$ 0.015,255,979,947,706,347,941, &c) $=$ 3661.435,187,449,523,505,840,000, &c. We must therefore now endeavour to resolve this biquadratick equation $22874z + 1091zz + 34z^3 + z^4 =$ 3661.435,187,449,523,505,840,000, &c.

25. Now we know that the root of this equation cannot differ much from the root of the last cubick equation $572z + 27zz + z^3 =$ 91.535,879,686, 238,087,646, &c, which was found to be 0.15883. We will therefore substitute 0.15883 instead of z in the compound quantity $22874z + 1091zz + 34z^3 + z^4$ (which forms the left-hand side of the said biquadratick equation,) in order to see whether the value of the said compound quantity resulting from such substitution will be greater, or less, than the absolute term, 3661.435,187,449, 523,505,840,000, &c, of the said biquadratick equation, and consequently whether 0.15883 is greater, or less, than the root of the said equation.

Now, if z be supposed to be $=$ 0.15883, we shall have $zz =$ 0.025,226, 968,9, and $z^3 =$ 0.004,006,799,470,387, and $z^4 =$ 0.000,636,399,959,881, 567,21, and

$$22,874z \; (= 22,874 \times 0.15883) = 3633.077,42,$$

$$\text{and } 1091zz \; (= 1091 \times 0.025,226,968,9) = 27.522,623,069,9,$$

$$\text{and } 34z^3 \; (= 34 \times 0.004,006,799,470,387) = 0.136,231,181,993,158,$$

and consequently $22,874z + 1091zz + 34z^3 + z^4 =$

$$\begin{aligned} & 3633.077,42 \\ + & \;\; 27.522,623,069,9 \\ + & \;\;\; 0.136,231,181,993,158 \\ + & \;\;\; 0.000,636,399,959,881,567,21 \end{aligned}$$

$=$ 3660.736,910,651,853,039,567,21; which is less than 3661.435,187,449, 523,505,840,000, &c, or the absolute term of the equation $22,874z + 1091zz + 34z^3 + z^4 =$ 3661.435,187,449,523,505,840,000, &c. Therefore 0.15883 is less than the true value of z in that equation.

26. Let us therefore suppose z to be $=$ 0.15883 $+ w$, and let this quantity be substituted instead of z in the terms of the said biquadratick equation, but with an omission of all the quantities that shall be found to involve either w^2, or w^3, or w^4. And we shall then have

$$\begin{aligned} zz \; (&= \overline{0.15883 + w}|^2 = \overline{0.15883}|^2 + 2 \times 0.15883 \times w + \&c \\ &= \overline{0.15883}|^2 + 0.31766 \times w + \&c) \\ &= 0.025,226,968,9 + 0.31766 \times w + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } z^3 \; (&= \overline{0.15883 + w}|^3 = \overline{0.15883}|^3 + 3 \times \overline{0.15883}|^2 \times w + \&c \\ &= \overline{0.15883}|^3 + 3 \times 0.025,226,968,9 \times w + \&c \\ &= \overline{0.15883}|^3 + 0.075,680,906,7 \times w + \&c) \\ &= 0.004,006,799,470,387 + 0.075,680,906,7 \times w + \&c, \end{aligned}$$

and

and z^4 ($= \overline{0.15883 + w}|^4 = \overline{0.15883}|^4 + 4 \times \overline{0.15883}|^3 \times w +$ &c

$= \overline{0.15883}|^4 + 4 \times 0.004,006,799,470,387 \times w +$ &c

$= \overline{0.15883}|^4 + 0.016,027,197,881,548 \times w +$ &c)

$= 0.000,636,399,959,881,567,21 + 0.016,027,197,881,548 \times w +$&c,

and $22874z$ ($= 22,874 \times \overline{0.15883 + w}$

$= 22,874 \times 0.15883 + 22,874 \times w$)

$= 3633.077,42 + 22,874 \times w$,

and $1091zz$ ($= 1091 \times 0.025,226,968,9 + 1091 \times 0.317,66 \times w$)

$= 27.522,623,069,9 + 346.567,06 \times w +$ &c

and $34z^3$ ($= 34 \times 0.004,006,799,470,387 + 34 \times 0.075,680,906,7 \times w +$ &c)

$= 0.136,231,181,993,158 + 2.573,150,827,8 \times w +$ &c,

and consequently $22,874z + 1091zz + 34z^3 + z^4 =$

$3633.077,42 + 22,874 \times w$

$+ \quad 27.522,623,069,9 + 346.567,06 \times w +$ &c

$+ \quad 0.136,231,181,993,158 + 2.573,150,827,8 \times w +$ &c

$+ \quad 0.000,636,399,959,881,567,21 + 0.016,027,197,881,548 \times w +$ &c

$= 3660.736,910,651,853,039,567,21 + 23,223.156,238,025,681,548 \times w +$ &c. Therefore this last quantity $3660.736,910,651,853,039,567,21 + 23,223.156,238,025,681,548 \times w +$ &c will be = the absolute term $3661.435,187,449,523,505,840,000$, &c; and consequently (subtracting $3660.736,910,651,853,039,567,21$ from both sides,) $23,223.156,238,025,681,548 \times w$ will be $= 0.698,276,797,670,466,272,790$, and therefore w will be $= \frac{0.698,276,797,670,466,272,790}{23,223.156,238,025,681,548} = 0.000,030,0$. Therefore z, or $0.15883 + w$, will be ($= 0.15883 + 0.000,030,0$) $= 0.158,860,0$, or 0.15886, and x, or $24 + z$, will be $= 24 + 0.15886$, or $24.158,86$; that is, the index x in the original equation $\overline{1 + \frac{1}{10}}|^x = 10$ is $=$ 24.15886.

27. This value of x is exact in the first six figures 24.1588, and differs only by an unit in the next figure from the true value of x, which is (as we have before observed,) $24.158,857,928,096,805,5$.

28. The labour of resolving the last equation was so considerable, and the progress made by it in our approach to the value of z was so small, (the value of z obtained by it, to wit, 0.15886, being not more exact than the next preceeding value of it, to wit, 0.15883, by so much as one figure, but only approaching nearer to the truth in the fifth, or last, figure,) that it would be by no means expedient to seek for a nearer approximation to the value of z by taking in more terms of the series $\frac{z}{10} - \frac{\overline{z-zz}}{200} + \frac{2z-3zz+z^3}{6000} - \frac{\overline{6z-11zz+6z^3-z^4}}{240,000}$ + &c, (as, for example, five terms, or six terms, or seven terms of it,) and supposing them to be equal to the whole series, and consequently to the absolute term $0.015,255,979,947,706,347,941$, &c, and resolving the equations resulting from such suppositions. But this purpose will be much better answered

 by

by dropping all further consideration of the equation $\frac{z}{10} - \frac{z-zz}{200} + \frac{2z-3zz+z^3}{6000} - \frac{6z-11zz+6z^3-z^4}{240,000} + \&c = 0.015,255,979,947,706,347,941$, &c, and deriving from the value of z already found, to wit, 24.15886, another equation in which the unknown quantity shall be a quantity much less than z, or 0.15886; as we before dropped the consideration of the first equation $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+6x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \&c = 9$, in order to enter upon that of the second equation $\frac{z}{10} - \frac{z-zz}{200} + \frac{2z-3zz+z^3}{6000} - \frac{6z-11zz+6z^3-z^4}{240,000} + \&c = 0.015,255,979,947,706,347,941$, &c. This may be done by computing the value of the 24.15886th power of the binomial quantity $1 + \frac{1}{10}$ by means of Sir Isaac Newton's binomial theorem, in order to discover whether the said power is greater, or less, than 10, or $\overline{1 + \frac{1}{10}}|^{x}$, and consequently whether the index 24.15886 is greater, or less, than the index x; after which we may suppose x to be equal either to $24.15886 - y$, or $24.15886 + y$, (that is, to $24.15886 - y$, if it is less than 24.15886, and to $24.15886 + y$, if it is greater than 24.15886) and thereby obtain another equation of which y shall be the root. This may be done in the manner following.

29. By the binomial theorem $\overline{1 + \frac{1}{10}}|^{24.15886}$ is = the series $1 + 24.15886 \times \frac{1}{10} + 24.15886 \times \frac{23.15886}{2} \times \frac{1}{100} + 24.15886 \times \frac{23.15886}{2} \times \frac{22.15886}{3} \times \frac{1}{1000} + \&c$ = (if the first, second, third, fourth, and other following terms of the series be denoted by the capital letters A, B, C, D, &c, respectively) $1 + \frac{24.15886}{10} \times A + \frac{23.15886}{2 \times 10} \times B + \frac{22.15886}{3 \times 10} \times C + \frac{21.15886}{4 \times 10} \times D + \frac{20.15886}{5 \times 10} \times E + \frac{19.15886}{6 \times 10} \times F + \frac{18.15886}{7 \times 10} \times G + \frac{17.15886}{8 \times 10} \times H + \frac{16.15886}{9 \times 10} \times I + \frac{15.15886}{10 \times 10} \times K + \frac{14.15886}{11 \times 10} \times L + \frac{13.15886}{12 \times 10} \times M + \frac{12.15886}{13 \times 10} \times N + \frac{11.15886}{14 \times 10} \times O + \frac{10.15886}{15 \times 10} \times P + \frac{9.15886}{16 \times 10} \times Q + \frac{8.15886}{17 \times 10} \times R + \frac{7.15886}{18 \times 10} \times S + \frac{6.15886}{19 \times 10} \times T + \frac{5.15886}{20 \times 10} \times V + \frac{4.15886}{21 \times 10} \times W + \frac{3.15886}{22 \times 10} \times X + \frac{2.15886}{23 \times 10} \times Y + \frac{1.15886}{24 \times 10} \times Z + \frac{0.15886}{25 \times 10} \times A' + \&c$ *ad infinitum*

$$\begin{aligned}
&= 1.000,000,000,000,000,000,\\
&+ 2.415,886,000,000,000,000,\\
&+ 2.797,458,282,498,000,000,\\
&+ 2.066,282,881,257,121,076,\\
&+ 1.093,004,755,122,901,221,\\
&+ 0.440,674,596,757,136,970,
\end{aligned}$$

+ 0.140,713,715,080,440,686,
+ 0.036,502,866,460,365,873,
+ 0.007,829,344,689,901,419,
+ 0.001,405,703,163,731,782,
+ 0.000,213,088,574,605,671,
+ 0.000,027,428,102,685,829,
+ 0.000,003,007,688,027,570,
+ 0.000,000,281,308,135,776,
+ 0.000,000,022,421,986,457,
+ 0.000,000,001,518,545,475,
+ 0.000,000,000,086,925,908,
+ 0.000,000,000,004,175,519,
+ 0.000,000,000,000,166,066,
+ 0.000,000,000,000,005,383,
+ 0.000,000,000,000,000,138,
+ 0.000,000,000,000,000,002,
+ &c

= 10.000,001,974,734,858,821, &c; which is ſomewhat greater than 10. Therefore 24.15886 is a little greater than the index x in the equation $\overline{1 + \frac{1}{10}}|^{x} = 10$.

30. In order to obtain the value of x to a ſtill greater degree of exactneſs, let us ſuppoſe y to be the difference by which 24.15886 exceeds it.

Then will $\overline{1 + \frac{1}{10}}|^{24.15886 - y}$ be $= \overline{1 + \frac{1}{10}}|^{x} = 10$.

But $\overline{1 + \frac{1}{10}}|^{24.15886 - y}$ is $= \overline{1 + \frac{1}{10}}|^{24.15886} \times \overline{1 + \frac{1}{10}}|^{-y} = \overline{1 + \frac{1}{10}}|^{24.15886} \times \frac{1}{\overline{1 + \frac{1}{10}}|^{y}}$. Therefore $\overline{1 + \frac{1}{10}}|^{24.15886} \times \frac{1}{\overline{1 + \frac{1}{10}}|^{y}}$ is $= 10$, and conſequently $\overline{1 + \frac{1}{10}}|^{24.15886}$ is $= 10 \times \overline{1 + \frac{1}{10}}|^{y}$.

But we have juſt ſeen that $\overline{1 + \frac{1}{10}}|^{24.15886}$ is = 10.000,001,974,734,858,821, &c. Therefore 10.000,001,974,734,858,821, &c, is $= 10 \times \overline{1 + \frac{1}{10}}|^{y}$, and conſequently $\overline{1 + \frac{1}{10}}|^{y}$ is $= \frac{10.000,001,974,734,858,821,\ \&c}{10} =$ 1.000,000,197,473,485,882,1, &c. We muſt therefore now endeavour to find the value of the index y in the equation $\overline{1 + \frac{1}{10}}|^{y} =$ 1.000,000,197,473,485,882,1, &c.

31. Now, by the binomial theorem, $\overline{1 + \frac{1}{10}}|^{y}$ is = the ſeries $1 + y \times \frac{1}{10} + y \times \frac{y-1}{2} \times \frac{1}{100} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{1}{1000} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \frac{1}{10,000} + \&c = 1 + \frac{y}{10} + \frac{yy - y}{200} + \frac{y^3 - 3yy + 2y}{6000} + \frac{y^4 - 6y^3 + 11yy - 6y}{240,000} + \&c =$

$1 + \frac{y}{10} - \frac{y - yy}{200} + \frac{2y - 3yy + y^3}{6000} - \frac{6y - 11yy + 6y^3 - y^4}{240{,}000}$ + &c. Therefore the series $1 + \frac{y}{10} - \frac{y - yy}{200} + \frac{2y - 3yy + y^3}{6000} - \frac{6y - 11yy + 6y^3 - y^4}{240{,}000}$ + &c will be = 1.000,000,197,473,485,882,1, &c; and consequently (subtracting 1 from both sides,) the series $\frac{y}{10} - \frac{y - yy}{200} + \frac{2y - 3yy + y^3}{6000} - \frac{6y - 11yy + 6y^3 - y^4}{240{,}000}$ + &c will be = 0.000,000,197,473,485,882,1, &c. This equation we must now endeavour to resolve.

32. Let us therefore, in the first place, suppose the first term $\frac{y}{10}$ alone of the last-mentioned series to be equal to the whole series, and consequently to the absolute term 0.000,000,197,473,485,882,1, &c. Then will y be (= 10 × 0.000,000,197, &c) = 0.000,001,97, &c, and consequently x, or 24.15886 − y, will be (= 24.15886 − 0.000,001,97) = 24.158,858,03; of which the first seven figures, 24.158,85, are exact: so that this first approximation to the value of y (which has been obtained by the resolution of an easy simple equation,) enables us to determine the value of 24.15886 − y, or x, to one figure more than we had found before.

33. In the second place, let us suppose the two first terms of the last-mentioned series, to wit, the terms $\frac{y}{10} - \frac{y - yy}{200}$, to be equal to the whole series, and consequently to 0.000,000,197,473,485,882,1, &c, and then seek the value of y resulting from this supposition.

We shall then have $\frac{y}{10} - \frac{y - yy}{200} = \frac{20y}{200} - \frac{y - yy}{200} = \frac{19y + yy}{200}$, and consequently $\frac{19y + yy}{200}$ = 0.000,000,197,473,485,882,1, &c, and $19y + yy$ (= 200 × 0.000,000,197,473,485,882,1, &c) = 0.000,039,494,697,176,420,0, &c. Therefore $\frac{361}{4} + 19y + yy$ will be ($= \frac{361}{4}$ + 0.000,039,494,697,176,420,0, &c $= \frac{361}{4} + \frac{4 \times 0.000{,}039{,}494{,}697{,}176{,}420{,}0,\ \&c}{4} = \frac{361}{4} + \frac{0.000{,}157{,}978{,}788{,}705{,}680{,}0}{4}$) $= \frac{361.000{,}157{,}978{,}788{,}705{,}680{,}0}{4}$; and consequently $\frac{19}{2} + y$ will be ($= \frac{\sqrt{361.000{,}157{,}978{,}788{,}705{,}680{,}0}}{2}$) $= \frac{19.000{,}004{,}157}{2}$, and y will be ($= \frac{19.000{,}004{,}157}{2} - \frac{19}{2} = \frac{0.000{,}004{,}157}{2}$) = 0.000,002,078; and consequently x, or 24.15886 − y, will be (= 24.158,860,000 − 0.000,002,078) = 24.158,857,922; of which number the first ten figures, 24.158,857,92, are exact, the more accurate value of the index x in the equation $\left(1 + \frac{1}{10}\right)^x = 10$ being (as we have already observed,) 24.158,857,928,096,805,5.

34. Of these ten figures, 24.158,857,92, which are exact, the 7th figure 5 was obtained by the resolution of the simple equation $\frac{y}{10}$ = 0.000,000,197,473,485,882,1, &c, and the three next figures 7,92 have been obtained by the resolu-

tion

tion of the quadratick equation $\frac{y}{10} - \frac{y-yy}{200} = 0.000,000,197,473,485,882,1,$ &c. And more figures of the value of x might in like manner be obtained by retaining three, or four, or more terms of the ſeries $\frac{y}{10} - \frac{y-yy}{200} + \frac{2y-3yy+y^3}{6000} - \frac{6y-11yy+6y^3-y^4}{240,000} +$ &c, and ſuppoſing them to be equal to the whole ſeries, and conſequently to 0.000,000,197,473,485,882,1, &c, and reſolving the equations reſulting from ſuch ſuppoſitions. But the labour of performing theſe reſolutions would be very conſiderable, and the number of new figures of the value of x which we ſhould thereby obtain would be but ſmall. For I have tried the two next equations, to wit, the cubick equation that reſults from a ſuppoſition that the three firſt terms of the ſaid ſeries are equal to the whole, and conſequently to 0.000,000,197,473,485,882,1, &c, and the biquadratick equation that reſults from a ſuppoſition that the four firſt terms of the ſaid ſeries are equal to the ſame quantity, and I have found that the root of ſuch cubick equation has been = 0.000,002,071,400, and that the root of ſuch biquadratick equation has been = 0.000,002,071,941,6, and conſequently that the value of x, or 24.15886 $-y$, obtained by means of the ſaid cubick equation, has been (= 24.158,860,000,000 − 0.000,002,071,400) = 24.158,857,928, 600, of which the firſt eleven figures, 24.158,857,928, are exact, and that the value of x, or 24.15886 $-y$, obtained by means of the ſaid biquadratick equation, has been (= 24.158,860,000,000,0 − 0.000,002,071,941,6) = 24.158, 857,928,058,4, of which the firſt twelve figures, 24.158,857,928,0, are exact. So that the reſolution of theſe two equations gives us only two figures of the true value of x more than we had before obtained by means of the foregoing quadratick equation. In order therefore to obtain the value of x exact to a few more figures than the ten figures 24.158,857,92, which were obtained by the reſolution of the foregoing quadratick equation, we will have recourſe to another method of proceeding, which will be leſs laborious than the reſolution of either the cubick or the biquadratick equation that have been juſt mentioned, and, *à fortiori*, leſs laborious than the reſolution of any of the higher equations that would reſult from the ſuppoſition that more than four terms of the foregoing ſeries were equal to 0.000,000,197,473,485,882,1, &c, and which will give us four additional figures of the true value of x above the ten figures, 24.158,857,92, already obtained by means of the foregoing quadratick equation. This method of proceeding is as follows:

35. Let all the quantities that form the left-hand ſide of the equation $\frac{y}{10} - \frac{y-yy}{200} + \frac{2y-3yy+y^3}{6000} - \frac{6y-11yy+6y^3-y^4}{240,000} +$ &c = 0.000,000,197,473,485, 882,1, &c, be ranged in ſeparate lines according to the ſeveral powers of y, thoſe involving the ſimple power of y being placed in the firſt, or higheſt line, and thoſe involving yy, or the ſquare of y, being placed in the ſecond line, and thoſe involving y^3, or the cube of y, being placed in the third line, and thoſe involving y^4, or the fourth power of y, being placed in the fourth line, and ſo on of the following powers of y. And we ſhall then have

$$\frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} - \frac{120y}{720,000,000}$$
$$+ \frac{yy}{200} - \frac{3yy}{6000} + \frac{11yy}{240,000} - \frac{50yy}{12,000,000} + \frac{274yy}{720,000,000}$$
$$+ \frac{y^3}{6000} - \frac{6y^3}{240,000} + \frac{35y^3}{12,000,000} - \frac{225y^3}{720,000,000}$$
$$+ \frac{y^4}{240,000} - \frac{10y^4}{12,000,000} + \frac{85y^4}{720,000,000}$$
$$+ \frac{y^5}{12,000,000} - \frac{15y^5}{720,000,000}$$
$$+ \frac{y^6}{720,000,000}$$

$= 0.000,000,197,473,485,882,1$, &c.

But, because y is an exceeding small quantity in comparison of unity, (being $=$ to about 0.000,002, or $\frac{2}{1000,000}$, or $\frac{1}{500,000}$,) it is evident that all the powers of y will be extremely small in comparison of y itself, and consequently that all the quantities contained in the second, third, fourth, fifth, sixth, and other following lines of the compound quantity that forms the left-hand side of the foregoing equation, will be extremely small in comparison of the quantities contained in the first, or upper, line of it, which involve only the simple power of y. We may therefore, without erring much from the truth, consider the quantities contained in the upper line alone as being equal to all the quantities contained in all the lines together, and consequently to the absolute term 0.000, 000,197,473,485,882,1, &c; and then the foregoing equation will be converted into the following one, to wit, $\frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} - \frac{120y}{720,000,000} + \text{\&c} = 0.000,000,197,473,485,882,1$, &c, or $\frac{y}{10} - \frac{y}{2 \times 100} + \frac{2y}{2 \times 3 \times 1000} - \frac{6y}{2 \times 3 \times 4 \times 10,000} + \frac{24y}{2 \times 3 \times 4 \times 5 \times 100,000} - \frac{120y}{2.3.4.5.6 \times 1000,000} + \text{\&c} = 0.000,000,197,473,485,882,1$, &c, or $\frac{y}{10} - \frac{y}{2 \times 100} + \frac{y}{3 \times 1000} - \frac{y}{4 \times 10,000} + \frac{y}{5 \times 100,000} - \frac{y}{6 \times 1000,000} + \text{\&c} = 0.000,000,197,473,485,882,1$, &c, or $y \times$ the infinite series $\frac{1}{10} - \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} - \frac{1}{4 \times 10,000} + \frac{1}{5 \times 100,000} - \frac{1}{6 \times 1000,000} + \text{\&c} = 0.000,000,197,473,485,882,1$, &c, or (if we denote the several terms $\frac{1}{10}$, $\frac{1}{2 \times 100}$, $\frac{1}{3 \times 1000}$, $\frac{1}{4 \times 10,000}$, $\frac{1}{5 \times 100,000}$, $\frac{1}{6 \times 1000,000}$, &c, of the foregoing infinite series by the capital letters A, B, C, D, E, F, &c, respectively,) $y \times$ the infinite series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B - \frac{3}{4 \times 10} \times C + \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E + \text{\&c} = 0.000,000,197,473,485,882,1$, &c. We must therefore, in the next place, compute the value of this infinite series to as great a degree of exactness as we shall think necessary; which will not be very difficult, because the said series evidently converges with a considerable degree of swiftness, every new term of it being less than a tenth part

part of the foregoing term. And when this value, which we will call S, is obtained, we shall have $y \times S = 0.000,000,197,473,485,882,1$, &c, and consequently $y = \frac{0.000,000,197,473,485,882,1, \&c}{S}$.

The Computation of the infinite Series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B - \frac{3}{4 \times 10} \times C + \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E +$ *&c ad infinitum.*

36. NOW the infinite series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B - \frac{3}{4 \times 10} \times C + \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E +$ &c may be computed in the following manner.

$$A \text{ is } = \frac{1}{10} = 0.100,000,000,000,000,000,$$

$$B \text{ is } = \frac{1}{2 \times 10} \times A = 0.005,000,000,000,000,000,$$

$$C = \frac{2}{3 \times 10} \times B = 0.000,333,333,333,333,333,$$

$$D = \frac{3}{4 \times 10} \times C = 0.000,024,999,999,999,999,$$

$$E = \frac{4}{5 \times 10} \times D = 0.000,001,999,999,999,999,$$

$$F = \frac{5}{6 \times 10} \times E = 0.000,000,166,666,666,666,$$

$$G = \frac{6}{7 \times 10} \times F = 0.000,000,014,285,714,285,$$

$$H = \frac{7}{8 \times 10} \times G = 0.000,000,001,249,999,999,$$

$$I = \frac{8}{9 \times 10} \times H = 0.000,000,000,111,111,111,$$

$$K = \frac{9 I}{10 \times 10} = 0.000,000,000,009,999,999,$$

$$L = \frac{10 K}{11 \times 10} = 0.000,000,000,000,909,090,$$

$$M = \frac{11 L}{12 \times 10} = 0.000,000,000,000,083,333,$$

$$N = \frac{12 M}{13 \times 10} = 0.000,000,000,000,007,692,$$

$$O = \frac{13 N}{14 \times 10} = 0.000,000,000,000,000,714,$$

$$P = \frac{14 O}{15 \times 10} = 0.000,000,000,000,000,066,$$

$$Q = \frac{15 P}{16 \times 10} = 0.000,000,000,000,000,006,$$

$$R = \frac{16 Q}{17 \times 10} = 0.000,000,000,000,000,000.$$

Therefore $A + C + E + G + I + L + N + P + R$ are =

0.100,000,000,000,000,000,

+ ....,333,333,333,333,333,

$$
\begin{array}{r}
+ \ldots\ldots 1,999,999,999,999, \\
+ \ldots\ldots 14,285,714,285, \\
+ \ldots\ldots 111,111,111, \\
+ \ldots\ldots 909,090, \\
+ \ldots\ldots 7,692, \\
+ \ldots\ldots 66, \\
+ \ldots\ldots\ldots
\end{array}
$$

$= 0.100,335,347,731,075,576$; and $B + D + F + H + K + M + O + Q$ are $=$

$$
\begin{array}{r}
0.005,000,000,000,000,000, \\
+ \ldots\ldots 24,999,999,999,999, \\
+ \ldots\ldots 166,666,666,666, \\
+ \ldots\ldots 1,249,999,999, \\
+ \ldots\ldots 9,999,999, \\
+ \ldots\ldots 83,333, \\
+ \ldots\ldots 714, \\
+ \ldots\ldots 6,
\end{array}
$$

$= 0.005,025,167,926,750,716$; and consequently $A + C + E + G + I + L + N + P + R - B - D - F - H - K - M - O - Q$ &c, or, its equal, $A - B + C - D + E - F + G - H + I - K + L - M + N - O + P - Q + R -$ &c, will be $=$

$$
\begin{array}{r}
0.100,335,347,731,075,576 \\
- \ 0.005,025,167,926,750,716 \\
= 0.095,310,179,804,324,860, \text{ \&c.}
\end{array}
$$

Therefore the infinite series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B - \frac{3}{4 \times 10} \times C + \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E +$ &c, or $\frac{1}{10} - \frac{A}{2 \times 10} + \frac{2B}{3 \times 10} - \frac{3C}{4 \times 10} + \frac{4D}{5 \times 10} - \frac{5E}{6 \times 10} +$ &c, or the series S, is $= 0.095,310,179,804,324,860$, &c.

Q. E. I.

37. Therefore $y \times$ the series S is $= y \times 0.095,310,179,804,324,860$, &c; and consequently $y \times 0.095,310,179,804,324,860$, &c is $= 0.000,000,197,473,485,882,1$, &c, and y is $= \frac{0.000,000,197,473,485,882,1, \text{ \&c}}{0.095,310,179,804,324,860, \text{ \&c}} = 0.000,002,071,903,403$. Therefore x, or $24.15886 - y$, is

$$
\begin{array}{r}
= 24.158,860,000,000,000 \\
- \ 0.000,002,071,903,403
\end{array}
$$

$= 24.158,857,928,096,597$, or the value of the index x in the original equation $\overline{1 + \frac{1}{10}}|^x = 10$ is $24.158,857,928,096,597$; of which number the first 14 figures $24.158,857,928,096$, are exact, the more accurate value of x being $24.158,857,928,096,805,5$.

38. Therefore the proportion of the ratio of 10 to 1 to the ratio of 11 to 10 is that of the number $24.158,857,928,096,597$ to 1, or (because 597 is nearly $=$ 600) that of $24.158,857,928,096,6$ to 1.

Q. E. I.

39. Coroll.

39. Coroll. The proportion of 24.158,857,928,096,6 to 1, is equal to that of 1 to $\frac{1}{24.158,857,928,096,6}$, or 0.041,392,685,158,228,29. Therefore, if 1 be taken for the representative of the magnitude of the ratio of 10 to 1, (as it is in Briggs's system of logarithms,) 0.041,392,685,158,228,29 will be the representative of the lesser ratio of 11 to 10; or, in other words, the logarithm of the small ratio of 11 to 10 in Briggs's system of logarithms will be 0.041,392,685, 158,228,29; of which number the first thirteen figures 0.041,392,685,158,22, are exact, the more accurate value of this logarithm (as computed by Mr. Abraham Sharp,) being 0.041,392,685,158,225,040,750.

40. And in the same manner we may compute Briggs's logarithms of the ratios of 10 to 9, or of $1 + \frac{1}{9}$ to 1, and of 81 to 80, or of $1 + \frac{1}{80}$ to 1, and of 121 to 120, or of $1 + \frac{1}{120}$ to 1, and of 2401 to 2400, or of $1 + \frac{1}{2400}$ to 1, and, in general, of $m + 1$ to m, (m being any whole number whatsoever,) or of $1 + \frac{1}{m}$ to 1, by first finding the values of the index x in the several equations $\overline{1 + \frac{1}{9}}\vert^{x} = 10$, $\overline{1 + \frac{1}{80}}\vert^{x} = 10$, $\overline{1 + \frac{1}{120}}\vert^{x} = 10$, $\overline{1 + \frac{1}{2400}}\vert^{x} = 10$, and, in general, $\overline{1 + \frac{1}{m}}\vert^{x} = 10$, and then dividing 1 by the values of x so found; the several values of $\frac{1}{x}$, or quotients of such divisions, being the logarithms of the said ratios of $1 + \frac{1}{9}$ to 1, and of $1 + \frac{1}{80}$ to 1, and of $1 + \frac{1}{120}$ to 1, and of $1 + \frac{1}{2400}$ to 1, and, in general, of $1 + \frac{1}{m}$ to 1, or of the ratios of 10 to 9, and of 81 to 80, and of 121 to 120, and of 2401 to 2400, and, in general, of $m + 1$ to m, in Briggs's system.

41. This method of computing logarithms is not to be compared, in point of ease and expedition, to either of the two logarithmick serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c *ad infinitum*, and $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c *ad infinitum*, which were invented by Mr. Mercator and Dr. Wallis. But it possesses the advantages recommended by Dr. Halley, of being derived from the abstract nature of ratios and the pure principles of arithmetick, without the assistance of the hyperbola, or any other geometrical figure, and without any recourse to the doctrine of infinitesimals, or of fluxions, or of the limits of ratios, or, in general, of the arithmetick of infinites in any of its modifications. And it is, I believe, considerably less laborious in the practice of it, than the methods by which Mr. Henry Briggs himself computed his logarithms; which were likewise purely arithmetical *. And it likewise serves as a notable instance of the

* See above, in page 73, Mr. Euclid Speidell's *Logarithmotechnia*, chapter viii. where, in speaking of Mr. Henry Briggs's computation of the logarithm of 2, he has these words: "*Whereby it is apparent that he did produce the logarithm for 2 to 15 places very true; though I have been told it was eight persons' work for a year's time after his method, which was by large and many extractions of the square-root.*"

 exten-

extensive utility of Sir Isaac Newton's wonderful theorem for raising the powers of a binomial quantity.

42. As the solution of the foregoing problem consists of a great number of steps, which, for the ease of the reader, have been set forth distinctly and at considerable length, it will not be amiss, before we conclude this discourse, to take a short review of all the foregoing processes, and of the gradual approximations obtained by means of them, to the value of the index x in the equation $\overline{1 + \frac{1}{10}}|^x = 10$, the discovery of which is the object of the Problem.

A Review of the several Steps of the foregoing Solution.

43. The object of the Problem is to find the value of the index x in the equation $\overline{1 + \frac{1}{10}}|^x = 10$.

The first step we took towards finding this value was to expand the quantity $\overline{1 + \frac{1}{10}}|^x$ into an infinite series by means of Sir Isaac Newton's binomial theorem, by which we obtained the equation $1 + \frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} +$ &c *ad infinitum* $= 10$, and consequently the equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} +$ &c $= 9$.

44. Then we supposed, first, one term, then two terms, then three terms, and, lastly, four terms of the series which forms the left-hand side of this last equation to be equal to the whole series, and consequently to the absolute term 9, and resolved the equations resulting from those suppositions.

By resolving the simple equation resulting from the first of these suppositions, to wit, the simple equation $\frac{x}{10} = 9$, we found x to be equal to 90; which was therefore our first approximation to the value of x in the original equation $\overline{1 + \frac{1}{10}}|^x = 10$. This approximation is very wide of the true value of x, being more than three times as great.

From the second supposition, to wit, that the two terms $\frac{x}{10} + \frac{xx - x}{200}$ were equal to the whole series, and consequently to 9, there resulted the quadratick equation $19x + xx = 1800$; by the resolution of which x appeared to be $= 33.97$; which was therefore our second approximation to the value of x in the original equation $\overline{1 + \frac{1}{10}}|^x = 10$. This approximation is much nearer to the truth than the former, but yet is considerably too large.

From the third supposition, to wit, that the three terms $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000}$ were equal to the whole series, and consequently to 9, there resulted the cubick equation $572x + 27xx + x^3 = 54,000$, by the resolution of which we had $x = 27.05$. This therefore was the third approximation to the

value of x in the original equation $\overline{1 + \frac{1}{10}}\big|^{x} = 10$; and it is true in the firſt, or higheſt, figure 2, the more accurate value of x being 24.158,857,928,096, 805,5.

From the fourth ſuppoſition, to wit, that the four terms $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000}$ are equal to the whole ſeries, and conſequently to 9, there reſulted the biquadratick equation $22,874x + 1091xx + 34x^3 + x^4 = 2,160,000$; by the reſolution of which we had $x = 25.07$. This therefore was our fourth approximation to the value of x in the original equation $\overline{1 + \frac{1}{10}}\big|^{x} = 10$.

45. As the difference of this fourth value of x, to wit, 25.07, from the third value of it, to wit, 27.05, is leſs than 2, whereas the difference between the third value, 27.05, and the ſecond value, 33.95, was more than 6, we concluded that it was highly probable that the exceſs of this fourth value, 25.05, above the true value of x was leſs than 1, and conſequently that x was greater than 24, but leſs than 25. We then tried whether x was greater, or leſs, than 24, by raiſing $1 + \frac{1}{10}$, or 1.1, to the 24th power; which we found to be ſomewhat leſs than 10, to wit, 9.849,732,675,807,611,094,711,841. We alſo raiſed 1.1 to the 25th power, which we found to be 10.834,705,943,388,372, 204,183,025,1; which is greater than 10. And we thereby diſcovered with certainty that the true value of the index x in the equation $\overline{1 + \frac{1}{10}}\big|^{x} = 10$ is greater than 24, but leſs than 25.

46. We then put z for the unknown difference by which the true value of x in the equation $\overline{1 + \frac{1}{10}}\big|^{x} = 10$ exceeds 24, ſo that x was $= 24 + z$. And we thereby had $\overline{1 + \frac{1}{10}}\big|^{24 + z}$ $(= \overline{1 + \frac{1}{10}}\big|^{x}) = 10$.

But $\overline{1 + \frac{1}{10}}\big|^{24 + z}$ is $= \overline{1 + \frac{1}{10}}\big|^{24} \times \overline{1 + \frac{1}{10}}\big|^{z}$; and $\overline{1 + \frac{1}{10}}\big|^{24}$ had been found to be $= 9.849,732,675,807,611,094,711,841$. Therefore $9.849,732,675,807,611,094,711,841 \times \overline{1 + \frac{1}{10}}\big|^{z}$ is $= 10$, and conſequently $\overline{1 + \frac{1}{10}}\big|^{z}$ is $\left(= \frac{10}{9.849,732,675,807,611,094,711,841}\right) = 1.015,255,979,947,706,347,941$, &c. And thus we obtained a new equation in which the unknown index z of the power of the binomial quantity $1 + \frac{1}{10}$ is leſs than a unit, inſtead of the original equation $\overline{1 + \frac{1}{10}}\big|^{x} = 10$, in which the index x of the power of the ſame quantity is greater than 24.

47. We then expanded $\overline{1 + \frac{1}{10}}\big|^{z}$ by the binomial theorem, and thereby obtained the equation $1 + \frac{z}{10} - \overline{\frac{z - zz}{200}} + \frac{2z - 3zz + z^3}{6000} - \overline{\frac{6z - 11zz + 6z^3 - z^4}{240,000}} + \text{\&c} = 1.015,$

$= 1.015,255,979,947,706,347,941$, &c, and (by subtracting 1 from both sides,) the equation $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} + \&c =$ $0.015,255,979,947,706,347,941$, &c.

48. We then proceeded to find approximations to the value of z in this new equation in the same manner as we had before found approximations to the value of x in the equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000}$ $+ \&c = 9$, by first supposing the first term $\frac{z}{10}$ alone of the series that forms the left-hand side of the equation, to be equal to the whole series, and consequently to the absolute term $0.015,255,979,947,706,347,941$, &c, and then supposing the two first terms, $\frac{z}{10} - \frac{z - zz}{200}$, to be equal to the same quantity, and then supposing the three first terms, $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000}$, to be equal to the same quantity, and, lastly, supposing the four first terms, $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000}$, to be equal to it, and by resolving the several equations resulting from those suppositions.

From the first of those suppositions we had the simple equation $\frac{z}{10} = 0.015,$ $255,979,947,706,347,941$, &c; by the resolution of which we had $z = 0.152$, &c, of which the two first figures 0.15 are exact.

From the second supposition there resulted the quadratick equation $19z + zz = 3.051,195,800$, &c; by the resolution of which we had $z = 0.1592$.

From the third supposition there resulted the cubick equation $572z + 27zz + z^3 = 91.535,879,686,238,087,646$, &c; by the resolution of which we had $z = 0.15883$, of which the four highest figures, 0.1588, are exact.

And from the fourth supposition there resulted the biquadratick equation $22874z + 1091zz + 34z^3 + z^4 = 3661.435,187,449,523,505,840,000$, &c; by the resolution of which we had $z = 0.15886$, of which the four highest figures, 0.1588, are exact, and the fifth, or last, figure, 6, differs but by an unit from the fifth figure of the true value of z, which is $= 0.158,857,928,$ $096,805,5$, &c. This fourth approximation to the value of z enabled us to conclude that x, or $24 + z$, would be very nearly equal to $24 + 0.15886$, or 24.15886.

49. We then dropped all further consideration of the equation $\frac{z}{10} - \frac{z - zz}{200}$ $+ \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} + \&c = 0.015,255,979,947,706,347,$ 941, &c, and made a trial of the exactness of the last value of x obtained by the foregoing processes, to wit, 24.15886, by raising the binomial quantity $1 + \frac{1}{10}$ to the power of which 24.15886 is the index, which was done by the help of Sir Isaac Newton's binomial theorem. And we found that the said power of $1 + \frac{1}{10}$ was $= 10,000,001,974,734,858,821$, &c, (which is a little greater

greater than 10) and consequently that 24.15886 is somewhat, though but a very little, greater than the true value of the index x in the equation $\overline{1+\frac{1}{10}}\vert^{x} = 10$.

50. We then supposed x to be $= 24.15886 - y$, and consequently $\overline{1+\frac{1}{10}}\vert^{24.15886-y}$ to be $= 10$.

Then, since $\overline{1+\frac{1}{10}}\vert^{24.15886-y}$ is $= \overline{1+\frac{1}{10}}\vert^{24.15886} \times \overline{1+\frac{1}{10}}\vert^{-y} = \overline{1+\frac{1}{10}}\vert^{24.15886} \times \frac{1}{\overline{1+\frac{1}{10}}\vert^{y}}$, we had $\overline{1+\frac{1}{10}}\vert^{24.15886} \times \frac{1}{\overline{1+\frac{1}{10}}\vert^{y}} = 10$, and consequently $\overline{1+\frac{1}{10}}\vert^{24.15886} = 10 \times \overline{1+\frac{1}{10}}\vert^{y}$, and $\overline{1+\frac{1}{10}}\vert^{y} = \frac{\overline{1+\frac{1}{10}}\vert^{24.15886}}{10}$

$= \frac{10.000,001,974,734,858,821, \&c}{10} = 1.000,000,197,473,485,882,1$, &c.

51. Having thus obtained a third equation $\overline{1+\frac{1}{10}}\vert^{y} = 1.000,000,197,473,485,882,1$, &c, in which y is much smaller than z in the former equation, we proceeded to expand $\overline{1+\frac{1}{10}}\vert^{y}$ into an infinite series by means of Sir Isaac Newton's binomial theorem, and thereby obtained the equation $1 + \frac{y}{10} - \frac{\overline{y-yy}}{200} + \frac{2y-3yy+y^3}{6000} - \frac{\overline{6y-11yy+6y^3-y^4}}{240,000} + \&c = 1.000,000,197,473,485,882,1$, &c, and (by subtracting 1 from both sides,) the equation $\frac{y}{10} - \frac{\overline{y-yy}}{200} + \frac{2y-3yy+y^3}{6000} - \frac{\overline{6y-11yy+6y^3-y^4}}{240,000} + \&c = 0.000,000,197,473,485,882,1$, &c.

52. We then proceeded to approximate to the value of y in this equation, by first supposing the first term $\frac{y}{10}$ alone of the series $\frac{y}{10} - \frac{\overline{y-yy}}{200} + \&c$, (which forms the left-hand side of this equation,) to be equal to the whole series, and consequently to the absolute term 0.000,000,197,473,485,882,1, &c, and, secondly, by supposing the two first terms $\frac{y}{10} - \frac{\overline{y-yy}}{200}$ of the said series to be equal to the same quantity, and resolving the equations resulting from these suppositions.

From the first of these suppositions we had the simple equation $\frac{y}{10} = 0.000,000,197,473,485,882,1$, &c; by the resolution of which we had $y = 0.000,001,97$, and consequently x ($= 24.15886 - y = 24.158,860,00 - 0.000,001,97$) $= 24.158,858,03$; of which number the first seven figures, 24.158,85, are exact.

And from the second of these suppositions there resulted the quadratick equation $19y + yy = 0.000,039,494,697,176,420,0$, &c; by the resolution of which

which we had $y = 0.000,002,078$, and consequently x $(= 24.15886 - y = 24.158,860,000 - 0.000,002,078) = 24.158,857,922$; of which number the first ten figures, 24.158,857,92, are exact.

53. This is a very considerable degree of exactness. But, in order to obtain the value of y exact to a few more places of figures, we had recourse to a different method of resolving the equation $\frac{y}{10} - \frac{y - yy}{200} + \frac{2y - 3yy + y^3}{6000} - \frac{6y - 11yy + 6y^3 - y^4}{240,000} + \&c = 0.000,000,197,473,485,882,1$, &c, which consisted in the omission of all the members in each term of the series that involved either yy, or y^3, or y^4, or any other power of y, except the simple power, or y itself: by which means the said equation was changed into the following simple equation, to wit, $\frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} - \frac{120y}{720,000,000} + \&c = 0.000,000,197,473,485,882,1$, &c, or $\frac{y}{10} - \frac{y}{2 \times 100} + \frac{2 \times y}{2 \times 3 \times 1000} - \frac{6 \times y}{2 \times 3 \times 4 \times 10,000} + \frac{24 \times y}{2 \times 3 \times 4 \times 5 \times 100,000} - \frac{120 \times y}{2 \times 3 \times 4 \times 5 \times 6 \times 1000,000} + \&c = 0.000,000,197,473,485,882,1$, &c, or $\frac{y}{10} - \frac{y}{2 \times 100} + \frac{y}{3 \times 1000} - \frac{y}{4 \times 10,000} + \frac{y}{5 \times 100,000} - \frac{y}{6 \times 1000,000} + \&c = 0.000,000,197,473,485,882,1$, &c, or $y \times$ the infinite series $\frac{1}{10} - \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} - \frac{1}{4 \times 10,000} + \frac{1}{5 \times 100,000} - \frac{1}{6 \times 1000,000} + \&c = 0.000,000,197,473,485,882,1$, &c. We then computed the value of the infinite series $\frac{1}{10} - \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} - \frac{1}{4 \times 10,000} + \frac{1}{5 \times 100,000} - \frac{1}{6 \times 1000,000} + \&c$, or $\frac{1}{10} - \frac{A}{2 \times 10} + \frac{2B}{3 \times 10} - \frac{3C}{4 \times 10} + \frac{4D}{5 \times 10} - \frac{5E}{6 \times 10} + \frac{6F}{7 \times 10} - \frac{7G}{8 \times 10} + \&c$, and found it to be $= 0.095,310,179,804,324,860$, &c; which gave us the simple equation $y \times 0.095,310,179,804,324,860$, &c $= 0.000,000,197,473,485,882,1$, &c, by the resolution of which we had $y = \frac{0.000,000,197,473,485,882,1, \&c}{0.095,310,179,804,324,860, \&c} = 0.000,002,071,903,403$. And from this value of y we concluded x, or $24.158,86 - y$, to be $(= 24.158,860,000,000,000 - 0.000,002,071,903,403) = 24.158,857,928,096,597$, or (because 597 is nearly $= 600$) 24.158,857,928,096,6; which number is true in the first fourteen figures, 24.158,857,928,096, the more acccurate value of x being (as has been before observed,) 24.158,857,928,096,805,5. And thus we obtained the value of the index x of the binomial quantity $1 + \frac{1}{10}$ in the original equation $\overline{1 + \frac{1}{10}}|^x = 10$ exact to fourteen places of figures. Q. E. I.

54. If any lover of this subject should be inclined to compute the value of x to still more than 14 figures, he may find it to nearly as many figures more, or 28 figures in all, by raising $1 + \frac{1}{10}$ to the 24.158,857,928,096th power by means of the binomial theorem, and then proceeding in the manner following.

following. Let the number that ſhall be found to be equal to this power of $1+\frac{1}{10}$ be denoted by the letter P. And, as this number will be ſomewhat leſs than 10, and conſequently the index 24.158,857,928,096 is ſomewhat leſs than the index x, let $24.158,857,928,096+v$ be ſubſtituted inſtead of x in the equation $\overline{1+\frac{1}{10}}|^{x}=10$; and the ſaid equation will be thereby converted into the equation $\overline{1+\frac{1}{10}}|^{24.158,857,928,096\,+\,v}=10$. But $\overline{1+\frac{1}{10}}|^{24.158,857,928,096\,+\,v}$ is $=\overline{1+\frac{1}{10}}|^{24.158,857,928,096}\times\overline{1+\frac{1}{10}}|^{v}=\mathrm{P}\times\overline{1+\frac{1}{10}}|^{v}$. Therefore $\mathrm{P}\times\overline{1+\frac{1}{10}}|^{v}$ will be $=10$, and $\overline{1+\frac{1}{10}}|^{v}$ will be $=\frac{10}{\mathrm{P}}$.

Now let $\overline{1+\frac{1}{10}}|^{v}$ be expanded into an infinite ſeries by means of the binomial theorem; and we ſhall have $1+\frac{v}{10}-\frac{\overline{v-vv}}{200}+\frac{2v-3vv+v^3}{6000}-\frac{\overline{6v-11vv+6v^3-v^4}}{240,000}+$ &c *ad infinitum* $\left(=\overline{1+\frac{1}{10}}|^{v}\right)=\frac{10}{\mathrm{P}}$, and $\frac{v}{10}-\frac{\overline{v-vv}}{200}+\frac{2v-3vv+v^3}{6000}-\frac{\overline{6v-11vv+6v^3-v^4}}{240,000}+$ &c $=\frac{10}{\mathrm{P}}-1$, and (omitting all the quantities that involve vv, or v^3, or v^4, or any of the powers of v except the ſimple power, or v itſelf, on account of their extreme ſmallneſs in compariſon of the terms that involve only the ſimple power of v,) $\frac{v}{10}-\frac{v}{200}+\frac{2v}{6000}-\frac{6v}{240,000}+$ &c, or $v\times$ the infinite ſeries $\frac{1}{10}-\frac{1}{2\times100}+\frac{1}{3\times1000}-\frac{1}{4\times10,000}+\frac{1}{5\times100,000}-\frac{1}{6\times1000,000}+$ &c, or $v\times$ the infinite ſeries $\frac{1}{10}-\frac{\mathrm{A}}{2\times10}+\frac{2\,\mathrm{B}}{3\times10}-\frac{3\,\mathrm{C}}{4\times10}+\frac{4\,\mathrm{D}}{5\times10}-\frac{5\,\mathrm{E}}{6\times10}+\frac{6\,\mathrm{F}}{7\times10}-\frac{7\,\mathrm{G}}{8\times10}+$ &c $=\frac{10}{\mathrm{P}}-1$, or (putting S for the value of the ſaid infinite ſeries computed to a ſufficient number of figures,) $v\times\mathrm{S}=\frac{10}{\mathrm{P}}-1$; and conſequently $v=\dfrac{\frac{10}{\mathrm{P}}-1}{\mathrm{S}}$, and x $(=24.158,857,928,096+v)=24.158,857,928,096+\dfrac{\frac{10}{\mathrm{P}}-1}{\mathrm{S}}$; which will be true to 27 or 28 places of figures.

55. Or, if it ſhould be required to find the value of x to only a few figures more than thoſe which were found in Art. 37, to wit, 24.158,857,928,096,597, (of which the firſt 14 figures, 24.158,857,928,096, were exact,) it may be done without raiſing the binomial quantity $1+\frac{1}{10}$ to the 24.158,857,928,096th power, (which is a very laborious operation,) by retaining ſome of the terms that involve the ſquare of y, as well as thoſe that involve the ſimple power of y, in the laſt equation $\frac{y}{10}-\frac{\overline{y-yy}}{200}+\frac{2y-3yy+y^3}{6000}-\frac{\overline{6y-11yy+6y^3-y^4}}{240,000}+$ &c $=0.000,000,197,473,485,882,1$, &c, and ſuppoſing the terms ſo retained to be

be equal to the whole series that forms the left-hand side of the said equation, and consequently to be equal to the absolute term 0.000,000,197,473,485,882, 1, &c, and then resolving the quadratick equation which will result from such supposition. This may be done in the manner following.

56. If we retain only the terms that involve y and yy in the foregoing equation, it will be converted into the following equation, to wit,

$$\frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} - \frac{120y}{720,000,000} + \&c$$
$$+ \frac{yy}{200} - \frac{3yy}{6000} + \frac{11yy}{240,000} - \frac{50yy}{12,000,000} + \frac{274yy}{720,000,000} - \&c$$
$$= 0.000,000,197,473,485,882,1, \&c,$$

$$\text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6000,000} + \&c$$
$$+ \frac{yy}{200} - \frac{yy}{2000} + \frac{11yy}{240,000} - \frac{yy}{240,000} + \frac{274yy}{720,000,000} - \&c$$
$$= 0.000,000,197,473,485,882,1, \&c,$$

$$\text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6000,000} + \&c$$
$$\begin{aligned} &+ 0.005,000,000,000,000,000 \times yy \\ &- 0.000,500,000,000,000,000 \times yy \\ &+ 0.000,045,833,333,333,333 \times yy \\ &- 0.000,004,166,666,666,666 \times yy \\ &+ 0.000,000,380,555,555,555 \times yy \\ &- 0.000,000,035,000,000,000 \times yy \\ &+ \&c \\ &= 0.000,000,197,473,485,882,1, \&c, \end{aligned}$$

$$\text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6000,000} + \&c$$
$$\begin{aligned} &+ 0.005,046,213,888,888,888 \times yy \\ &- 0.000,504,201,666,666,666 \times yy \\ &= 0.000,000,197,473,485,882,1, \&c \end{aligned}$$

$$\text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6,000,000} + \&c$$
$$\begin{aligned} &+ 0.004,542,012,222,222,222 \times yy \\ &= 0.000,000,197,473,485,882,1, \&c, \text{ or} \\ &\quad 0.095,310,179,804,324,860, \&c \times y \\ &+ 0.004,542,012,222,222,222 \times yy \end{aligned}$$

$= 0.000,000,197,473,485,882,1$, &c, or (neglecting all but the four highest figures of the co-efficient of yy,) $0.095,310,179,804,324,860, \&c \times y + 0.004, 542 \times yy = 0.000,000,197,473,485,882,1$, &c.

57. This quadratick equation may be most conveniently resolved by approximation, by substituting, instead of y, in the quantity $0.004,542 \times yy$ the value of y derived from the simple equation $0.095,310,179,804,324,860, \&c \times y = 0.000,000,197,473,485,882,1, \&c$, which is $= (\frac{0.000,197,473,485,882,1, \&c}{0.095,310,179,804,324,860, \&c}$, or) $0.000,002,071,903,403$. We shall then have $yy = \overline{0.000,002,071,903,403}|^2 = 0.000,000,000,004,292,783,711,362,980,409$, and consequently $0.004,542 \times yy = 0.004,542 \times 0.000,000,000,004,292,783, \&c = 0.000,000,000,000,$ 019,

019,497,820, &c. Therefore 0.095,310,179,804,324,860 $\times y +$ 0.000,000,000,000,019,497,820, &c, will be = 0.000,000,197,473,485,882,1, &c, and (dividing all the terms by 0.095,310,179,804,324,860,) $y +$ 0.000,000,000,000,204, will be = 0.000,002,071,903,403, and (subtracting 0.000,000,000,000,204, from both sides) y will = 0.000,002,071,903,199; and consequently x, or 24.158,86 $- y$, will be =

$$\begin{array}{r} 24.158,860,000,000,000 \\ -\ 0.000,002,071,903,199 \end{array}$$

= 24.158,857,928,096,801; of which all the figures, except the last, are exact, the more accurate value of x being (as we have before observed in Art. 6,) 24.158,857,928,096,805,5. We have therefore now found the value of the index x in the equation $\overline{1 + \frac{1}{10}}\,\big|^{x} = 10$ exact to sixteen places of figures.

Q. E. I.

58. The value of x just now obtained, to wit, 24.158,857,928,096,801, is to 1, as 1 is to $\frac{1.000,000,000,000,000,000,\ \&c}{24.158,857,928,096,801}$ = 0.041,392,685,158,225,484, &c. Therefore the proportion of the ratio of 10 to 1 to the ratio of $1 + \frac{1}{10}$ to 1, or to the ratio of 11 to 10, is that of 1 to 0.041,392,685,158,225,484, &c, or, in other words, Briggs's logarithm of the ratio of 11 to 10 is = 0.041,392,685,158,225,484, &c; of which number the first fifteen figures, (reckoning from the place of units,) to wit, 0.041,392,685,158,225, are exact, the more accurate value of this logarithm (as computed by Mr. Abraham Sharp,) being 0.041,392,685,158,225,040,750.

A DEMON-

A

DEMONSTRATION

OF

SIR ISAAC NEWTON'S

BINOMIAL THEOREM

In the Case of Integral Powers, or Powers of which the Indexes are whole Numbers.

BY *FRANCIS MASERES*, ESQ.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

ARTICLE I.

THE binomial theorem of the great Sir Isaac Newton has been so much resorted to in the preceeding Discourse of Dr. Halley, and in the last foregoing Tract, intitled, " An Appendix to the said Tract," for the purpose of computing logarithms, and is so closely connected with that subject, that it will probably be agreeable to the readers of this collection to see a demonstration given of it in the same volume with the other tracts, of which it is the chief foundation. I shall therefore now proceed to give a demonstration of it in the first and simplest case, or that in which the indexes of the powers, to which the binomial quantity is raised, are positive whole numbers; which may be done with sufficient perspicuity and exactness within the compass of a few pages.

2. This theorem is as follows.

THEOREM.

If m be any whole number whatsoever, $\overline{a+b}|^m$, or the mth power of the binomial quantity $a+b$, will be equal to the following series of terms, to wit, $a^m + \frac{m}{1} \times a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} \times a^{m-2} b^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} b^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times a^{m-5} b^5 +$ &c continued to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$

$\times \frac{m-3}{4} \times \frac{m-4}{5} \times$ &c $\times \frac{m-\overline{m-1}}{m} \times a^{m-m} \times b^m$, or to $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ &c $\times \frac{1}{m} \times 1 \times b^m$, or to $\frac{1}{m} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ &c $\times \frac{b^m}{m}$; in which feries the law by which the fevaral terms, after the firft term a^m, are generated from the faid firft term, is as follows. The literal parts of the fecond, and third, and fourth, and other following terms, to wit, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, &c, are generated from the firft term a^m by the continual multiplication of the fraction $\frac{b}{a}$, the index of the power of a (the firft term of the binomial quantity $a+b$) in every new term decreafing continually by an unit, and the index of the power of b (the fecond term of the binomial quantity $a+b$,) increafing by an unit at the fame time. And the feveral co-efficients of the fecond, third, fourth, and other following terms of the feries, to wit, $\frac{m}{1}$, $\frac{m}{1} \times \frac{m-1}{2}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, are generated from 1, (the co-efficient of the firft term a^m, or $1 \times a^m$,) by the continual multiplication of the fractions $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, of which the denominators are the numbers 1, 2, 3, 4, 5, &c, in their natural order, and the numerator of the firft fraction $\frac{m}{1}$ is the index m, and the numerators of the following fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, are derived from that of the firft, to wit, m, by the continual fubtraction of an unit.

Examples of the faid Theorem.

3. Thus, for example, if m is $=2$, or it is required to find, by means of this theorem, the fquare of the binomial quantity $a+b$, we fhall have $\overline{a+b}|^2 = a^2 + \frac{2}{1} \times a^{2-1}b + \frac{2}{1} \times \frac{2-1}{2} \times a^{2-2}b^2 = a^2 + \frac{2}{1} \times a^1 b + \frac{2}{1} \times \frac{1}{2} \times a^0 \times b^2 = a^2 + 2ab + \frac{2}{2} \times 1 \times b^2 = a^2 + 2ab + b^2$.

And, in like manner, if m is $=3$, we fhall have $\overline{a+b}|^3 = a^3 + \frac{3}{1} \times a^{3-1} b + \frac{3}{1} \times \frac{3-1}{2} \times a^{3-2}b^2 + \frac{3}{1} \times \frac{3-1}{2} \times \frac{3-2}{3} \times a^{3-3}b^3 = a^3 + \frac{3}{1} \times a^2 b + \frac{3}{1} \times \frac{2}{2} \times a^1 b^2 + \frac{3}{1} \times \frac{2}{2} \times \frac{1}{3} \times a^0 b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$.

And, if m is $=4$, we fhall have $\overline{a+b}|^4 = a^4 + \frac{4}{1} \times a^{4-1}b + \frac{4}{1} \times \frac{4-1}{2} \times a^{4-2}b^2 + \frac{4}{1} \times \frac{4-1}{2} \times \frac{4-2}{3} \times a^{4-3}b^3 + \frac{4}{1} \times \frac{4-1}{2} \times \frac{4-2}{3} \times \frac{4-3}{4} \times a^{4-4}b^4 = a^4 + \frac{4}{1} \times a^3 b + \frac{4}{1} \times \frac{3}{2} \times a^2 b^2 + \frac{4}{1} \times \frac{3}{2} \times$

$\frac{2}{3} \times a^1 b^3 + \frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} \times 1 \times b^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$.

And, if m is $= 5$, we ſhall have $\overline{a+b}|^5 = a^5 + \frac{5}{1} \times a^{5-1} b + \frac{5}{1} \times \frac{5-1}{2} \times a^{5-2} b^2 + \frac{5}{1} \times \frac{5-1}{2} \times \frac{5-2}{3} \times a^{5-3} b^3 + \frac{5}{1} \times \frac{5-1}{2} \times \frac{5-2}{3} \times \frac{5-3}{4} \times a^{5-4} b^4 + \frac{5}{1} \times \frac{5-1}{2} \times \frac{5-2}{3} \times \frac{5-3}{4} \times \frac{5-4}{5} \times a^{5-5} b^5 = a^5 + \frac{5}{1} \times a^4 b + \frac{5}{1} \times \frac{4}{2} \times a^3 b^2 + \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times a^2 b^3 + \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times a^1 b^4 + \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times \frac{1}{5} \times a^0 b^5 = a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5$.

4. Theſe examples are ſufficient both to illuſtrate the meaning of the foregoing theorem, and to prove the truth of it in theſe few eaſy caſes. For "that "the theorem is true in theſe caſes," will appear by raiſing the ſecond, third, fourth, and fifth, powers of the binomial quantity $a + b$ in the common way by multiplication; which will produce the very ſame quantities for the ſaid powers as have been juſt now obtained by means of the foregoing theorem. This may be done in the manner following.

$$\begin{array}{l}
a + b \\
a + b \\
\hline
aa + ab \\
\quad + ab + bb \\
\hline
aa + 2ab + bb = \overline{a+b}|^2 \\
a + b \\
\hline
a^3 + 2aab + abb \\
\quad + aab + 2abb + b^3 \\
\hline
a^3 + 3aab + 3abb + b^3 = \overline{a+b}|^3 \\
a + b \\
\hline
a^4 + 3a^3 b + 3a^2 b^2 + ab^3 \\
\quad + a^3 b + 3a^2 b^2 + 3ab^3 + b^4 \\
\hline
a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4 = \overline{a+b}|^4 \\
a + b \\
\hline
a^5 + 4a^4 b + 6a^3 b^2 + 4a^2 b^3 + ab^4 \\
\quad + a^4 b + 4a^3 b^2 + 6a^2 b^3 + 4ab^4 + b^5 \\
\hline
a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5 = \overline{a+b}|^5
\end{array}$$

Theſe values of the ſeveral powers $\overline{a+b}|^2$, $\overline{a+b}|^3$, $\overline{a+b}|^4$, and $\overline{a+b}|^5$, are the very ſame with thoſe that have been obtained juſt above in Art. 3 by means of the foregoing theorem. And conſequently the ſaid theorem is true in theſe four inſtances.

Of the literal Parts of the Terms of the foregoing Products, and the Law of their Generation one from another.

5. The law by which the literal parts of the ſecond and other following terms of the ſeries that is equal to $\overline{a+b}|^m$ are derived from the firſt term a^m, and from

from each other, to wit, "that they are generated by the continual multiplication of the fraction $\frac{b}{a}$," will be sufficiently evident from the foregoing multiplications of $a + b$ and its powers by $a + b$. For it is evident that the indexes of the powers of a in those several products decrease continually by an unit, and that the indexes of the powers of b in the same products increase by an unit at the same time; so that the literal part of the second and every following term in each of the said products is generated from the term next before it by multiplying the said preceeding term into $\frac{b}{a}$. And it is easy to see that the same thing must take place in any higher powers of $a + b$ whatsoever, if the said multiplications by $a + b$ were to be continued till such higher powers were produced. This part, therefore, of the aforesaid binomial theorem stands in need of no further demonstration.

Of the numeral Co-efficients of the two first Terms of the said Products.

6. And with respect to the numeral co-efficients of the several terms of these products obtained by the foregoing multiplications by $a + b$, (which co-efficients are in the square 1, 2, and 1, and in the cube 1, 3, 3, 1, and in the fourth power 1, 4, 6, 4, 1, and in the fifth power 1, 5, 10, 10, 5, 1,) it is easy to see that the co-efficient of the first term of every new power of $a + b$ must always be 1; because it arises by multiplying the next preceeding power of a by a, or $1 \times a$, which cannot alter the co-efficient of the said next preceeding power, which at first was 1. And it is also manifest that the co-efficient of the second term of every new power of $a + b$ must always be the index of the said new power, or, in our present notation, must be m; because it is produced by adding the product of the multiplication of the first term of the next preceeding power of $a + b$, (of which first term 1 is always the co-efficient,) by b to the product of the multiplication of the second term of the said next preceeding power of $a + b$ by a; the effect of which addition is, to increase the co-efficient of the second term of every new power of $a+b$ by an unit. And this, it is easy to see, must be the case in any higher powers whatsoever of $a + b$, if we were to continue the said multiplications by $a + b$ till such higher powers were produced. We may, therefore, conclude that the co-efficients of the two first terms of the series that is equal to $\overline{a + b}|^m$ must always be 1 and m, and consequently that the two first terms of the said series must always be $1 \times a^m$ and $m \times a^{m-1} b$, or a_m and $\frac{m}{1} \times a^{m-1} b$.

7. It remains that we shew that the co-efficients of the third, fourth, fifth, sixth, and other following terms of the series that is equal to $\overline{a + b}|^m$ are generated from the co-efficient m, or $\frac{m}{1}$, of the second term by the continual multiplication of the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$ &c; which is indeed by no means obvious.

Of the numeral Co-efficients of the third and fourth, and other following Terms of the said Products, and the Law of their Generation from the second Term, and from each other.

8. Now, in order to demonstrate the law of the generation of these co-efficients, it will be convenient to get rid of the powers of a and b in the terms of the

the series that is equal to $\overline{a+b}|^{m}$, and to fix our attention only on the generation of the numeral co-efficients of the third, fourth, fifth, sixth, and other following terms of the said series. This may be done by supposing a and b to be, each of them, equal to 1, and consequently $a+b$ to be equal to $1+1$, and $\overline{a+b}|^{m}$ to be equal to $\overline{1+1}|^{m}$. For, as all the powers of both a and b will, on this supposition, be equal to 1, our theorem will then be reduced to this, to wit, that $\overline{1+1}|^{m}$ will be equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} +$ &c, continued to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ &c $\times \frac{m-\overline{m-1}}{m}$, or to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ &c $\times \frac{1}{m}$, or to the term 1. For the last term of this series must always be 1; because the numerators of the several factors in it form a decreasing progression of numbers from m to 1, and the denominators of the same factors form an increasing progression of numbers from 1 to m, and consequently, the product of all the denominators is equal to the product of all the numerators, and therefore the product of all the said factors, or the said last term, must always be equal to 1; which we have seen to be the case in the last terms of the values of $\overline{a+b}|^{2}$, $\overline{a+b}|^{3}$, $\overline{a+b}|^{4}$, $\overline{a+b}|^{5}$, as derived from the series in the theorem in Art. 3, which last terms were $\frac{2}{1} \times \frac{1}{2}$, and $\frac{3}{1} \times \frac{2}{2} \times \frac{1}{3}$, and $\frac{4}{1} \times \frac{3}{2} \times \times \frac{2}{3} \times \frac{1}{4}$, and $\frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times \frac{1}{5}$, which are, each of them, equal to 1. We are therefore now to demonstrate that $\overline{1+1}|^{m}$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-2}{1} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} +$ &c $+ 1$.

Of Mr. James Bernouilli's Demonstration of the Law of the Generation of the said numeral Co-efficients.

9. Now the clearest and best demonstration that I have ever seen of this useful proposition is, that which is given us by the learned and sagacious *Mr. James Bernouilli* in the third chapter of the second book of his excellent Treatise on the Doctrine of Chances, intitled, "*De Arte Conjectandi*," which is written in Latin, and was published in a small quarto volume at *Basil*, or *Basle*, in Switzerland, in the year 1713, eight years after the author's death. This demonstration is founded on the doctrine of combinations and the properties of the figurate numbers, which are there shewn to involve in them the generation of these co-efficients. And the most important properties of the said numbers are in the same chapter set forth and demonstrated by that great author in a very perspicuous and masterly manner, though with rather too much conciseness to be easily understood by beginners in these studies. Those readers, therefore, who are desirous of seeing this theorem demonstrated from its natural and fundamental principles, and in the clearest and most satisfactory

manner

manner that, as I believe, the nature of the thing will admit of, muſt be referred to the ſaid third chapter of the ſecond book of that learned treatiſe; which, together with the two preceeding chapters of the ſame book, I would adviſe them to peruſe with the cloſeſt attention, and to make themſelves thorough maſters of their contents; to do which they will find, will require a conſiderable exertion of their diligence.

Another Demonſtration will be here given of the ſaid Law.

10. But, though that is the beſt and moſt ſatisfactory demonſtration that has been given, and, probably that can be given, of this theorem, yet it may alſo be demonſtrated in a ſhorter and eaſier manner, and with the ſame degree of certainty as by that method of Mr. James Bernouilli, though not with the ſame degree of originality and elegance. And ſuch a demonſtration I now propoſe to give of it in the courſe of the following pages.

11. The demonſtration which is here intended to be given of this important theorem is founded on the obſervation "that the ſaid theorem is found by "trial to be true in ſome of the loweſt powers of the binomial quantity "$1 + 1$;" as has been ſeen above, in Art. 4, in the caſes of the ſecond, third, fourth, and fifth, powers of $a + b$. For, if this theorem is true when the index m is of any particular value, as, for example, when it is equal to 5, it may be ſhewn by abſtract and general reaſonings, derived from the nature of multiplication, that it muſt likewiſe be true when the index m is increaſed by an unit, or that, if n be taken $= m + 1$, the quantity $\overline{1 + 1}|^n$, or the nth power of the binomial quantity $1 + 1$, will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} +$ &c continued to the term $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$ &c $\times \frac{1}{n}$, or 1. And this is what I ſhall now endeavour to demonſtrate.

12. To facilitate the demonſtration of this propoſition it will be convenient to premiſe the following Lemma.

A LEMMA.

If the terms of the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} +$ &c $+ 1$ (in which m repreſents any whole number whatſoever,) be ſet down twice together in two parallel lines, or rows, one under the other, but with the terms in the lower row advanced one ſtep further to the right-hand than the terms in the upper row, ſo that the firſt term in the lower row ſhall ſtand under the ſecond term of the upper row, and the ſecond term in the lower row ſhall ſtand under the third term in the upper row, and the third, fourth, fifth, ſixth, and other following terms in the lower row ſhall ſtand under the fourth, fifth, ſixth, ſeventh, and other following terms in the upper row, reſpectively; and

and both rows are continued to the ſame number of terms, namely, to the whole number of terms in the ſaid ſeries, or to $m + 1$ terms; and then the terms in the lower row, (which, it is evident, will conſiſt of one factor leſs than the correſponding terms, or terms ſtanding immediately above them in the upper row,) be reduced to the ſame denomination as the terms that ſtand immediately above them in the upper row, and, after being ſo reduced, are added to the ſaid terms that ſtand immediately above them in the ſaid upper row;—upon theſe ſuppoſitions the new ſeries of terms ariſing from this addition of the ſaid two rows of terms to each other, will be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m+1}{5}$ + &c + 1; in which ſeries the laſt term is 1, as well as in the two ſerieſes from the addition of which this ſeries ariſes; and the numerators of the laſt factors in all the terms, except the laſt, are always equal to $m + 1$, inſtead of being equal to $m - 1$, $m - 2$, $m - 3$, $m - 4$, &c, as in the two foregoing ſerieſes; and the number of terms in the ſaid new ſeries is $m + 2$, inſtead of $m + 1$, which is the number of terms in each of the ſaid foregoing ſerieſes.

DEMONSTRATION.

13. This will appear by ſetting down the ſaid ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ + &c + 1 twice over, in the manner that has been juſt deſcribed; which may be done as follows:

$$\begin{array}{lllll} 1 + \frac{m}{1} & + \frac{m}{1} \times \frac{m-1}{2} & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} & + \&c \\ \quad 1 + \frac{m}{1} & & + \frac{m}{1} \times \frac{m-1}{2} & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} & + \&c. \end{array}$$

In theſe two rows of terms it is evident, in the firſt place, that the terms in the upper row, after the two firſt terms 1 and $\frac{m}{1}$, conſiſt of two, three, and four, and more, factors, every new term having one more factor than the term next before it; and, 2dly, that the terms in the lower row that ſtand immediately under the third, fourth, fifth, and other following terms in the upper row, conſiſt of one factor leſs than the correſponding terms, or terms immediately over them in the upper row; and, 3dly, that the terms in the lower row conſiſt of the very ſame factors as the correſponding terms in the upper row, excepting that they want the laſt factors of the ſaid terms in the upper row. And hence it follows, that, in order to reduce the terms in the lower row to the ſame denomination as the terms in the upper row, we muſt multiply them by factors that ſhall have the ſame denominators as the laſt, or additional factors in the upper row, and which muſt have their numerators equal to their denominators, ſo as to make each of them equal to 1, to the end that the magnitudes of the ſaid lower terms may not be altered by the multiplication of them by the ſaid

ſaid new factors. Thus, for example, the ſecond term of the lower row, to wit, $\frac{m}{1}$, muſt be multiplied into the factor $\frac{2}{2}$, in order to bring it to the ſame denomination as the third term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, without altering its magnitude; and the third term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, muſt be multiplied into the factor $\frac{3}{3}$, in order to bring it to the ſame denomination as the fourth term of the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, without altering its magnitude; and the fourth term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, muſt be multiplied into the factor $\frac{4}{4}$, in order to bring it to the ſame denomination as the fifth term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, without altering its magnitude; and, for the like reaſon, the fifth, and ſixth, and ſeventh, and other following terms in the lower row muſt be multiplied into the ſeveral factors $\frac{5}{5}$, and $\frac{6}{6}$, and $\frac{7}{7}$, &c, reſpectively; after which multiplications the two rows of terms that are to be added to each other, will be as follows, to wit,

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$$

$$+ 1 + \frac{m}{1} \times \frac{2}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{3}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{4}{4} + \&c.$$

14. And, if theſe two rows of terms, (being now brought to the ſame denominations,) are added together in the manner above deſcribed; that is, every term in the lower row to the term that is immediately above it, the ſum thence reſulting will be the ſeries

$$1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c,$$

in which the numerator of the laſt factor in every term is always $m + 1$, inſtead of $m - 1$, $m - 2$, $m - 3$, $m - 4$, &c.

And "That this muſt be the caſe in all the following terms of the ſaid new "ſeries as well as in the few terms of it that have been here ſet down," will be evident from this conſideration, to wit, That the denominator of the laſt factor of every term in the upper of the two rows of terms that are added together is always greater by an unit than the number which is ſubtracted from m in the numerator of the ſame factor. For from thence it follows that the denominator of the new multiplying fraction in the correſponding term of the lower row (which is always equal to the denominator of the ſaid laſt factor in the upper row,) muſt always be greater by an unit than the number which is ſubtracted from m in the numerator of the laſt factor of the ſaid upper term. And, therefore, the numerator of the ſaid new multiplying fraction in the lower row (which is always equal to its denominator,) muſt alſo always be greater by an unit than the number which is ſubtracted from m in the numerator of the laſt factor of the ſaid upper term; the conſequence of which, in adding the lower

lower term to the upper term, is to convert the numerator of the laſt factor in the upper term from $m-1$, or $m-2$, or $m-3$, or the exceſs of m above ſome other number, into $m+1$. Q. E. D.

15. And the number of terms in the new ſeries, ariſing from the addition of the two former in the manner that has been deſcribed, will be greater by one than the number of the terms in either of the two added ſerieſes: becauſe the lower row of terms, conſiſting of the ſame number of terms as the upper row, and being placed one term further to the right-hand, muſt extend one term beyond it; and conſequently, as the number of terms in each of the two rows of terms is $m+1$, the number of terms in the new ſeries, ariſing from the addition of the two rows together, muſt be $m+2$. Q. E. D.

16. And, laſtly, the laſt term of the ſaid new ſeries muſt be the ſame as the laſt term of the old ſeries, or of the lower row of terms; becauſe, as the lower row of terms extends one term beyond the upper row, the laſt term in the lower row will not have any term over it in the upper row to which it is to be added, and conſequently will continue the ſame in the new ſeries $1+\frac{m+1}{1}+\frac{m}{1}\times\frac{m+1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m+1}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m+1}{4}+$ &c as in the old ſeries $1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+$ &c. But we have ſeen above, in Art. 8, that the laſt term of the ſeries $1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+$ &c is 1. Therefore the laſt term in the new ſeries $1+\frac{m+1}{1}+\frac{m}{1}\times\frac{m+1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m+1}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m+3}{4}+$ &c will alſo be 1. Q. E. D.

17. Coroll. 1. Now let the order of the numerators m, $m-1$, $m-2$, $m-3$, $m-4$, &c, and $m+1$, of the factors of the ſeveral terms of the laſt ſeries $1+\frac{m+1}{1}+\frac{m}{1}\times\frac{m+1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m+1}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m+1}{4}+$ &c, after the two firſt terms, be changed, by making $m+1$ the numerator of the firſt factor of every term inſtead of being the numerator of the laſt factor. The ſaid ſeries will then be as follows, to wit, $1+\frac{m+1}{1}+\frac{m+1}{1}\times\frac{m}{2}+\frac{m+1}{1}\times\frac{m}{2}\times\frac{m-1}{3}+\frac{m+1}{1}\times\frac{m}{2}\times\frac{m-1}{3}\times\frac{m-2}{4}+$ &c. Now this change in the order of the numerators of the ſeveral factors of the terms will create no change in the values, or magnitudes, of the ſeveral terms themſelves; becauſe the products ariſing from the multiplication of the ſame numbers are always the ſame, in whatever order the numbers are multiplied. Therefore the foregoing ſeries, after this change in the order of the numerators of

of the several factors of its terms, will still be of the same magnitude as before, and consequently will be equal to the sum that arises from the addition of the aforesaid two rows of terms in the manner above described; that is, the series $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$ &c $+ 1$ will be equal to the sum that arises from the addition of the aforesaid two rows of terms in the manner above described.

18. Coroll. 2. Now let n be $= m + 1$. Then will $n - 1$ be $= m$, and $n - 2$ will be $= m - 1$, and $n - 3$ will be $= m - 2$, and $n - 4$ will be $= m - 3$, and, in like manner, $n - 5$, $n - 6$, $n - 7$, &c, will be equal to $m - 4$, $m - 5$, $m - 6$, &c. respectively. And consequently the series obtained in the foregoing Corollary, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$ &c $+ 1$, consisting of $m + 2$ terms, will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, consisting of $n + 1$ terms. Therefore the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, consisting of $n + 1$ terms, will be equal to the sum that arises by adding the two aforesaid rows of terms together in the manner above described.

The Demonstration of the principal Proposition.

19. These things being premised, the main proposition stated at the end of Art. 8, to wit, that, if m denote any whole number whatsoever, the quantity $\overline{1 + 1}|^m$, or the mth power of the binomial quantity $1 + 1$, will be equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} +$ &c continued to $m + 1$ terms, or to the term 1, may be demonstrated in the manner following.

20. The product that arises by multiplying the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c into $1 + 1$ is the sum that arises by setting down the said series twice following in two parallel rows, one under the other, with the terms in the lower row advanced one term further to the right-hand than the terms in the upper row, in the manner above described, and then adding the terms in the lower row to the corresponding terms in the upper row. And the $\overline{m + 1}|^{th}$ power of $1 + 1$ is the product of the multiplication of the mth power of $1 + 1$ into $1 + 1$. Therefore, if in any particular value of m the mth power of $1 + 1$ is equal to the

the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c $+ 1$, conſiſting of $m + 1$ terms, the $\overline{m+1}|^{th}$ power of $1 + 1$ will be equal to the ſum that ariſes by ſetting down the ſaid ſeries twice following in two parallel rows in the manner above deſcribed, and adding the ſaid two rows of terms together. But, by the ſecond Corollary of the foregoing Lemma, if n be $= m + 1$, the ſum ariſing from the addition of the ſaid two rows of terms is the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$ conſiſting of $n + 1$ terms. Therefore, if in any particular value of m the mth power of $1 + 1$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c $+ 1$, conſiſting of $m + 1$ terms, it will follow that the $\overline{m+1}|^{th}$, or nth, or next higher power, of $1 + 1$ will be equal to the ſeries $1 + \frac{n}{1} \times \frac{n}{1} + \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, conſiſting of $n + 1$ terms. But it has been ſhewn in Art. 3 and 4 that, when m is equal either to 2, or to 3, or to 4, or to 5, the the mth power of $1 + 1$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c $+ 1$ conſiſting of $m + 1$ terms. Therefore, if n be equal to $5 + 1$, or 6, the $\overline{5+1}|^{th}$ power, or 6th power, or nth power, of $1 + 1$ will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, conſiſting of $n + 1$, or $6 + 1$, or 7, terms. And in the ſame manner it may be proved that, ſince, when m is $= 6$, the mth power of $1 + 1$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c $+ 1$, conſiſting of $m + 1$, or $6 + 1$, or 7, terms, the $\overline{m+1}|^{th}$, or $\overline{6+1}|^{th}$, or 7th, or (putting $n = m + 1 = 6 + 1 = 7$) the nth, power of $1 + 1$ will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, conſiſting of $n + 1$, or $7 + 1$, or 8, terms. And ſo we may proceed from number to number *ad infinitum*. And conſequently, whatever be the whole number denoted by m, it will always be true that $\overline{1+1}|^{m}$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} +$ &c $+ 1$, conſiſting of $m + 1$ terms. Q. E. D.

The

The foregoing Demonstration expressed in a more concise Manner.

21. The foregoing reasonings may be expressed in a more concise manner as follows. If n be $= m + 1$, and it be true in any particular value of m that $\overline{1+1}|^{m}$ is $=$ the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ + &c, it will also be true that $\overline{1+1}|^{n}$ will be $= 1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ + &c.

For $\overline{1+1}|^{n}$ is $= \overline{1+1}|^{m+1} = \overline{1+1}|^{m} \times \overline{1+1} =$ the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ + &c multiplied into $1 + 1 =$

$$\begin{aligned}
&1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c \\
&\quad + 1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \&c. \\
= {}&1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c \\
&\quad + 1 + \frac{m}{1} \times \frac{2}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{3}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{4}{4} + \&c \\
= {}&1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c \\
= {}&1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} + \&c \\
= {}&1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c.
\end{aligned}$$

But it has been shewn in Art. 3 and 4 that, when m is equal either to 2, or to 3, or to 4, or to 5, $\overline{1+1}|^{m}$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ + &c. Therefore, if n be $= 5 + 1$, or 6, $\overline{1+1}|^{n}$, or $\overline{1+1}|^{6}$, will be $=$ the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ + &c. And it may be shewn in like manner that, if n be put for 7, 8, 9, 10, &c *ad infinitum* successively, $\overline{1+1}|^{n}$ will in all these suppositions be always equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{m-3}{4}$ + &c; and therefore the proposition is universally true, whatever be the whole number denoted by the letter n. Q. E. D.

22. This demonstration of the binomial theorem in the case of integral powers, is nearly the same with that given by Mr. John Stewart, of Aberdeen, in the 6th Section of his Commentary on Sir Isaac Newton's curious little Tract, intitled, *Analysis by Equations of an infinite number of Terms.* See his edition of

of Newton's Treatise *on the Quadrature of Curves*, and of the said Tract intitled *Analysis, &c*, with his learned Comments on both, in one volume, quarto, published at London, in the year 1745, page 471, Art. 155.

Of the Invention of the foregoing Theorem.

23. This famous theorem is usually ascribed to Sir Isaac Newton: and in the case of roots, or fractional powers, of a binomial quantity, it is generally agreed that he was the first inventor of it. But in the case of integral powers, (which has been the subject of this Discourse,) it seems to have been discovered many years before by Mr. Henry Briggs, the ingenious computer of the logarithms that are called by his name. For it has been lately observed by Dr. Hutton, (the learned Professor of Mathematicks at Woolwich Academy,) in his very curious historical Introduction to his new edition of Sherwin's Mathematical Tables, that there is a passage in the said Mr. Briggs's valuable Treatise, intitled, *Arithmetica Logarithmica*, (which was published in the year 1624,) in which Mr. Briggs shews how to derive the third, and fourth, and fifth, and other following terms of any integral power of a binomial quantity from the second term of it without raising all the intermediate, or lower powers of the binomial by continual multiplication; which is the operation of the binomial theorem. But Mr. Briggs has only described the method of doing this in words, and has not expressed it in Algebräick symbols, as Sir Isaac Newton has done, by assigning the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$ &c, as the factors, by the continual multiplication of which the co-efficients of the said terms may be produced. The merit, therefore, of being the first inventor, or publisher, of this useful discovery in the case of integral powers, must be allowed to Mr. Briggs, and only that of giving the invention a more convenient form, by expressing it by a short and suitable algebraick notation, must be ascribed to Sir Isaac Newton, together with that of extending it by his sagacious conjectures from the case of powers of which the indexes are positive whole numbers to the various other cases of it in which the indexes of the powers are either fractions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{17}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{17}$, and $\frac{17}{4}$, or, in general, $\frac{m}{n}$, or negative whole numbers, as -1, -2, -3, -4, -5, &c, or, in general, $-m$, or negative fractions, as $\frac{-1}{2}$, $\frac{-1}{3}$, $\frac{-1}{17}$, $\frac{-2}{3}$, $\frac{-3}{2}$, $\frac{-4}{17}$, $\frac{-17}{4}$, or, in general, $\frac{-m}{n}$: which extension of it is of wonderful utility in various branches of the higher parts of Algebra and mathematicks, as we have seen in the remarks that have been published in the former volume of these Tracts on the Logarithmick Serieses invented by Mr. Mercator and Dr. Wallis, and in the foregoing Tract concerning the resolution of the equation $\overline{1+\frac{1}{10}}|^{x} = 10$, or the investigation of the proportion of the ratio of 10 to 1 to the lesser ratio of $1+\frac{1}{10}$ to 1, or of 11 to 10.

24. Yet it may reasonably be conjectured that Sir Isaac Newton was likewise *an inventor*, as well as Mr. Briggs, of this useful theorem even in the case of integral powers, though *not the first inventor* of it. For it is well known that he was

was not an extensive reader of mathematical works; and he appears to have applied himself principally in his younger years to the study of Des Cartes's Geometry, with Schooten's Commentary on it, and the other Tracts published by Schooten with it, and of Dr. Wallis's *Arithmetica Infinitorum*, and his other works on mathematical subjects then published. And therefore he may well be supposed not to have seen Mr. Briggs's *Arithmetica Logarithmica*, in which this method of deriving the co-efficients of the terms of the powers of a binomial quantity one from another is contained, at the time of his discovering this famous theorem himself, which was about the year 1665, or when he was only 23 years old. And, if he had seen that book, and observed this discovery to be contained in it, it is hardly to be conceived that, when he was speaking of this theorem, he would have omitted to make mention of this discovery, and to acknowledge that it contained the substance of the said binomial theorem in the case of integral powers, though not expressed in algebraick symbols. For these reasons I am inclined to think that Sir Isaac Newton had not seen Mr. Briggs's *Arithmetica Logarithmica*, when he invented the binomial theorem, and therefore, that he was truly ***an inventor*** of it even in the case of integral powers, though ***not the first inventor***.

25. But it seems more surprising that Dr. Wallis, who was a much more copious reader of mathematical works than Sir Isaac Newton, and who actually had seen Mr. Briggs's *Arithmetica Logarithmica*, and makes mention of it in his Algebra, chapter xii, page 60, should not have attended to the contents of that ingenious treatise enough to have observed that it contained this most useful theorem. Yet this appears to have been the fact, from what the Doctor tells us in the 85th chapter of his Algebra, page 319, where, in speaking of Mr. Newton's method of generating, or deriving, the co-efficients of the third and fourth, and other following terms of the mth power of a binomial quantity from m, the co-efficient of the second term of it, by the continual multiplication of the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, he confesses that he had sought after this method of generating, or deriving, these co-efficients himself, but without success. His words are these, after speaking of some other excellent inventions in the mathematicks contained in a letter of Mr. Isaac Newton (at that time Professor of Mathematicks in the University of Cambridge, and who was afterwards better known by the title of Sir Isaac Newton,) to Mr. Oldenburgh, (the Secretary of the Royal Society at London,) dated October 24, 1676. "*He* [*Mr. Newton*] *then observes* (*what I had formerly sought after, but unsuccessfully,*) *that the following numbers are, from the two first, to be found by continual multiplication of this series* $1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ &c." These are the words of Dr. Wallis in his Algebra, page 319; from which, I think, we may conclude that, though he had seen Mr. Briggs's *Arithmetica Logarithmica*, he had not read it with sufficient attention to discover that this method of generating the co-efficients of the terms of the mth power of a binomial quantity, when m was any whole number whatsoever, was contained in it: though it seems indeed unaccountably strange that he should not have taken notice of it.

I 26. Dr.

26. Dr. Wallis's Algebra was publiſhed in the year 1685. And that was the firſt time, after Sir Iſaac Newton's diſcovery of it, that the binomial theorem (in Sir Iſaac Newton's manner of expreſſing it) was publiſhed in print, and made known to the learned world in general, though Mr. Oldenburgh and Mr. Leibnitz, and probably Dr. Barrow, (who was Sir Iſaac Newton's great friend and patron in his youth,) and ſome other learned mathematicians of that time, had ſeen it in that letter to Mr. Oldenburgh, of October 24, 1676, ſoon after the ſaid letter was written. But, how Sir Iſaac diſcovered this theorem, is not known; nor is any demonſtration of it, even in this eaſieſt caſe of it, (in which the index m of the power to which the binomial quantity is to be raiſed, is a whole number,) any where to be found in all his works.

Of the Powers of a Reſidual Quantity $a - b$, when their Indexes are whole Numbers.

27. We have hitherto been conſidering the integral powers of a *binomial* quantity $a + b$, or of the *ſum* of two ſingle quantities a and b; and we have ſeen that, if the ſaid binomial quantity $a + b$ be raiſed to any power of which a whole number denoted by m is the index, the quantity $\overline{a+b}|^m$, or the ſaid mth power of $a + b$, will be equal to the ſeries $a^m + \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c + b^m$, or (if we put $A = 1$, $B = \frac{m}{1} A$, $C = \frac{m-1}{2} B$, $D = \frac{m-2}{3} C$, $E = \frac{m-3}{4} D$, $F = \frac{m-4}{5} E$, and G, H, I, K, L, &c, $= \frac{m-5}{6} F$, $\frac{m-6}{7} G$, $\frac{m-7}{8} H$, $\frac{m-8}{9} I$, $\frac{m-9}{10} K$, &c, reſpectively,) to the ſeries $a^m + \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 + \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 + \frac{m-4}{5} E a^{m-5} b^5 + \&c + b^m$; in which all the terms after the firſt term a^m are marked with the ſign +, or are added to the ſaid firſt term. We will now proceed to conſider the value of $\overline{a-b}|^m$, or the mth power of the *reſidual* quantity $a - b$, or the *difference* of the two quantities a and b, upon a ſuppoſition that a is the greater of the two.

28. Now, if a be ſuppoſed to be greater than b, and m be any whole number whatſoever, the quantity $\overline{a-b}|^m$, or the mth power of the reſidual quantity, or difference, $a - b$, will be equal to the ſeries $a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c$, or (if we put A, as before, $= 1$, and $B = \frac{m}{1} A$, and $C = \frac{m-1}{2} B$, and $D = \frac{m-2}{3} C$, and $E = \frac{m-3}{4} D$, and $F = \frac{m-4}{5} E$, and G, H, I, K, L, &c, $= \frac{m-5}{6} F$, $\frac{m-6}{7}$

G, $\frac{m-7}{8}$ H, $\frac{m-8}{9}$ I, $\frac{m-9}{10}$ K, &c, reſpectively,) to the ſeries $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5$ + &c, which conſiſts of exactly the ſame terms as the ſeries that is equal to $\overline{a+b}|^m$, or the ſame power of the binomial quantity $a + b$, but with the ſign $-$ prefixed to the ſecond, and fourth, and ſixth, and every following even term in the ſeries, which denotes that the ſaid terms are not to be added to the firſt term a^m, and to the third, and fifth, and other following odd terms, (as they were in the former ſeries, which was equal to $\overline{a+b}|^m$,) but to be ſubtracted from them.

29. That this muſt be ſo, will be evident from conſidering the manner in which the ſeveral powers of the reſidual quantity $a - b$ are generated from each other by the continual multiplication of $a - b$, of which we will now exhibit a ſpecimen with reſpect to a few of its loweſt powers. The ſecond, third, fourth, and fifth powers of $a - b$ are derived from $a - b$ itſelf by the following multiplications.

$$\begin{array}{l}
a - b \\
\underline{a - b} \\
aa - ab \\
\underline{\quad - ab + bb} \\
aa - 2ab + bb = \overline{a-b}|^2 \\
\underline{\qquad a - b} \\
a^3 - 2a^2b + ab^2 \\
\underline{\quad - a^2b + 2ab^2 - b^3} \\
a^3 - 3a^2b + 3ab^2 - b^3 = \overline{a-b}|^3 \\
\underline{\qquad\qquad a - b} \\
a^4 - 3a^3b + 3a^2b^2 - ab^3 \\
\underline{\quad - a^3b + 3a^2b^2 - 3ab^3 + b^4} \\
a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 = \overline{a-b}|^4 \\
\underline{\qquad\qquad\qquad a - b} \\
a^5 - 4a^4b + 6a^3b^2 - 4a^2b^3 + ab^4 \\
\underline{\quad - a^4b + 4a^3b^2 - 6a^2b^3 + 4ab^4 - b^5} \\
a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 = \overline{a-b}|^5.
\end{array}$$

30. From theſe operations it is evident that, wherever the odd powers of b occur in the ſaid powers of $a - b$, the terms are marked with the ſign $-$, and that, wherever the even powers of b occur in the ſaid powers of $a - b$, the terms are marked with the ſign $+$. And the ſame thing, it is evident, muſt happen in all higher powers of $a - b$ whatſoever, as well in thoſe that have been here ſet down, becauſe b is marked with the ſign $-$ in the two original factors $a - b$ and $a - b$; whence it follows, from the nature of algebraick multiplication, that, whenever b is multiplied into itſelf an even number of times, the product will be marked with the ſign $+$; and, whenever it is multiplied into itſelf an odd number of times, the product will be marked with the ſign $-$. And it is further evident, from the foregoing multiplications, that the odd powers of b occur in the ſecond, and fourth, and ſixth, terms of the fore-

foregoing products, and that the even powers of b occur in the third and fifth terms of them. And it is eaſy to ſee that the odd powers of b will occur in like manner in the eighth, and tenth, and twelfth, and other following even terms of all higher powers of $a - b$ whatſoever, and that the even powers of b will occur in like manner in the ſeventh, and ninth, and eleventh, and other following odd terms of the ſaid higher powers of $a - b$. And it is alſo evident, from the foregoing multiplications, that the terms themſelves of which the ſeveral powers of $a - b$ will be compoſed, are exactly the ſame with the terms of which the ſame powers of $a + b$ are compoſed. And hence it follows that the ſeries which is equal to $\overline{a - b}|^m$ will be the ſame with the ſeries which is equal to $\overline{a + b}|^m$, when the ſign $-$ has been prefixed to the ſecond, and fourth, and ſixth, and other following even terms of it, inſtead of the ſign $+$, or that $\overline{a - b}|^m$, or the mth power of the reſidual quantity $a - b$, will be equal to the ſeries $a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 +$ &c, or $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5 +$ &c. Q. E. D.

A

DEMONSTRATION

OF

Sir ISAAC NEWTON'S

BINOMIAL THEOREM,

In the Cafes of Roots and the Powers of Roots, as well as in the Cafe of Integral Powers; publifhed by Mr. John Landen in the Year 1758.

IN the year 1758 the very learned Mr. John Landen, of Walton, near Peterborough, in Northamptonfhire, publifhed a mathematical Tract in quarto, containing 43 pages, intitled, *A Difcourfe concerning the Refidual Analyfis: a new Branch of the Algebraïck Art, of very extenfive Ufe both in Pure Mathematicks and Natural Philofophy*. In this Difcourfe he has given us a demonftration of the famous binomial theorem of Sir Ifaac Newton, that extends to the cafes of roots and the powers of roots of a binomial quantity, as well as to the cafe of its integral powers; and this, without having recourfe to the doctrine of fluxions, which had ufually been employed for this purpofe by the writers that had gone before him. The propofition he demonftrates is as follows, to wit, "That, if m and n be any two whole numbers whatfoever, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or that power of the binomial quantity $1+x$ which is denoted by the fractional index $\frac{m}{n}$, will be equal to the feries $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 +$ &c, or (putting $A = 1$, and $B = \frac{m}{n}$, and $C = \frac{m-n}{2n}B$, and $D = \frac{m-2n}{3n}C$, and $E = \frac{m-3n}{4n}D$, and $F = \frac{m-4n}{5n}E$, and G, H, I, K, L, &c, for $\frac{m-5n}{6n}F$, $\frac{m-6n}{7n}G$, $\frac{m-7n}{8n}H$, $\frac{m-8n}{9n}I$, $\frac{m-9n}{10n}K$, &c, refpectively,)

ſpectively,) to the ſeries $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4 + \frac{m-4n}{5n} E x^5 + \frac{m-5n}{6n} F x^6 + \frac{m-6n}{7n} G x^7 +$ &c. And it is the firſt propoſition contained in the ſaid Diſcourſe. I ſhall therefore here give the reader both the propoſition itſelf and the demonſtration of it, in the learned author's own words, together with the reflections with which he introduces it; and, as the demonſtration is in itſelf rather ſubtle and difficult, and is alſo expreſſed in a very conciſe manner, I ſhall afterwards ſubjoin ſuch an explanation of it as will, I hope, enable the reader to comprehend it eaſily. The author's words are as follows:

" Before Sir Iſaac Newton invented the *Method of Fluxions*, mathematicians " had made conſiderable improvements in the algebraic art, and had deviſed " ſeveral curious rules for reſolving certain problems relating to the greateſt " and leaſt ordinates, points of inflexion, tangents, curvature, and quadrature " of curve lines, and the cubature of ſolids, &c. Thoſe rules, however, were " eſteemed of little value, upon the appearance of the fluxionary method; " which, being of far more extenſive uſe, was, by the mathematical world, " received with great admiration, and ſtudied with great eagerneſs. Highly " indeed has that method been extolled by many writers; yea, a certain " gentleman has gone ſo far as to ſay, The method of fluxions is capable of " reſolving ſuch difficulties as raiſe the wonder and ſurpriſe of all mankind, and " which would in vain be attempted by any other method whatſoever. So that " it is juſtly eſteemed the greateſt work of genius, and the nobleſt thought that " ever entered the human mind. *Pref. to* EMERS. *Flux.*

" Yet, notwithſtanding the method of fluxions is ſo greatly applauded, I " am induced to think it is not the moſt natural method of reſolving many " problems to which it is uſually applied.—The operations therein being chiefly " performed with algebraic quantities, it is, in fact, a branch of the algebraic " art, or an improvement thereof, made by the help of ſome peculiar principles " borrowed from the doctrine of motion: which principles, I muſt confeſs, to " me ſeem not ſo properly applicable to algebra as thoſe on which that art was, " before, very naturally founded. We may indeed very naturally conceive a " line to be generated by motion; but there are quantities of various kinds, " which we cannot conceive to be ſo generated. It is only in a figurative ſenſe " that an algebraic quantity can be ſaid to increaſe or decreaſe with ſome " velocity or degree of ſwiftneſs; and, by the fluxion of a quantity of that " kind, we muſt, I preſume, to have a clear idea of its meaning, underſtand " the velocity of a point ſuppoſed to deſcribe a line denoting ſuch quantity. " Fluxions therefore are not immediately applicable to algebraic quantities; " but in fluxionary computations made by means of ſuch quantities, we, to " proceed with perſpicuity, muſt have recourſe to the ſuppoſition of lines being " put to denote thoſe quantities, and the generation of thoſe lines by motion. " It therefore, to me, ſeems more proper, in the inveſtigation of propoſitions by " algebra, to proceed upon the *anciently received* principles of that art, than to " introduce therein, without any neceſſity, the new fluxionary principles, de- " rived from a conſideration of motion; and the rather, as the introduction of

" those new principles is not attended with any peculiar advantage.—That the " borrowing principles from the doctrine of motion, with a view to improve " the analytic art, was done, not only without any necessity, but even without " any peculiar advantage, will appear by shewing that, whatever can be done " by the method of computation, which is founded on those borrowed princi- " ples, may be done as well by another method founded entirely on the " *anciently-received* principles of algebra: And that I shall endeavour to shew, " as soon as I have leisure, in the treatise I lately proposed to publish by sub- " scription.—In the mean time, this essay is intended to give the inquisitive " reader some notion of the new method of computation, which is the subject " of that treatise.—Which method I call the *Residual Analysis*; because, in all " the enquiries wherein it is made use of, the conclusions are obtained by means " of residual quantities.

" In the application of the *Residual Analysis*, a geometrical or physical pro- " blem is naturally reduced to another purely algebraical; and the solution is " then readily obtained, without any supposition of motion, and without con- " sidering quantities as composed of infinitely small particles.

" It is by means of the following theorem, *viz.*

$$\text{``}\;\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times \frac{1 + \frac{v}{x} + \overline{\frac{v}{x}}\Big|^{2} + \overline{\frac{v}{x}}\Big|^{3} \quad (m)}{1 + \overline{\frac{v}{x}}\Big|^{\frac{m}{n}} + \overline{\frac{v}{x}}\Big|^{\frac{2m}{n}} + \overline{\frac{v}{x}}\Big|^{\frac{3m}{n}} \quad (n)},$$

(where m and n are any integers,)

" that we are enabled to perform all the principal operations in our said Ana- " lysis; and I am not a little surprised, that a theorem so obvious, and of such " vast use, should so long escape the notice of algebraists!

" I have no objection against the truth of the method of fluxions, being " fully satisfied, that even a problem purely algebraical may be very clearly " resolved by that method, by bringing into consideration lines, and their " generation by motion. But I must own, I am inclined to think such a " problem would be more naturally resolved by pure algebra, without any such " consideration of lines and motion.—Suppose it required to investigate the " binomial theorem; *i. e.* to expand $\overline{1 + x}\,\vert^{\frac{m}{n}}$ into a series of terms of x, and " known co-efficients.

" To do this by the method of fluxions, we first assume $\overline{1 + x}\,\vert^{\frac{m}{n}} = 1 + ax + $ " $bx^2 + cx^3 + dx^4$ &c. We, to proceed with perspicuity, are next to conceive " x, and each term of that assumed equation, to be denoted by some line, and " that line to be described by the motion of a point: then, supposing $\dot{x}$ to be " the velocity of the point describing the line x, and taking, by the rules " taught by those who have treated of the said method, the several contemporary " velocities of the other describing points, or the fluxions of the several terms " in the said equation*, we get $\frac{m}{n} \times \overline{1 + x}\,\vert^{\frac{m}{n}-1} \times \dot{x} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} +$ " $4dx^3\dot{x}$ &c. because, when the space described by a motion is always equal to " the

* See Mac Laurin's Fluxions, vol. 2, page 585, Art. 714.

"the ſum of the ſpaces deſcribed in the ſame time by any other motions, the "velocity of the firſt motion is always equal to the ſum of the velocities of the "other motions.

"From which laſt equation, by dividing by $\dot{x}$, or ſuppoſing $\dot{x}$ equal to unity, "we have $\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^3$ &c. Conſequently, "multiplying by $1+x$, we have $\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}}$, or its equal $\frac{m}{n} + \frac{m}{n}ax + \frac{m}{n}$ "$bx^2 + \frac{m}{n}cx^3$ &c $= a + \left.\begin{matrix}2b\\a\end{matrix}\right\} x + \left.\begin{matrix}3c\\2b\end{matrix}\right\} x^2 + \left.\begin{matrix}4d\\3c\end{matrix}\right\} x^3$ &c. From whence, by "comparing the homologous terms, the co-efficients a, b, c, &c. will be found.

"The ſame theorem is inveſtigated by the *Reſidual Analyſis*, in the following "manner:

"Aſſuming, as above, $\overline{1+x}^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3$ &c, we have $\overline{1+y}^{\frac{m}{n}} =$ "$1 + ay + by^2 + cy^3$ &c; and, by ſubtraction, $\overline{1+x}^{\frac{m}{n}} - \overline{1+y}^{\frac{m}{n}} = a.\overline{x-y}$ "$+ b.\overline{x^2-y^2} + c.\overline{x^3-y^3} + d.\overline{x^4-y^4}$ &c.

"If, now, we divide by the reſidual $x-y$, we ſhall get

$$\text{"}\ \overline{1+x}^{\frac{m}{n}-1} \times \frac{1 + \frac{1+y}{1+x} + \left.\overline{\frac{1+y}{1+x}}\right|^2 + \left.\overline{\frac{1+y}{1+x}}\right|^3 \quad (m)}{1 + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{m}{n}} + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{2m}{n}} + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{3m}{n}} \quad (n)} *$$

"$= a + b.\overline{x+y} + c.\overline{x^2+xy+y^2} + d.\overline{x^3+x^2y+xy^2+y^3}$ &c; which "equation muſt hold true, let y be what it will: from whence, by taking y "equal to x, we find, as before, $\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^3$ "&c. The reſt of the operation will therefore be as above ſpecified.

"Now, as to either of theſe methods of inveſtigation, I ſhall not take upon "me to ſay any thing in particular; it is ſubmitted to the reader to compare "one with the other, and judge which of the two is moſt natural."

Upon this paſſage, at the place marked with an aſteriſk *, the author has ſubjoined, at the end of his diſcourſe, in page 41, the following note.

$$\text{"}\ \frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x-v} \text{ being } = x^{\frac{m}{n}-1} \times \frac{1 + \frac{v}{x} + \left.\overline{\frac{v}{x}}\right|^2 + \left.\overline{\frac{v}{x}}\right|^3 \quad (m)}{1 + \left.\overline{\frac{v}{x}}\right|^{\frac{m}{n}} + \left.\overline{\frac{v}{x}}\right|^{\frac{2m}{n}} + \left.\overline{\frac{v}{x}}\right|^{\frac{3m}{n}} \quad (n)},$$

"(as is obſerved in page 5) we have, by writing $1+x$ and $1+y$ reſpectively,

$$\text{"}\ \frac{\left.\overline{1+x}\right|^{\frac{m}{n}} - \left.\overline{1+y}\right|^{\frac{m}{n}}}{1+x - \overline{1+y}} \left(= \frac{\left.\overline{1+x}\right|^{\frac{m}{n}} - \left.\overline{1+y}\right|^{\frac{m}{n}}}{x-y}\right)$$

$$\text{“} = \overline{1+x}\,|^{\frac{m}{n}-1} \times \frac{1 + \frac{1+y}{1+x} + \overline{\frac{1+y}{1+x}}\,|^{2} \qquad (m)}{1 + \overline{\frac{1+y}{1+x}}\,|^{\frac{m}{n}} + \overline{\frac{1+y}{1+x}}\,|^{\frac{2m}{n}} \qquad (n)}.\text{”}$$

This is Mr. Landen's demonstration of the foregoing celebrated theorem, expressed in his own words in the above-mentioned Tract, intitled, *A Discourse concerning the Residual Analysis, &c*, which was published in the year 1758. Six years afterwards, to wit, in the year 1764, Mr. Landen published a second Tract on the same subject, containing 218 pages in quarto, intitled, *The Residual Analysis*; Book I. To which the former Tract, intitled, *A Discourse concerning the Residual Analysis, &c*, had been only an introduction. In this second Tract, pages 5 and 6, Mr. Landen informs us that one of the theorems which chiefly enabled him to perform a certain division of one residual quantity by another, which is of great use in his Residual Analysis, was the following, to wit,

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w} = \frac{v^{m-1} + v^{m-2}w + v^{m-3}w^2 + v^{m-4}w^3 \qquad (m)}{v^{m-\frac{m}{r}} + v^{m-\frac{2m}{r}}w^{\frac{m}{r}} + v^{m-\frac{3m}{r}}w^{\frac{2m}{r}} + v^{m-\frac{4m}{r}}w^{\frac{3m}{r}} \qquad (r)}$$

$$= v^{\frac{m}{r}-1} \times \frac{1 + \frac{w}{v} + \overline{\frac{w}{v}}\,|^{2} + \overline{\frac{w}{v}}\,|^{3} \qquad (m)}{1 + \overline{\frac{w}{v}}\,|^{\frac{m}{r}} + \overline{\frac{w}{v}}\,|^{\frac{2m}{r}} + \overline{\frac{w}{v}}\,|^{\frac{3m}{r}} \qquad (r)},$$

m and r being positive integers. And in the bottom of the said page 6 he subjoins the following note, containing an investigation of the said theorem.

NOTE.——This theorem may be investigated as follows:

It is well known, that

$$\frac{v^m - w^m}{v - w} \text{ is } = v^{m-1} + v^{m-2}w + v^{m-3}w^2 \qquad (m);$$

and that

$$\frac{a^r - b^r}{a - b} \text{ is } = a^{r-1} + a^{r-2}b + a^{r-3}b^2 \qquad (r),$$

m and r being positive integers.

In the second equation write $v^{\frac{m}{r}}$ and $w^{\frac{m}{r}}$ instead of a and b respectively; and you will have

$$\frac{v^m - w^m}{v^{\frac{m}{r}} - w^{\frac{m}{r}}} = v^{m-\frac{m}{r}} + v^{m-\frac{2m}{r}}w^{\frac{m}{r}} + v^{m-\frac{3m}{r}}w^{\frac{2m}{r}} \; (r).$$

Then, from the first and third equations, it will appear by division, that

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w} \text{ is } = \frac{v^{m-1} + v^{m-2}w \qquad (m)}{v^{m-\frac{m}{r}} + v^{m-\frac{2m}{r}}w^{\frac{m}{r}} \qquad (r)},$$

which being ſo obvious, it is matter of ſurpriſe to me, that algebraiſts have not before obſerved it, and ſhewn its ſingular uſe in analytics.

Theſe are all the paſſages that I have found in theſe two Tracts of Mr. Landen concerning the foregoing demonſtration, of which I will now proceed to give as full and clear an explanation as I can.

AN

EXPLANATION

OF THE

FOREGOING DEMONSTRATION

OF THE

BINOMIAL THEOREM,

In the Case of the fractional Index $\frac{m}{n}$, invented by Mr. John Landen.

BY *FRANCIS MASERES*, ESQ.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

ARTICLE I.

THE proposition demonstrated by Mr. John Landen is as follows: If m and n be any whole numbers whatsoever, and x be any quantity not greater than 1, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or that power of the binomial quantity $1+x$ which has for its index the fraction $\frac{m}{n}$, will be equal to the series $1 + \frac{m}{n} x + \frac{m}{n} \times \frac{m-n}{2n} x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5 +$ &c, or (putting $A = 1$, and $B = \frac{m}{n} A$, and $C = \frac{m-n}{2n} B$, and $D = \frac{m-2n}{3n} C$, and $E = \frac{m-3n}{4n} D$, and $F = \frac{m-4n}{5n} E$, and G, H, I, K, L, &c $= \frac{m-5n}{6n} F$, and $\frac{m-6n}{7n} G$, and $\frac{m-7n}{8n} H$, and $\frac{m-8n}{9n} I$, and $\frac{m-9n}{10n} K$, &c, respectively,) to the series $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4 + \frac{m-4n}{5n} E x^5 + \frac{m-5n}{6n} F x^6 + \frac{m-6n}{7n} G x^7$

$+ \frac{m-7n}{8n} H x^8 + \frac{m-8n}{9n} I x^9 + \frac{m-9n}{10n} K x^{10}$ + &c. His demonſtration of this propoſition is deduced from the two following Lemmas.

LEMMA I.

2. If a and b be any two quantities whatſoever, of which a is the greater, and m be any whole number whatſoever, the difference of the mth powers of a and b, to wit, $a^m - b^m$, will be capable of being exactly divided by $a - b$, or the difference of the ſaid quantities themſelves, ſo as to leave no remainder; and the quotient that ariſes by ſuch diviſion, or the value of the fraction $\frac{a^m - b^m}{a - b}$, will be a ſeries of terms in continued geometrical proportion, conſiſting of m terms, of which a^{m-1} will be the firſt term, and b^{m-1} will be the laſt term, and the common ratio of the terms will be that of a to b.

EXAMPLES.

Thus, for example, if we divide $a^2 - b^2$ by $a - b$, the quotient will be $a + b$. The diviſion is performed as follows:

$$
\begin{array}{rllll}
a - b) & a^2 & * & - b^2 & (a + b \\
 & a^2 & - ab & & \\
\hline
 & * & + ab & - b^2 & \\
 & & + ab & - b^2 & \\
\hline
 & & * & * &
\end{array}
$$

And, if we divide $a^3 - b^3$ by $a - b$, the quotient will be $a^2 + ab + b^2$. The diviſion is performed as follows:

$$
\begin{array}{rlllll}
a - b) & a^3 & * & * & - b^3 & (a^2 + ab + b^2 \\
 & a^3 & - a^2 b & & & \\
\hline
 & * & + a^2 b & * & & \\
 & & + a^2 b & - ab^2 & & \\
\hline
 & & * & + ab^2 & - b^3 & \\
 & & & + ab^2 & - b^3 & \\
\hline
 & & & * & * &
\end{array}
$$

And, if we divide $a^4 - b^4$ by $a - b$, the quotient will be $a^3 + a^2 b + ab^2 + b^3$. The diviſion is performed as follows:

$$
\begin{array}{rllllll}
a - b) & a^4 & * & * & * & - b^4 & (a^3 + a^2 b + ab^2 + b^3 \\
 & a^4 & - a^3 b & & & & \\
\hline
 & * & + a^3 b & * & & & \\
 & & + a^3 b & - a^2 b^2 & & & \\
\hline
 & & * & + a^2 b^2 & * & & \\
 & & & + a^2 b^2 & - ab^3 & & \\
\hline
 & & & * & + ab^3 & - b^4 & \\
 & & & & + ab^3 & - b^4 & \\
\hline
 & & & & * & * &
\end{array}
$$

And,

And, if we divide $a^5 - b^5$ by $a - b$, the quotient will be $a^4 + a^3 b + a^2 b^2 + ab^3 + b^4$. The division is performed as follows:

$$\begin{array}{l}
a - b)\ a^5 \quad * \quad * \quad * \quad * \quad - b^5\ (a^4 + a^3 b + a^2 b^2 + ab^3 + b^4. \\
\quad a^5 - a^4 b \\
\quad * \ + a^4 b \ * \\
\quad \quad + a^4 b - a^3 b^2 \\
\quad \quad * \ + a^3 b^2 \ * \\
\quad \quad \quad + a^3 b^2 - a^2 b^3 \\
\quad \quad \quad * \ + a^2 b^3 \ * \\
\quad \quad \quad \quad + a^2 b^3 - ab^4 \\
\quad \quad \quad \quad * \ + ab^4 - b^5 \\
\quad \quad \quad \quad \quad + ab^4 - b^5 \\
\quad \quad \quad \quad \quad \quad * \quad *
\end{array}$$

3. In all these examples we see that this Lemma is true; since each of these quotients $a + b$, $aa + ab + bb$, $a^2 + a^2 b + ab^2 + b^3$, and $a^4 + a^3 b + a^2 b^2 + ab^3 + b^4$, is a geometrical progression of terms, of which the common ratio is that of a to b, and the number of terms in each series is equal to the number of units in m, or the index of the powers of a and b in the dividend; and the first term of each series is a^{m-1}, to wit, in the first example, a^{2-1}, or a, and in the second example, a^{3-1}, or a^2, and in the third example, a^{4-1}, or a^3, and in the fourth example, a^{5-1}, or a^4; and the last term of each series is b^{m-1}, to wit, in the first example, b^{2-1}, or b, and in the second example, b^{3-1}, or b^2, and in the third example, b^{4-1}, or b^3; and in the fourth example, b^{5-1}, or b^4.

DEMONSTRATION.

4. And that the same thing must take place when m is equal to any greater whole number whatsoever, will appear by performing the division of $a^m - b^m$ by $a - b$ in general terms; which may be done in the manner following.

$$\begin{array}{l}
a - b)\ a^m \quad * \quad * \quad * \quad * \quad * \quad - b^m\ (a^{m-1} + a^{m-2} b + a^{m-3} b^2 + a^{m-4} b^3 + a^{m-5} b^4 + a^{m-6} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad b^5 + \&c + b^{m-1}. \\
\quad a^m - a^{m-1} b \\
\quad * \ + a^{m-1} b \\
\quad \quad + a^{m-1} b - a^{m-2} b^2 \\
\quad \quad * \ + a^{m-2} b^2 \\
\quad \quad \quad + a^{m-2} b^2 - a^{m-3} b^3 \\
\quad \quad \quad * \ + a^{m-3} b^3 \\
\quad \quad \quad \quad + a^{m-3} b^3 - a^{m-4} b^4 \\
\quad \quad \quad \quad * \ + a^{m-4} b^4 \\
\quad \quad \quad \quad \quad + a^{m-4} b^4 - a^{m-5} b^5 \\
\quad \quad \quad \quad \quad * \ + a^{m-5} b^5 \\
\quad \quad \quad \quad \quad \quad + a^{m-5} b^5.
\end{array}$$

Now it is evident that the terms of this quotient, $a^{m-1} + a^{m-2}b + a^{m-3}b^2 + a^{m-4}b^3 + a^{m-5}b^4 + a^{m-6}b^5 +$ &c, decrease in the continual proportion of a to b. For the index of the power of b in every new term of it is greater by an unit than the index of the power of b in the term next before it; and the index of the power of a in every new term of it is less by an unit than the index of the power of a in the term next before it; so that every new term is equal to the term next before it multiplied into the fraction $\frac{b}{a}$. And, further, the first term of this series, or quotient, is a^{m-1}; and its last term will be b^{m-1} for the following reason. The index of the power of b in every term is less by an unit than the number subtracted from m in the index of the power of a in the same term. Thus, for example, in the fourth term, $a^{m-4}b^3$, the index of b is 3, which is less by an unit than the number 4, which is subtracted from m in the index $m - 4$ of the power of a. It follows therefore that, by continuing the division, and obtaining more and more terms of the quotient, we shall come at last (whatever be the magnitude of the number m,) to the term $a^{m-m} \times b^{m-1}$, or $a^0 \times b^{m-1}$, or $1 \times b^{m-1}$, or b^{m-1}; and this term, being multiplied into the divisor $a - b$, will produce the terms $ab^{m-1} - b^m$, which being subtracted from the last dividend, (which will also be $ab^{m-1} - b^m$,) will leave no remainder. Therefore b^{m-1} will be the last term of the quotient, as is asserted in the proposition. And, lastly, since b^{m-1} is the last term of the said quotient, and the terms of the said quotient, beginning with the second term $a^{m-2}b$, involve all the powers of b, from b, or b^1, to b^{m-1}, in their regular order, it follows, that there will be as many terms in the said quotient that will involve some power of b, as there are units in the index $m - 1$, that is, $m - 1$ such terms; and consequently, the whole quotient, including the first term a^{m-1}, (which does not involve b in it,) will consist of $m - 1$ terms and one term, that is, of m terms. It appears therefore, in the first place, that the terms of this quotient will constitute a decreasing geometrical progression, in which the common ratio of the terms will be that of a to b; and secondly, that the first term of the said progression will be a^{m-1}, and that the last term of it will be b^{m-1}; and, thirdly, that the number of the terms of the said progression will be m; which are the several points which were to be demonstrated.

LEMMA II.

5. If m and n be any whole numbers whatsoever, and a and b be any two quantities of which a is the greater, and p and q be any two quantities of which

p is

p is the greater, the fraction $\dfrac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$, or the quotient of the division of $p^{\frac{m}{n}} - q^{\frac{m}{n}}$ by $p - q$, will be equal to the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + \&c, \text{ continued to } m \text{ terms,}}{p^{m-\frac{m}{n}} + p^{m-\frac{2m}{n}}q^{\frac{m}{n}} + p^{m-\frac{3m}{n}} \times q^{\frac{2m}{n}} + p^{m-\frac{4m}{n}} \times q^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

DEMONSTRATION.

By the foregoing Lemma we shall have $\dfrac{a^n - b^n}{a - b}$ = the series $a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + a^{n-5}b^4 +$ &c, continued to n terms. Therefore, if we substitute $p^{\frac{m}{n}}$ instead of a, and $q^{\frac{m}{n}}$ instead of b in this equation, we shall have $\dfrac{\overline{p^{\frac{m}{n}}}\big|^{n} - \overline{q^{\frac{m}{n}}}\big|^{n}}{p^{\frac{m}{n}} - q^{\frac{m}{n}}}$ = the series $\overline{p^{\frac{m}{n}}}\big|^{n-1} + \overline{p^{\frac{m}{n}}}\big|^{n-2} \times q^{\frac{m}{n}} + \overline{p^{\frac{m}{n}}}\big|^{n-3} \times \overline{q^{\frac{m}{n}}}\big|^{2} + \overline{p^{\frac{m}{n}}}\big|^{n-4} \times \overline{q^{\frac{m}{n}}}\big|^{3} + \overline{p^{\frac{m}{n}}}\big|^{n-5} \times \overline{q^{\frac{m}{n}}}\big|^{4} +$ &c, continued to n terms; that is, $\dfrac{p^m - q^m}{p^{\frac{m}{n}} - q^{\frac{m}{n}}}$ will be = the series $p^{m-\frac{m}{n}} + p^{m-\frac{2m}{n}} \times q^{\frac{m}{n}} + p^{m-\frac{3m}{n}} \times q^{\frac{2m}{n}} + p^{m-\frac{4m}{n}} \times q^{\frac{3m}{n}} + p^{m-\frac{5m}{n}} \times q^{\frac{4m}{n}} +$ &c, continued to n terms. Therefore, the reciprocal of this fraction, or the fraction $\dfrac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^m - q^m}$, will be equal to the quotient that arises by dividing 1 by the series $p^{m-\frac{m}{n}} + p^{m-\frac{2m}{n}} \times q^{\frac{m}{n}} + p^{m-\frac{3m}{n}} \times q^{\frac{2m}{n}} + p^{m-\frac{4m}{n}} \times q^{\frac{3m}{n}} + p^{m-\frac{5m}{n}} \times q^{\frac{4m}{n}} +$ &c, continued to n terms; that is, the fraction $\dfrac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^m - q^m}$ will be equal to the fraction

$$\frac{1}{p^{m-\frac{m}{n}} + p^{m-\frac{2m}{n}} \times q^{\frac{m}{n}} + p^{m-\frac{3m}{n}} \times q^{\frac{2m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

Further, by the foregoing Lemma, the fraction $\dfrac{p^m - q^m}{p - q}$ is equal to the series $p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 +$ &c, continued to m terms.

m terms. Therefore, if we multiply the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^m - q^m}$ by the fraction $\frac{p^m - q^m}{p - q}$, and the fraction

$$\frac{1}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} \times q^{\frac{2m}{n}} + \text{\&c, continued to } n \text{ terms}}$$

by the series $p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 +$ &c, continued to m terms, the products thence arising will be equal. But the product of the multiplication of the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^m - q^m}$ by the fraction $\frac{p^m - q^m}{p - q}$ is the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$; and the product of the multiplication of the fraction

$$\frac{1}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{2m}{n}} + \text{\&c, continued to } n \text{ terms}}$$

by the series $p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 +$ &c, continued to m terms, is the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \text{\&c, continued to } m \text{ terms,}}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} q^{\frac{2m}{n}} + \text{\&c, continued to } n \text{ terms.}}$$

Therefore the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$ will be equal to the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \text{\&c, continued to } m \text{ terms,}}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}} q^{\frac{3m}{n}} + \text{\&c, continued to } n \text{ terms.}}$$

Q. E. D.

6. Coroll. The said fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$ is also equal to $p^{\frac{m}{n} - 1} \times$ the fraction

$$\frac{1 + \frac{q}{p} + \overline{\frac{q}{p}}\big|^2 + \overline{\frac{q}{p}}\big|^3 + \overline{\frac{q}{p}}\big|^4 + \overline{\frac{q}{p}}\big|^5 + \text{\&c, continued to } m \text{ terms,}}{1 + \overline{\frac{q}{p}}\big|^{\frac{m}{n}} + \overline{\frac{q}{p}}\big|^{\frac{2m}{n}} + \overline{\frac{q}{p}}\big|^{\frac{3m}{n}} + \overline{\frac{q}{p}}\big|^{\frac{4m}{n}} + \text{\&c, continued to } n \text{ terms.}}$$

For the series $p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 +$ &c, continued to m terms, is $= p^{m-1} \times$ the series $1 + p^{-1}q + p^{-2}q^2 + p^{-3}q^3 + p^{-4}q^4 +$ &c, continued to m terms,

$= p^{m-1} \times$ the series $1 + \frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \frac{q^4}{p^4} +$ &c, continued to m terms,

$= p^{m-1} \times$ the series $1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \left(\frac{q}{p}\right)^4 +$ &c, continued to m terms.

And the series $p^{m-\frac{m}{n}} + p^{m-\frac{2m}{n}} q^{\frac{m}{n}} + p^{m-\frac{3m}{n}} q^{\frac{2m}{n}} + p^{m-\frac{4m}{n}} q^{\frac{3m}{n}} +$ &c, continued to n terms, is $=$

$p^{m-\frac{m}{n}} \times$ the series $1 + p^{-\frac{m}{n}} q^{\frac{m}{n}} + p^{-\frac{2m}{n}} q^{\frac{2m}{n}} + p^{-\frac{3m}{n}} q^{\frac{3m}{n}} +$ &c, continued to n terms, $=$

$p^{m-\frac{m}{n}} \times$ the series $1 + \frac{q^{\frac{m}{n}}}{p^{\frac{m}{n}}} + \frac{q^{\frac{2m}{n}}}{p^{\frac{2m}{n}}} + \frac{q^{\frac{3m}{n}}}{p^{\frac{3m}{n}}} +$ &c, continued to n terms, $=$

$p^{m-\frac{m}{n}} \times$ the series $1 + \left(\frac{q}{p}\right)^{\frac{m}{n}} + \left(\frac{q}{p}\right)^{\frac{2m}{n}} + \left(\frac{q}{p}\right)^{\frac{3m}{n}} +$ &c, continued to n terms.

Therefore the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \text{\&c, continued to } m \text{ terms,}}{p^{m-\frac{m}{n}} + p^{m-\frac{2m}{n}} q^{\frac{m}{n}} + p^{m-\frac{3m}{n}} q^{\frac{2m}{n}} + p^{m-\frac{4m}{n}} q^{\frac{3m}{n}} + \text{\&c, continued to } n \text{ terms,}}$$

will be $=$ the fraction $\frac{p^{m-1}}{p^{m-\frac{m}{n}}} \times$ the fraction

$$\frac{1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \left(\frac{q}{p}\right)^4 + \text{\&c, continued to } m \text{ terms,}}{1 + \left(\frac{q}{p}\right)^{\frac{m}{n}} + \left(\frac{q}{p}\right)^{\frac{2m}{n}} + \left(\frac{q}{q}\right)^{\frac{3m}{n}} + \text{\&c, continued to } n \text{ terms.}}$$

But the fraction $\frac{p^{m-1}}{p^{m-\frac{m}{n}}}$ is $= p^{m-1} \times \frac{1}{p^{m-\frac{m}{n}}} = p^{m-1} \times p^{-m+\frac{m}{n}} = p^{m-1-m+\frac{m}{n}}$

$= p^{\frac{m}{n}-1}$.

Therefore, if we substitute $p^{\frac{m}{n}-1}$ instead of $\frac{p^{m-1}}{p^{m-\frac{m}{n}}}$ in the last equation, we shall have the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \text{\&c, continued to } m \text{ terms,}}{p^{m-\frac{m}{n}} + p^{m-\frac{2m}{n}} q^{\frac{m}{n}} + p^{m-\frac{3m}{n}} q^{\frac{2m}{n}} + p^{m-\frac{4m}{n}} q^{\frac{3m}{n}} + \text{\&c, continued to } n \text{ terms,}}$$

$= p^{\frac{m}{n}-1} \times$ the fraction.

$$\frac{1+\frac{q}{p}+\overline{\frac{q}{p}}|^{2}+\overline{\frac{q}{p}}|^{3}+\overline{\frac{q}{p}}|^{4}+\text{ \&c, continued to } m \text{ terms,}}{1+\overline{\frac{q}{p}}|^{\frac{m}{n}}+\overline{\frac{q}{p}}|^{\frac{2m}{n}}+\overline{\frac{q}{p}}|^{\frac{3m}{n}}+\overline{\frac{q}{p}}|^{\frac{4m}{n}}+\text{ \&c, continued to } n \text{ terms.}}$$

Therefore, the fraction $\frac{p^{\frac{m}{n}}-q^{\frac{m}{n}}}{p-q}$, (which is shewn in the foregoing Lemma to be equal to the fraction

$$\frac{p^{m-1}+p^{m-2}q+p^{m-3}q^{2}+p^{m-4}q^{3}+p^{m-5}q^{4}+\text{ \&c, continued to } m \text{ terms,}}{p^{m-\frac{m}{n}}+p^{m-\frac{2m}{n}}q^{\frac{m}{n}}+p^{m-\frac{3m}{n}}q^{\frac{2m}{n}}+p^{m-\frac{4m}{n}}q^{\frac{3m}{n}}+\text{ \&c, continued to } n \text{ terms,}}\Big)$$

will be equal to $p^{\frac{m}{n}-1}$ $\times$ the fraction

$$\frac{1+\frac{q}{p}+\overline{\frac{q}{p}}|^{2}+\overline{\frac{q}{p}}|^{3}+\overline{\frac{q}{p}}|^{4}+\text{ \&c, continued to } m \text{ terms,}}{1+\overline{\frac{q}{p}}|^{\frac{m}{n}}+\overline{\frac{q}{p}}|^{\frac{2m}{n}}+\overline{\frac{q}{p}}|^{\frac{3m}{n}}+\overline{\frac{q}{p}}|^{\frac{4m}{n}}+\text{ \&c, continued to } n \text{ terms.}}$$

Q. E. D.

These things being premised, Mr. Landen's Demonstration of the Binomial Theorem will be as follows:

7. In the first place, Mr. Landen supposes that the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the $\frac{m}{n}$th power of the binomial quantity $1+x$, will be equal to a certain series of quantities, of which the first term will be 1, and the following terms will involve the several powers of x in their natural order, to wit, x, x^2, x^3, x^4, x^5, &c, multiplied by certain fixt numbers, as co-efficients, which may be denoted by the letters B, C, D, E, F, G, H, &c, or that $\overline{1+x}|^{\frac{m}{n}}$ is $=$ the series $1+Bx+Cx^2+Dx^3+Ex^4+Fx^5+$ &c, in which the co-efficients B, C, D, E, F, &c, are hitherto unknown, and are now to be investigated. And this he supposes to be true, of whatever magnitude, not greater than 1, the quantity x be taken.

8. Now, if $\overline{1+x}|^{\frac{m}{n}}$ is always equal to the series $1+Bx+Cx^2+Dx^3+Ex^4+Fx^5$ &c, whatever be the magnitude of x, (so long as it is not greater than 1,) it follows that, if we suppose x to decrease from its first magnitude (which we will suppose to be denoted by x,) to a magnitude somewhat less than x, which we will call y, we shall have $\overline{1+y}|^{\frac{m}{n}}$ equal to a series containing the same powers of y, combined with the same co-efficients B, C, D, E, F, &c, respectively, as there were powers of x in the former series which was equal to $\overline{1+x}|^{\frac{m}{n}}$; or $\overline{1+y}|^{\frac{m}{n}}$ will be equal to the series $1+By+Cy^2+Dy^3+Ey^4+Fy^5+$ &c. Therefore, if we subtract this latter equation from the former, we

we shall have $\overline{1+x}|^{\frac{m}{n}} - \overline{1+y}|^{\frac{m}{n}}$ = the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c $-$ the series $1 + By + Cy^2 + Dy^3 + Ey^4 + Fy^5 +$ &c $= B \times \overline{x-y} + C \times \overline{x^2-y^2} + D \times \overline{x^3-y^3} + E \times \overline{x^4-y^4} + F \times \overline{x^5-y^5}$ + &c; and consequently, if we divide both sides by $x-y$, we shall have

$$\frac{\overline{1+x}|^{\frac{m}{n}} - \overline{1+y}|^{\frac{m}{n}}}{x-y} = B + C \times \frac{x^2-y^2}{x-y} + D \times \frac{x^3-y^3}{x-y} + E \times \frac{x^4-y^4}{x-y} + F \times \frac{x^5-y^5}{x-y} + \&c$$

= (by the first Lemma) $B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3} + F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4}$ + &c.

But $x-y$ is $= 1+x-\overline{1+y}$. Therefore $\dfrac{\overline{1+x}|^{\frac{m}{n}} - \overline{1+y}|^{\frac{m}{n}}}{1+x-\overline{1+y}}$ will be $=$ $B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3} + F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4}$ + &c. Now it has been shewn in the Corollary to Lemma 2, that $\dfrac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p-q}$ is $= p^{\frac{m}{n}-1}$ × the fraction

$$\frac{1 + \frac{q}{p} + \overline{\frac{q}{p}}|^2 + \overline{\frac{q}{p}}|^3 + \overline{\frac{q}{p}}|^4 + \text{\&c, continued to } m \text{ terms,}}{1 + \overline{\frac{q}{p}}|^{\frac{m}{n}} + \overline{\frac{q}{p}}|^{\frac{2m}{n}} + \overline{\frac{q}{p}}|^{\frac{3m}{n}} + \overline{\frac{q}{p}}|^{\frac{4m}{n}} + \text{\&c, continued to } n \text{ terms.}}$$

Therefore, if we substitute $1+x$ for p, and $1+y$ for q, it will follow that $\dfrac{\overline{1+x}|^{\frac{m}{n}} - \overline{1+y}|^{\frac{m}{n}}}{1+x-\overline{1+y}}$ will be $= \overline{1+x}|^{\frac{m}{n}-1}$ × the fraction

$$\frac{1 + \frac{1+y}{1+x} + \overline{\frac{1+y}{1+x}}|^2 + \overline{\frac{1+y}{1+x}}|^3 + \text{\&c, continued to } m \text{ terms,}}{1 + \overline{\frac{1+y}{1+x}}|^{\frac{m}{n}} + \overline{\frac{1+y}{1+x}}|^{\frac{2m}{n}} + \overline{\frac{1+y}{1+x}}|^{\frac{3m}{n}} + \text{\&c, continued to } n \text{ terms.}}$$

But it has been shewn that $\dfrac{\overline{1+x}|^{\frac{m}{n}} - \overline{1+y}|^{\frac{m}{n}}}{1+x-\overline{1+y}}$ is $= B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3} + F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4}$ + &c. Therefore $\overline{1+x}|^{\frac{m}{n}-1}$ × the fraction

$$\frac{1 + \frac{1+y}{1+x} + \overline{\frac{1+y}{1+x}}|^2 + \overline{\frac{1+y}{1+x}}|^3 + \text{\&c, continued fo } m \text{ terms,}}{1 + \overline{\frac{1+y}{1+x}}|^{\frac{m}{n}} + \overline{\frac{1+y}{1+x}}|^{\frac{2m}{n}} + \overline{\frac{1+y}{1+x}}|^{\frac{3m}{n}} + \text{\&c, continued to } n \text{ terms,}}$$

will

will be $= B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3}$ $+ F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4}$ + &c.

9. Now let y (which was ſuppoſed to be leſs than x,) be ſuppoſed to increaſe gradually till it becomes equal to x.

Then, ſince the laſt equation is always true, however little y may differ from x, it will alſo be true when y becomes abſolutely equal to x; as might eaſily be proved by a demonſtration *ex abſurdo*, if the matter were not too evident to make ſuch a demonſtration neceſſary. But, when y is equal to x, we ſhall have $x + y = 2x$, and $x^2 + xy + y^2 = 3x^2$, and $x^3 + x^2y + xy^2 + y^3 = 4x^3$, and $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 5x^4$; and, in like manner, in the following terms of the ſeries $B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3}$ $+ F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4}$ + &c we ſhall have $6x^5$, $7x^6$, $8x^7$, &c. And $\frac{1+y}{1+x}$ will, in this caſe, be $= \frac{1+x}{1+x} = 1$, and conſequently, $\overline{\frac{1+y}{1+x}}\Big|^2$ and $\overline{\frac{1+y}{1+x}}\Big|^3$ and $\overline{\frac{1+y}{1+x}}\Big|^4$, and all the following integral powers of $\frac{1+y}{1+x}$, and likewiſe $\overline{\frac{1+y}{1+x}}\Big|^{\frac{m}{n}}$ and $\overline{\frac{1+y}{1+x}}\Big|^{\frac{2m}{n}}$ and $\overline{\frac{1+y}{1+x}}\Big|^{\frac{3m}{n}}$, and all the following fractional powers of $\frac{1+y}{1+x}$, will become equal to 1. Therefore the laſt equation, to wit, $\overline{1+x}\big|^{\frac{m}{n}-1}$ $\times$ the fraction

$$\frac{1 + \frac{1+y}{1+x} + \overline{\frac{1+y}{1+x}}\Big|^2 + \overline{\frac{1+y}{1+x}}\Big|^3 + \text{\&c, continued to } m \text{ terms,}}{1 + \overline{\frac{1+y}{1+x}}\Big|^{\frac{m}{n}} + \overline{\frac{1+y}{1+x}}\Big|^{\frac{2m}{n}} + \overline{\frac{1+y}{1+x}}\Big|^{\frac{3m}{n}} + \text{\&c, continued to } n \text{ terms,}}$$

$= B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3}$ $+ F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4}$ + &c, will, in this caſe, be transformed into the equation $\overline{1+x}\big|^{\frac{m}{n}-1}$ $\times$ the fraction

$$\frac{1+1+1+1+ \text{\&c, continued to } m \text{ terms,}}{1+1+1+1+ \text{\&c, continued to } n \text{ terms,}} = B + C \times 2x + D \times 3x^2$$

$+ E \times 4x^3 + F \times 5x^4$ + &c, or $\overline{1+x}\big|^{\frac{m}{n}-1} \times \frac{m}{n} = B + C \times 2x + D \times$ $3x^2 + E \times 4x^3 + F \times 5x^4$ + &c, or $\frac{m}{n} \times \overline{1+x}\big|^{\frac{m}{n}-1} = B + C \times 2x +$ $D \times 3x^2 + E \times 4x^3 + F \times 5x^4$ + &c. Therefore, (multiplying both ſides of this laſt equation by $1 + x$,) we ſhall have $\frac{m}{n} \times \overline{1+x}\big|^{\frac{m}{n}} =$

$$\begin{aligned} &B + C \times 2x + D \times 3x^2 + E \times 4x^3 + F \times 5x^4 + \text{\&c} \\ &\quad + B \times x + C \times 2x^2 + D \times 3x^3 + E \times 4x^4 + \text{\&c} \\ &= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \text{\&c} \\ &\quad + Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \text{\&c.} \end{aligned}$$

 But

But $\overline{1+x}\big|^{\frac{m}{n}}$ is $=$ the ſeries $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c, and conſequently $\frac{m}{n} \times \overline{1+x}\big|^{\frac{m}{n}}$ is $= \frac{m}{n} \times$ the ſeries $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c $=$ the ſeries $\frac{m}{n} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \frac{m}{n}Ex^4 + \frac{m}{n}Fx^5 +$ &c. Therefore the ſeries $\frac{m}{n} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \frac{m}{n}Ex^4 + \frac{m}{n}Fx^5 +$ &c will be $=$

$$\begin{array}{l} B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ \quad + \;\; Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c. \end{array}$$

10. By the help of this equation $\frac{m}{n} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \frac{m}{n}Ex^4 + \frac{m}{n}Fx^5 +$ &c $=$

$$\begin{array}{l} B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ \quad + \;\; Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c \end{array}$$

we may determine the values of B, C, D, E, F, &c, or the co-efficients of the powers of x in the aſſumed ſeries $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c (which is $= \overline{1+x}\big|^{\frac{m}{n}}$,) by proceeding in the following manner:

11. This equation is always true, how ſmall ſoever we ſuppoſe the magnitude of x to be: and therefore, it will alſo be true when x is $= 0$. But, when x is $= 0$, all the terms on both ſides of the equation $\frac{m}{n} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \frac{m}{n}Ex^4 + \frac{m}{n}Fx^5 +$ &c $=$

$$\begin{array}{l} B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ \quad + \;\; Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c \end{array}$$

that involve any power of x, will become equal to 0 likewiſe, that is, all the terms, except $\frac{m}{n}$ on the left ſide of the equation, and B on the right ſide of it, will become equal to 0; and conſequently, $\frac{m}{n}$ and B will be the only remaining terms of the equation. Therefore B will be $= \frac{m}{n}$; that is, the co-efficient of x in Bx, the ſecond term of the aſſumed ſeries $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c, which is $= \overline{1+x}\big|^{\frac{m}{n}}$, will be $= \frac{m}{n}$. Q. E. I.

12. To find the value of C, the co-efficient of x^2 in the third term Cx^2 of the ſaid aſſumed ſeries, we muſt proceed as follows:

Since $\frac{m}{n} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \frac{m}{n}Ex^4 +$ &c, is $=$

$$\begin{array}{l} B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ \quad + \;\; Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c, \end{array}$$

and B has been ſhewn to be equal to $\frac{m}{n}$, it follows that, if we ſubtract $\frac{m}{n}$ and B on

on the opposite sides of the equation, the remaining quantities will be equal to each other; that is, the series $\frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \frac{m}{n}Ex^4 +$ &c will be equal to the series

$$\begin{array}{l} 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ +Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c. \end{array}$$

Therefore (dividing all the terms by x,) we shall have $\frac{m}{n}B + \frac{m}{n}Cx + \frac{m}{n}Dx^2 + \frac{m}{n}Ex^3 +$ &c $=$

$$\begin{array}{l} 2C + 3Dx + 4Ex^2 + 5Fx^3 + \&c \\ +B + 2Cx + 3Dx^2 + 4Ex^3 + \&c, \end{array}$$

and (supposing x to become $= 0$) $\frac{m}{n}B = 2C + B$. Therefore $2C$ will be $= \frac{m}{n}B - B = \overline{\frac{m}{n} - 1} \times B = \frac{m-n}{n} \times B$; and C will be $= \frac{m-n}{2n}B = \frac{m-n}{2n} \times \frac{m}{n}$, or $\frac{m}{n} \times \frac{m-n}{2n}$; that is, the co-efficient of x^2 in the third term Cx^2 of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c, (which is $= \overline{1+x}|^{\frac{m}{n}}$,) will be $\frac{m-n}{2n} \times B$, or $\frac{m}{n} \times \frac{m-n}{2n}$. Q. E. I.

13. To find the value of D, the co-efficient of x^3 in the fourth term of the said assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c, we must proceed as follows:

It has been seen in the last article that $\frac{m}{n}B + \frac{m}{n}Cx + \frac{m}{n}Dx^2 + \frac{m}{n}Ex^3 +$ &c is $= \left\{\begin{array}{l} 2C + 3Dx + 4Ex^2 + 5Fx^3 + \&c \\ +B + 2Cx + 3Dx^2 + 4Ex^3 + \&c \end{array}\right\}$, and likewise that $\frac{m}{n}B$ is $= 2C + B$. Therefore, if we subtract this last equation from the former, we shall have $\frac{m}{n}Cx + \frac{m}{n}Dx^2 + \frac{m}{n}Ex^3 +$ &c $=$ $\left.\begin{array}{l} 3Dx + 4Ex^2 + 5Fx^3 + \&c \\ + 2Cx + 3Dx^2 + 4Ex^3 + \&c \end{array}\right\}$, and consequently (dividing all the terms by x,) $\frac{m}{n}C + \frac{m}{n}Dx + \frac{m}{n}Ex^2 +$ &c $=$ $\left.\begin{array}{l} 3D + 4Ex + 5Fx^2 + \&c \\ + 2C + 3Dx + 4Ex^2 + \&c \end{array}\right\}$, and (supposing x to become $= 0$,) $\frac{m}{n}C = 3D + 2C$. Therefore $3D$ will be $= \frac{m}{n}C - 2C = \overline{\frac{m}{n} - 2} \times C = \frac{m-2n}{n} \times C$, and D will be $= \frac{m-2n}{3n} \times C$, or $\frac{m-2n}{3n} + \frac{m-n}{2n} \times \frac{m}{n}$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}$; that is, the co-efficient of x^3 in the fourth term of the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c (which is $= \overline{1+x}|^{\frac{m}{n}}$,) will be $= \frac{m-2n}{3n} \times C$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}$. Q. E. I.

14. To find the value of E, the co-efficient of x^4 in the fifth term of the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$ (which is $= \overline{1+x}|^{\frac{m}{n}}$,) we must proceed as follows:

It has been seen in the last article that $\frac{m}{n}C + \frac{m}{n}Dx + \frac{m}{n}Ex^2 + \&c$ is $= \left.\begin{array}{l} 3D + 4Ex + 5Fx^2 + \&c \\ + 2C + 3Dx + 4Ex^2 + \&c \end{array}\right\}$, and likewise that $\frac{m}{n}C$ is $= 3D + 2C$. Therefore, if we subtract this last equation from the former, we shall have $\frac{m}{n}Dx + \frac{m}{n}Ex^2 + \&c = \left\{\begin{array}{l} 4Ex + 5Fx^2 + \&c \\ + 3Dx + 4Ex^2 + \&c \end{array}\right\}$, and consequently (dividing all the terms by x,) $\frac{m}{n}D + \frac{m}{n}Ex + \&c =$ $\left.\begin{array}{l} 4E + 5Fx + \&c \\ + 3D + 4Ex + \&c \end{array}\right\}$, and (supposing x to become $= 0$,) $\frac{m}{n}D = 4E + 3D$. Therefore $4E$ will be $= \frac{m}{n}D - 3D = \overline{\frac{m}{n} - 3}| \times D = \frac{m-3n}{n} \times D$; and consequently E will be $= \frac{m-3n}{4n} \times D$; that is, the co-efficient of x^4 in the fifth term of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, (which is $= \overline{1+x}|^{\frac{m}{n}}$,) will be $\frac{m-3n}{4n} \times D$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}$.

Q. E. I.

15. Lastly, to find the value of F, the co-efficient of x^5 in the sixth term of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, we must proceed as follows:

It has been seen in the last article that $\frac{m}{n}D + \frac{m}{n}Ex + \&c$ is $=$ $\left.\begin{array}{l} 4E + 5Fx + \&c \\ + 3D + 4Ex + \&c \end{array}\right\}$, and likewise that $\frac{m}{n}D$ is $= 4E + 3D$. Therefore, if we subtract this last equation from the former, we shall have $\frac{m}{n}Ex + \frac{m}{n}Fx^2 + \frac{m}{n}Gx^3 + \&c = \left\{\begin{array}{l} 5Fx + 6Gx^2 + 7Hx^3 + \&c \\ + 4Ex + 5Fx^2 + 6Gx^3 + \&c \end{array}\right\}$, and consequently (dividing all the terms by x,) $\frac{m}{n}E + \frac{m}{n}Fx + \frac{m}{n}Gx^2 + \&c =$ $\left.\begin{array}{l} 5F + 6Gx + 7Hx^2 + \&c \\ + 4E + 5Fx + 6Gx^2 + \&c \end{array}\right\}$, and (supposing x to become $= 0$,) $\frac{m}{n}E = 5F + 4E$. Therefore $5F$ will be $= \frac{m}{n}E - 4E = \overline{\frac{m}{n} - 4}| \times E = \frac{m-4n}{n} \times E$, and consequently F will be $= \frac{m-4n}{5n} \times E$; that is, the co-efficient of x^5 in the sixth term of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$ (which is $= \overline{1+x}|^{\frac{m}{n}}$,) will be $\frac{m-4n}{5n} \times E$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}$.

Q. E. I.

16. And,

16. And, in like manner, we ſhall have for the determination of the values of the following co-efficients G, H, I, K, L, &c, the following equations, to wit,

$$\frac{m}{n}F = 6G + 5F,$$
$$\frac{m}{n}G = 7H + 6G,$$
$$\frac{m}{n}H = 8I + 7H,$$
$$\frac{m}{n}I = 9K + 8I,$$
$$\text{and } \frac{m}{n}K = 10L + 9K,$$
&c;

And conſequently

$$6G = \frac{m}{n}F - 5F = \overline{\frac{m}{n} - 5} \times F = \frac{m - 5n}{n} \times F,$$
$$\text{and } 7H = \frac{m}{n}G - 6G = \overline{\frac{m}{n} - 6} \times G = \frac{m - 6n}{n} \times G,$$
$$\text{and } 8I = \frac{m}{n}H - 7H = \overline{\frac{m}{n} - 7} \times H = \frac{m - 7n}{n} \times H,$$
$$\text{and } 9K = \frac{m}{n}I - 8I = \overline{\frac{m}{n} - 8} \times I = \frac{m - 8n}{n} \times I,$$
$$\text{and } 10L = \frac{m}{n}K - 9K = \overline{\frac{m}{n} - 9} \times K = \frac{m - 9n}{n} \times K,$$

&c; and conſequently

$$G = \frac{m - 5n}{6n} \times F,$$
$$\text{and } H = \frac{m - 6n}{7n} \times G,$$
$$\text{and } I = \frac{m - 7n}{8n} \times H,$$
$$\text{and } K = \frac{m - 8n}{9n} \times I,$$
$$\text{and } L = \frac{m - 9n}{10n} \times K.$$

17. And in the ſame manner we may determine as many more of theſe co-efficients as we think proper, the law of their generation, or continuation, from each other being manifeſt from the manner in which the final equation $\frac{m}{n} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \frac{m}{n}Ex^4 + \frac{m}{n}Ex^5 + \&c = \left.\begin{matrix} B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ + \quad Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c \end{matrix}\right\}$ in Art. 9, (by means of which they are inveſtigated,) was obtained.

18. We may therefore conclude that, if m and n be any whole numbers whatſoever, $\overline{1 + x}|^{\frac{m}{n}}$ will be equal to the ſeries $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m - n}{2n}x^2 + \frac{m}{n} \times \frac{m - n}{2n} \times \frac{m - 2n}{3n}x^3 + \frac{m}{n} \times \frac{m - n}{2n} \times \frac{m - 2n}{3n} \times \frac{m - 3n}{4n}x^4 + \frac{m}{n} \times \frac{m - n}{2n} \times$

$\frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5 +$ &c, or (putting $A = 1$, $B = \frac{m}{n} A$, $C = \frac{m-n}{2n} B$, $D = \frac{m-2n}{3n} C$, $E = \frac{m-3n}{4n} D$, $F = \frac{m-4n}{5n} E$, and G, H, I, K, L, &c, for the co-efficients of the following powers of x, to wit, x^6, x^7, x^8, x^9, x^{10}, &c, derived from the former co-efficients by the ſame law of generation or continuation,) to the ſeries $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4 + \frac{m-4n}{5n} E x^5 + \frac{m-5n}{6n} F x^6 + \frac{m-6n}{7n} G x^7 + \frac{m-7n}{8n} H x^8 + \frac{m-8n}{9n} I x^9 + \frac{m-9n}{10n} K x^{10} +$ &c. Q. E. D.

19. Coroll. 1. If m is leſs than n, (as, for example, if m is 6 and n is 17,) $m - n$ will be a negative quantity, and therefore the third term of the ſeries, to wit, $\frac{m}{n} \times \frac{m-n}{2n} x^2$, will be a negative quantity likewiſe, and will be equal to $-\frac{m}{n} \times \frac{n-m}{2n} x^2$, and conſequently muſt be ſubtracted from the two firſt terms 1 and $\frac{m}{n} x$, inſtead of being added to them. And, in like manner, $\frac{m-2n}{3n}$ and $\frac{m-3n}{4n}$ and $\frac{m-4n}{5n}$, and all the following factors in the ſeveral co-efficients of the powers of x in the following terms of the ſeries, will alſo be negative quantities, on account of the exceſs of $2n$, $3n$, $4n$, &c, above m: and conſequently, thoſe terms in which theſe negative factors occur an odd number of times, will be negative, and muſt be ſubtracted from 1 and $\frac{m}{n} x$, the two firſt terms of the ſeries; and thoſe terms in which the ſaid negative factors occur an even number of times, will be poſitive, and muſt be added to the ſaid two firſt terms; that is, the third, fifth, ſeventh, and other following odd terms of the ſeries will be negative, and muſt be ſubtracted from 1 and $\frac{m}{n} x$; and the fourth, ſixth, eighth, and other following even terms of the ſeries will be poſitive, and muſt be added to the ſaid two firſt terms. Therefore, when m is leſs than n, the foregoing ſeries, when its terms are ſet down correctly, with the proper ſigns prefixed to them, will be as follows, to wit, $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n} x^5 -$ &c, or (if A be put $= 1$, $B = \frac{m}{n}$, $C = \frac{n-m}{2n} B$, $D = \frac{2n-m}{3n} C$, $E = \frac{3n-m}{4n} D$, $F = \frac{4n-m}{5n} E$, and G, H, I, K, L, &c, to $\frac{5n-m}{6n} F$, $\frac{6n-m}{7n} G$, $\frac{7n-m}{8n} H$, $\frac{8n-m}{9n} I$, $\frac{9n-m}{10n} K$, &c, reſpectively, $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 - \frac{5n-m}{6n} F x^6 + \frac{6n-m}{7n} G x^7 - \frac{7n-m}{8n} H x^8 + \frac{8n-m}{9n} I x^9 - \frac{9n-m}{10n} K x^{10} +$ &c.

20. Coroll. 2. If m is an exact aliquot part of n, and is contained in it p times, or, if n is $= pm$, p being put for ſome whole number, the foregoing ſeries will exhibit the pth root of the binomial quantity $1 + x$.

For we ſhall then have $\frac{m}{n} = \frac{1}{p}$, and $\frac{n-m}{2n} = \frac{pm-m}{2pm} = \frac{p-1}{2p}$, and $\frac{2n-m}{3n} = \frac{2pm-m}{3pm} = \frac{2p+1}{3p}$, and $\frac{3n-m}{4n} = \frac{3pm-m}{4pm} = \frac{3p-1}{4p}$, and $\frac{4n-m}{5n} = \frac{4pm-m}{5pm} = \frac{4p-1}{5p}$, and, in like manner, $\frac{5n-m}{6n}$, and $\frac{6n-m}{7n}$, and $\frac{7n-m}{8n}$, and $\frac{8n-m}{9n}$, and $\frac{9n-m}{10n}$, &c, equal to $\frac{5p-1}{6p}$, $\frac{6p-1}{7p}$, $\frac{7p-1}{8p}$, $\frac{8p-1}{9p}$, and $\frac{9p-1}{10p}$, &c, reſpectively; and conſequently $\overline{1+x}|^{\frac{m}{n}}$ will be $= \overline{1+x}|^{\frac{1}{p}}$, and the ſeries $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n} x^5 -$ &c will be $= 1 + \frac{1}{p}x - \frac{1}{p} \times \frac{p-1}{2p} x^2 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} x^3 - \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} x^4 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} \times \frac{4p-1}{5p} x^5 -$ &c. Therefore $\overline{1+x}|^{\frac{1}{p}}$, or the pth root of the binomial quantity $1 + x$, will be $=$ the ſeries $1 + \frac{x}{p} - \frac{1}{p} \times \frac{p-1}{2p} x^2 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} x^3 - \frac{1}{p} \times \frac{p+1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} x^4 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} \times \frac{4p-1}{5p} x^5 -$ &c, or $1 + \frac{1}{p}Ax - \frac{p-1}{2p}Bx^2 + \frac{2p-1}{3p}Cx^3 - \frac{3p-1}{4p}Dx^4 + \frac{4p-1}{5p}Ex^5 -$ &c. Q. E. I.

An Example to the foregoing Corollary.

21. Let it be required to find by means of the laſt-mentioned ſeries the cube root of the binomial quantity $1 + x$, or the value of $\overline{1+x}|^{\frac{1}{3}}$.

Here p is $= 3$. We ſhall therefore have

$$B = \frac{1}{p} \times A = \frac{1}{3}A,$$

$$\text{and } C = \frac{p-1}{2p}B = \frac{3-1}{2 \times 3}B = \frac{2}{6}B,$$

$$\text{and } D = \frac{2p-1}{3p}C = \frac{6-1}{9}C = \frac{5}{9}C,$$

$$\text{and } E = \frac{3p-1}{4p}D = \frac{9-1}{12}D = \frac{8}{12}D,$$

$$\text{and } F = \frac{4p-1}{5p}E = \frac{12-1}{15}E = \frac{11}{15}E,$$

$$\text{and } G = \frac{5p-1}{6p}F = \frac{15-1}{18}F = \frac{14}{18}F,$$

$$\text{and } H = \frac{6p-1}{7p}G = \frac{18-1}{21}G = \frac{17}{21}G,$$

$$\text{and } I = \frac{7p-1}{8p}H = \frac{21-1}{24}H = \frac{20}{24}H,$$

$$\text{and } K = \frac{8p-1}{9p}\,I = \frac{24-1}{27}\,I = \frac{23}{27}\,I,$$
$$\text{and } L = \frac{9p-1}{10p}\,K = \frac{27-1}{30}\,K = \frac{26}{30}\,K,$$
$$\text{and } M = \frac{10p-1}{11p}\,L = \frac{30-1}{33}\,L = \frac{29}{33}\,L,$$
$$\text{and } N = \frac{11p-1}{12p}\,M = \frac{33-1}{36}\,M = \frac{32}{36}\,M,$$
$$\text{and } O = \frac{12p-1}{13p}\,N = \frac{36-1}{39}\,N = \frac{35}{39}\,N,$$
$$\text{and } P = \frac{13p-1}{14p}\,O = \frac{39-1}{42}\,O = \frac{38}{42}\,O,$$
$$\text{and } Q = \frac{14p-1}{15p}\,P = \frac{42-1}{45}\,P = \frac{41}{45}\,P,$$
$$\text{and } R = \frac{15p-1}{16p}\,Q = \frac{45-1}{48}\,Q = \frac{44}{48}\,Q,$$
$$\text{and } S = \frac{16p-1}{17p}\,R = \frac{48-1}{51}\,R = \frac{47}{51}\,R,$$
$$\text{and } T = \frac{17p-1}{18p}\,S = \frac{51-1}{54}\,S = \frac{50}{54}\,S.$$

Therefore the series $1 + \frac{1}{p}Ax - \overline{\frac{p-1}{2p}}Bx^2 + \frac{2p-1}{3p}Cx^3 - \overline{\frac{3p-1}{4p}}Dx^4 + \frac{4p-1}{5p}Ex^5 - \overline{\frac{5p-1}{6p}}Fx^6 + \&c$ will be $= 1 + \frac{1}{3}Ax - \frac{2}{6}Bx^2 + \frac{5}{9}Cx^3 - \frac{8}{12}Dx^4 + \frac{11}{15}Ex^5 - \frac{14}{18}Fx^6 + \frac{17}{21}Gx^7 - \frac{20}{24}Hx^8 + \frac{23}{27}Ix^9 - \frac{26}{30}Kx^{10} + \frac{29}{33}Lx^{11} - \frac{32}{36}Mx^{12} + \frac{35}{39}Nx^{13} - \frac{38}{42}Ox^{14} + \frac{41}{45}Px^{15} - \frac{44}{48}Qx^{16} + \frac{47}{51}Rx^{17} - \frac{50}{54}Sx^{18} + \&c = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{154x^6}{6561} + \frac{374x^7}{19683} - \frac{935x^8}{59049} + \frac{21505x^9}{1,594,323} - \frac{55,913x^{10}}{4,782,969} + \frac{147,407x^{11}}{14,348,907} - \frac{1,179,256x^{12}}{129,140,163} + \frac{41,273,960x^{13}}{5,036,466,357} - \frac{112,029,320x^{14}}{15,109,399,071} + \frac{918,640,424x^{15}}{135,984,591,639} - \frac{2,526,261,166x^{16}}{407,953,774,917} + \frac{118,734,274,802x^{17}}{20,805,642,520,767} - \frac{2,968,356,870,050x^{18}}{561,752,348,060,709} + \&c$, which therefore is $= \overline{1+x}|^{\frac{1}{3}}$, or the cube root of the binomial quantity $1 + x$. Q. E. I.

22. Coroll. 3. The foregoing proposition demonstrated by Mr. Landen, to wit, that, if m and n be any whole numbers whatsoever, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the $\frac{m}{n}$th power of the binomial quantity $1 + x$, is equal to the series $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 + \&c$, includes within it the case of the integral powers of $1 + x$ which was the subject of the preceeding discourse. For we need only suppose m to be greater than n, and to contain it p times,

p times, or to be equal to pn, and we ſhall have $\overline{1+x}|^{\frac{m}{n}} = \overline{1+x}|^{p}$. Therefore in this caſe the foregoing ſeries will be $= \overline{1+x}|^{p}$. But, when $\frac{m}{n}$ is $= p$, and m is $= pn$, we ſhall have

$$\frac{m-n}{2n} = \frac{pn-n}{2n} = \frac{p-1}{2},$$
$$\text{and } \frac{m-2n}{3n} = \frac{pn-2n}{3n} = \frac{p-2}{3},$$
$$\text{and } \frac{m-3n}{4n} = \frac{pn-3n}{4n} = \frac{p-3}{4},$$
$$\text{and } \frac{m-4n}{5n} = \frac{pn-4n}{5n} = \frac{p-4}{5},$$

and, in like manner, $\frac{m-5n}{6n}$, $\frac{m-6n}{7n}$, $\frac{m-7n}{8n}$, $\frac{m-8n}{9n}$, and $\frac{m-9n}{10n}$, &c $=$ to $\frac{pn-5n}{6n}$, $\frac{pn-6n}{7n}$, $\frac{pn-7n}{8n}$, $\frac{pn-8n}{9n}$, and $\frac{pn-9n}{10n}$, &c, reſpectively, or to $\frac{p-5}{6}$, $\frac{p-6}{7}$, $\frac{p-7}{8}$, $\frac{p-8}{9}$, and $\frac{p-9}{10}$, &c, reſpectively; and conſequently the ſeries $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 +$ &c, will be $=$ the ſeries $1 + px + p \times \frac{p-1}{2}x^2 + p \times \frac{p-1}{2} \times \frac{p-2}{3}x^3 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}x^4 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}x^5 +$ &c. Therefore $\overline{1+x}|^{p}$ will be $=$ the ſeries $1 + px + p \times \frac{p-1}{2}x^2 + p \times \frac{p-1}{2} \times \frac{p-2}{3}x^3 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}x^4 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}x^5 +$ &c; which is the binomial theorem in the caſe of integral powers, which was demonſtrated in the foregoing diſcourſe.

A
DISCOURSE
CONCERNING THE
BINOMIAL THEOREM,
In the Case of fractional Powers, or Powers of which the Indexes are Fractions.

By *FRANCIS MASERES*, Esq. F.R.S.
CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

ARTICLE I.

THE Proposition known by the name of the *Binomial Theorem*, to wit, "that $\overline{1+x}|^m$, or the mth power of the binomial quantity $1+x$, is equal to the series $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 +$ &c," is found to be true, not only in the case of the integral powers of $1+x$, or when the index m of the power to which the said quantity is to be raised, is a whole number, but also in the case of its roots, or when the index m is a fraction, of which 1 is the numerator, and any whole number whatsoever is the denominator, such as the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ &c; and likewise in the case of the powers of any roots of the said binomial quantity $1+x$, or the roots of any powers of it, (which comes to the same thing,) or when m is equal to a fraction of which any whole number whatsoever is the numerator, and any other whole number whatsoever is the denominator, such as the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, &c, or $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{6}{5}$, $\frac{7}{6}$, &c. The truth of this theorem in the former case, or when the index m is a whole number, has been already shewn in the last tract but one in this collection, from page 153 to page 169;

169; and therefore will here be taken for granted, and made ufe of as a ftep in the reafonings by which we fhall endeavour in the prefent Difcourfe to eftablifh the truth of it in the other cafes. But, before we attempt to demonftrate the faid theorem in thofe other cafes, it will be proper to fet it forth in the expreffions and forms in which it will appear in thofe cafes, upon a fuppofition that the faid theorem extends to the faid cafes, or to fet down thofe expreffions of the firft theorem above-mentioned, which will arife from a fuppofition that it continues to be true when the index of the power to which the binomial quantity $1 + x$ is to be raifed, is a fraction, as well as when it is a whole number. And, in doing this, it will be convenient to divide the fubject into three cafes, according as the index of the power, to which $1 + x$ is to be raifed, fhall be a fraction of which 1 is the numerator, and any whole number whatfoever is the denominator, or a fraction of which any whole number whatfoever is the numerator, and any *greater* whole number is the denominator, or a fraction of which any whole number whatfoever is the numerator, and any *leffer* whole number is the denominator. The expreffions that will be derived from the original feries $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 +$ &c, according to thefe different fuppofitions, will be as follows:

Of the Binomial Theorem in the Cafe of Roots, as derived from the fame Theorem in the Cafe of Integral Powers.

2. When the index m of the power to which the binomial quantity $1 + x$ is to be raifed, is equal to a fraction of which 1 is the numerator and a whole number is the denominator, let n be put for the faid denominator, or let m be fuppofed to be $= \frac{1}{n}$. Then, by fubftituting $\frac{1}{n}$ inftead of m in the terms of the foregoing theorem, $\overline{1+x}|^m = 1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 +$ &c, we fhall (if the faid theorem extends to the cafe of roots,) have $\overline{1+x}|^{\frac{1}{n}} =$ to the feries

$$1 + \frac{\frac{1}{n}}{1}x + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2}x^2 + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3}x^3$$
$$+ \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3} \times \frac{\frac{1}{n}-3}{4}x^4$$
$$+ \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3} \times \frac{\frac{1}{n}-3}{4} \times \frac{\frac{1}{n}-4}{5}x^5 + \&c$$

$$= 1 + \frac{1}{n}x + \frac{1}{n} \times \frac{\frac{1-n}{n}}{2}x^2 + \frac{1}{n} \times \frac{\frac{1-n}{n}}{2} \times \frac{\frac{1-2n}{n}}{3}x^3$$

$$+ \frac{1}{n} \times \frac{\frac{1-n}{n}}{2} \times \frac{\frac{1-2n}{n}}{3} \times \frac{\frac{1-3n}{n}}{4} \times x^4$$

$$+ \frac{1}{n} \times \frac{\frac{1-n}{n}}{2} \times \frac{\frac{1-2n}{n}}{3} \times \frac{\frac{1-3n}{n}}{4} \times \frac{\frac{1-4n}{n}}{5} \times x^5 + \&c$$

$$= 1 + \frac{1}{n}x + \frac{1}{n} \times \frac{1-n}{2n}x^2 + \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n}x^3$$

$$+ \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n} \times \frac{1-3n}{4n}x^4$$

$$+ \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n} \times \frac{1-3n}{4n} \times \frac{1-4n}{5n}x^5 + \&c$$

$$= 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3$$

$$- \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4$$

$$+ \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5 - \&c;$$

that is, $\overline{1+x}|^{\frac{1}{n}}$, or the nth root of the binomial quantity $1 + x$, will (if the binomial theorem extends to the case of roots,) be equal to the series $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5 + \&c$, or (if, for the sake of brevity, we put $A = 1$, and $B = \frac{1}{n}A$, and $C = \frac{n-1}{2n}B$, and $D = \frac{2n-1}{3n}C$, and $E = \frac{3n-1}{4n}D$, and $F = \frac{4n-1}{5n}E$, and G, H, I, K, L, &c $= \frac{5n-1}{6n}F$, $\frac{6n-1}{7n}G$, $\frac{7n-1}{8n}H$, $\frac{8n-1}{9n}I$, $\frac{9n-1}{10n}K$, &c, respectively,) to the series $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \frac{5n-1}{6n}Fx^6 + \frac{6n-1}{7n}Gx^7 - \frac{7n-1}{8n}Hx^8 + \frac{8n-1}{9n}Ix^9 - \frac{9n-1}{10n}Kx^{10} + \&c$; in which the second term $\frac{1}{n}Ax$, or $\frac{1}{n}x$, is marked with the sign $+$, or is to be added to the first term 1, but all the following terms are marked alternately with the signs $-$ and $+$, or are to be subtracted from, and added to, the said first term 1, or the two first terms $1 + \frac{1}{n}x$, alternately.

This is, as I apprehend, the clearest and most convenient way of expressing the binomial theorem in the case of roots, or of setting down the series which is equal to $\overline{1+x}|^{\frac{1}{n}}$, or to $\sqrt[n]{1+x}$, or the nth root of the binomial quantity $1 + x$.

Of the Binomial Theorem in the Case of the Powers of Roots, when the Index of the Power of the Root is less than the Index of the Root.

3. When the index m stands for a fraction, of which a whole number is the numerator, and another whole number, greater than the former, is the denominator, we must substitute $\frac{m}{n}$ instead of m in the theorem stated in Art. 1, to wit, the theorem, $\overline{1+x}|^m = 1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} x^5 + \&c$; and then (if the binomial theorem extends to this case,) we shall have $\overline{1+x}|^{\frac{m}{n}}$ = the series $1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5} x^5 + \&c = 1 + \frac{m}{n}x + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2}x^2 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3}x^3 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3} \times \frac{\frac{m-3n}{n}}{4}x^4 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3} \times \frac{\frac{m-3n}{n}}{4} \times \frac{\frac{m-4n}{n}}{5}x^5 + \&c = 1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 + \&c = 1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5 + \&c$, or (if, for the sake of brevity, we put $A = 1$, and $B = \frac{m}{n}$, or $\frac{m}{n}A$, and $C = \frac{n-m}{2n}B$, and $D = \frac{2n-m}{3n}C$, and $E = \frac{3n-m}{4n}D$, and $F = \frac{4n-m}{5n}E$, and $G, H, I, K, L, \&c = \frac{5n-m}{6n}F, \frac{6n-m}{7n}G, \frac{7n-m}{8n}H, \frac{8n-m}{9n}I, \frac{9n-m}{10n}K$, &c, respectively, to the series $1 + \frac{m}{n}Ax - \overline{\frac{n-m}{2n}}Bx^2 + \frac{2n-m}{3n}Cx^3 - \overline{\frac{3n-m}{4n}}Dx^4 + \frac{4n-m}{5n}Ex^5 - \overline{\frac{5n-m}{6n}}Fx^6 + \frac{6n-m}{7n}Gx^7 - \overline{\frac{7n-m}{8n}}Hx^8 + \frac{8n-m}{9n}Ix^9 - \overline{\frac{9n-m}{10n}}Kx^{10} + \&c$; in which the second term $\frac{m}{n}Ax$, or $\frac{m}{n}x$, is always to be marked with the sign $+$, or to be added to the first term 1, but the third, and fourth, and fifth, and other following terms, are marked alternately with the sign $-$ and the sign $+$, or are to be subtracted from, and added to, the said first term 1, or the two first terms $1 + \frac{m}{n}x$, alternately.

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This is, as I apprehend, the cleareſt and moſt convenient way of expreſſing the binomial theorem in this caſe of the power of a root of the binomial quantity $1 + x$, when m, or the index of the power to which the root of $1 + x$ is to be raiſed is leſs than n, or the index of the root.

Of the Binomial Theorem in the Caſe of the Powers of Roots, when the Index of the Power of the Root is greater than the Index of the Root.

4. When the index m ſtands for a fraction, of which a whole number is the numerator, and another whole number, leſs than the former, is the denominator, we muſt, as in the laſt caſe, ſubſtitute $\frac{m}{n}$ inſtead of m in the theorem ſtated Art. 1, to wit, the theorem, $\overline{1+x}|^{m} = 1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c$; and then (if the binomial theorem extends to this caſe,) we ſhall have

$$\overline{1+x}|^{\frac{m}{n}} = 1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c = 1 + \frac{m}{n}x + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2}x^2 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3}x^3 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3} \times \frac{\frac{m-3n}{n}}{4}x^4 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3} \times \frac{\frac{m-3n}{n}}{4} \times \frac{\frac{m-4n}{n}}{5}x^5 + \&c = 1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 + \&c.$$

5. Now, ſince m is in this caſe greater than n, it will be poſſible to ſubtract n from m, and conſequently the term $\frac{m}{n} \times \frac{m-n}{2n}x^2$ will be a poſitive, or affirmative, term, and muſt be added to the firſt term 1; and therefore the three firſt terms of this ſeries, to wit, 1, $\frac{m}{n}x$, and $\frac{m}{n} \times \frac{m-n}{2n}x^2$, will in this caſe be added together, or connected by the ſign +. And, if m is not only greater than n, but likewiſe than $2n$, the fourth term $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3$, will alſo be a poſitive or affirmative quantity, and muſt be marked with the ſign +, and added to the three former terms. And, in like manner, if m is greater than $3n$, but leſs than $4n$, the five firſt terms of the ſeries will be poſitive, or added to each other; and, if m is greater than $4n$, but leſs than $5n$, the firſt ſix terms of the ſeries will be poſitive, or added to each other; and, if m is greater than $5n$, but leſs than $6n$, or greater than $6n$, but leſs than $7n$, or greater than $7n$, but leſs than

than $8n$, the first seven, or eight, or nine, terms of the series will, respectively, be positive, or added to each other. But, however great we may suppose m to be in comparison to n, it is evident that there will always be some term or other of the series in which the multiple of n that is to be subtracted from m, will become equal to, or greater than m. Let pn denote the greatest multiple of n, that is less than m. Then it is evident that the next multiple of n, namely, $\overline{p+1} \times n$, or $pn + n$, must either be equal to, or greater than m. If it is equal to m, $m - pn - n$ will be $= m - m = 0$, and consequently that term of the series in the numerator of which the quantity $m - pn - n$ occurs, will be $= 0$ likewise; and so will all the subsequent terms of the series, because they are derived from the said term by multiplication. Therefore, the series will in this case consist of a finite number of terms, of which the term in which the quantity $m - pn$ first occurs will be the last. But, if $pn + n$ is greater than m, it cannot be subtracted from m, but m must be subtracted from it, and the term in which $m - pn - n$ occurs must, (instead of being added to the first term 1, as the foregoing terms were,) be subtracted from it; after which all the following terms of the series will be alternately added to, and subtracted from, the said first term 1, just as in the last case, (which is set down in Art. 3) all the terms after the second term $\frac{m}{n}x$ are so alternately subtracted and added. Therefore (if, for the sake of brevity, we put $A = 1$, and $B = \frac{m}{n}$, and $C = \frac{m-n}{2n}B$, and $D = \frac{m-2n}{3n}C$, or $\frac{2n-m}{3n}$, according as m is greater, or less, than $2n$; and $E = \frac{m-3n}{4n}D$, or $\frac{3n-m}{4n}D$, according as m is greater, or less, than $3n$; and $F = \frac{m-4n}{5n}E$, or $\frac{4n-m}{5n}E$, according as m is greater, or less, than $4n$; and, in like manner, G, H, I, K, L, &c =, respectively, to $\frac{m-5n}{6n}F$, $\frac{m-6n}{7n}G$, $\frac{m-7n}{8n}H$, $\frac{m-8n}{9n}I$, $\frac{m-9n}{10n}K$, &c, or to $\frac{5n-m}{6n}F$, $\frac{6n-m}{7n}G$, $\frac{7n-m}{8n}H$, $\frac{8n-m}{9n}I$, $\frac{9n-m}{10n}K$, &c,) we shall have $\overline{1+x}|^{\frac{m}{n}} = 1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 +$ &c, till we come to the term in which the generating fraction is $\frac{m-pn}{\overline{p+1}|\times n}$, or $\frac{m-pn}{pn+n}$, after which the terms will be marked with the signs $-$ and $+$ alternately, or will be alternately subtracted from, and added to, the first term 1, as in the former case set forth in Art. 3; and the numerators of the generating fractions in the following terms will be the excesses of the successive multiples of n, to wit, $pn + n$, $pn + 2n$, $pn + 3n$, $pn + 4n$, $pn + 5n$, &c, above m, as they before were the excesses of m above the preceeding lower multiples of n, to wit, $2n$, $3n$, $4n$, $5n$, $6n$, &c.

6. This law of the continuation of the co-efficients C, D, E, F, G, H, I, K, L, &c, in the terms of the series that is equal to $\overline{1+x}|^{\frac{m}{n}}$, when m is greater than n, will be more clearly understood by setting down a few examples of this series according to different relative magnitudes of m and n. This may be done as follows:

If m is greater than n, but leſs than $2n$, we ſhall have $\overline{1+x}|^{\frac{m}{n}}$ = the ſeries

$$1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}x^4 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5 + \&c = 1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \frac{2n-m}{3n}Cx^3 + \frac{3n-m}{4n}Dx^4 - \frac{4n-m}{5n}Ex^5 + \frac{5n-m}{6n}Fx^6 - \frac{6n-m}{7n}Gx^7 + \frac{7n-m}{8n}Hx^8 - \frac{8n-m}{9n}Ix^9 + \frac{9n-m}{10n}Kx^{10} - \&c.$$

If m is greater than $2n$, but leſs than $3n$, we ſhall have $\overline{1+x}|^{\frac{m}{n}}$ = the ſeries

$$1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{3n-m}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5 - \&c = 1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 - \frac{3n-m}{4n}Dx^4 + \frac{4n-m}{5n}Ex^5 - \frac{5n-m}{6n}Fx^6 + \frac{6n-m}{7n}Gx^7 - \frac{7n-m}{8n}Hx^8 + \frac{8n-m}{9n}Ix^9 - \frac{9n-m}{10n}Kx^{10} + \&c.$$

If m is greater than $3n$, but leſs than $4n$, we ſhall have $\overline{1+x}|^{\frac{m}{n}}$ = the ſeries

$$1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{4n-m}{5n}x^5 + \&c = 1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 - \frac{4n-m}{5n}Ex^5 + \frac{5n-m}{6n}Fx^6 - \frac{6n-m}{7n}Gx^7 + \frac{7n-m}{8n}Hx^8 - \frac{8n-m}{9n}Ix^9 + \frac{9n-m}{10n}Kx^{10} - \&c.$$

And, if m is greater than $4n$, but leſs than $5n$, we ſhall have $\overline{1+x}|^{\frac{m}{n}}$ =

$$1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} \times \frac{5n-m}{6n}x^6 + \&c = 1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 - \frac{5n-m}{6n}Fx^6 + \frac{6n-m}{7n}Gx^7 - \frac{7n-m}{8n}Hx^8 + \frac{8n-m}{9n}Ix^9 - \frac{9n-m}{10n}Kx^{10} + \&c.$$

Theſe examples will, I preſume, be ſufficient to illuſtrate the courſe of the co-efficients C, D, E, F, G, H, I, K, L, &c, in all the different relative magnitudes of m and n that can be ſuppoſed.

7. Before we enter upon the inveſtigations of theſe important theorems, it may not be amiſs to apply the three foregoing ſerieſes, (which have been derived from the original ſeries ſet down in Art. 1, by ſubſtituting $\frac{1}{n}$ and $\frac{m}{n}$ in its

its terms inſtead of m,) to the determination of the values of $\overline{1+x}\big|^{\frac{1}{n}}$ and $\overline{1+x}\big|^{\frac{m}{n}}$ in a few particular inſtances: after which we ſhall prove, by raiſing the particular ſerieſes thereby obtained to the proper powers, that in thoſe inſtances, at leaſt, the ſeveral theorems are true; which will be a good introduction to the general inveſtigations of the ſaid theorems in all other values of m and n, which will be preſented to the reader's conſideration in the ſubſequent part of this Diſcourſe.

Examples of the Extraction of ſome particular Roots of the Binomial Quantity $1 + x$ by means of the Series given above in Art. 2.

8. In the firſt place we will extract the ſquare root of the binomial quantity $1 + x$ by means of the ſeries given in Art. 2, to wit, the ſeries $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c.

Now, in this caſe, $\overline{1+x}\big|^{\frac{1}{n}}$ is $= \overline{1+x}\big|^{\frac{1}{2}}$, or n is $= 2$. Therefore $2n$ is $(= 2 \times 2) = 4$, and $3n$ is $(= 3 \times 2) = 6$, and $4n$ is $(= 4 \times 2) = 8$, and $5n$ is $(= 5 \times 2) = 10$, and conſequently $n - 1$ is $(= 2 - 1) = 1$, and $2n - 1$ is $(= 4 - 1) = 3$, and $3n - 1$ is $(= 6 - 1) = 5$, and $4n - 1$ is $(= 8 - 1) = 7$. We ſhall therefore have $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 - \&c = 1 + \frac{1}{2}Ax - \frac{1}{4}Bx^2 + \frac{3}{6}Cx^3 - \frac{5}{8}Dx^4 + \frac{7}{10}Ex^5 - \&c = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c.$ Therefore, if the binomial theorem is true in the caſe of roots, $\overline{1+x}\big|^{\frac{1}{2}}$, or the ſquare-root of the binomial quantity $1 + x$, will be equal to the ſeries $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}$ — &c.

9. Now that this ſeries is really equal to the ſquare-root of $1 + x$, will appear by multiplying it into itſelf. For we ſhall find that the product of the ſaid multiplication will be $1 + x$. This may be done in the manner following:

$$
\begin{array}{ccccccccccccc}
1 & + & \frac{x}{2} & - & \frac{x^2}{8} & + & \frac{x^3}{16} & - & \frac{5x^4}{128} & + & \frac{7x^5}{256} & - & \&c \\
1 & + & \frac{x}{2} & - & \frac{x^2}{8} & + & \frac{x^3}{16} & - & \frac{5x^4}{128} & + & \frac{7x^5}{256} & - & \&c \\
\hline
1 & + & \frac{x}{2} & - & \frac{x^2}{8} & + & \frac{x^3}{16} & - & \frac{5x^4}{128} & + & \frac{7x^5}{256} & - & \&c \\
 & + & \frac{x}{2} & + & \frac{x^2}{4} & - & \frac{x^3}{16} & + & \frac{x^4}{32} & - & \frac{5x^5}{256} & + & \&c \\
 & & & - & \frac{x^2}{8} & - & \frac{x^3}{16} & + & \frac{x^4}{64} & - & \frac{x^5}{128} & + & \&c \\
 & & & & & + & \frac{x^3}{16} & + & \frac{x^4}{32} & - & \frac{x^5}{128} & + & \&c \\
 & & & & & & & - & \frac{5x^4}{128} & - & \frac{5x^5}{256} & + & \&c \\
 & & & & & & & & & + & \frac{7x^5}{256} & + & \&c \\
\hline
1 & + & x & & * & & * & & * & & * & & \&c.
\end{array}
$$

It appears therefore that the series $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} -$ &c is really and truly, as far as relates to the said six first terms of it, the square-root of the binomial quantity $1 + x$, and consequently that the binomial theorem is true in the case of square-roots.

10. In the next place we will investigate the value of $\overline{1+x}|^{\frac{1}{3}}$, or the cube-root of the binomial quantity $1 + x$, by means of the same series $1 + \frac{1}{n} Ax$ $- \overline{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \overline{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 -$ &c.

Now in this case n is $= 3$, and consequently we shall have $2n = 6$, $3n = 9$, $4n = 12$, and $5n = 15$, and $n - 1 = 2$, $2n - 1 = 5$, $3n - 1 = 8$, and $4n - 1 = 11$. And therefore the series $1 + \frac{1}{n} Ax - \overline{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \overline{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 -$ &c will, in this case, be $= 1 + \frac{1}{3} Ax - \frac{2}{6} Bx^2 + \frac{5}{9} Cx^3 - \frac{8}{12} Dx^4 + \frac{11}{15} Ex^5 -$ &c $= 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729}$ $-$ &c.

11. Now that this series is really equal to the cube-root + of $1x$, will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be equal to $1 + x$. These multiplications will be as follows:

$$
\begin{array}{ccccccccccccc}
1 & + \frac{x}{3} & - & \frac{x^2}{9} & + & \frac{5x^3}{81} & - & \frac{10x^4}{243} & + & \frac{22x^5}{729} & - & \text{\&c} \\
1 & + \frac{x}{3} & - & \frac{x^2}{9} & + & \frac{5x^3}{81} & - & \frac{10x^4}{243} & + & \frac{22x^5}{729} & - & \text{\&c} \\
\hline
1 & + \frac{x}{3} & - & \frac{x^2}{9} & + & \frac{5x^3}{81} & - & \frac{10x^4}{243} & + & \frac{22x^5}{729} & - & \text{\&c} \\
 & + \frac{x}{3} & + & \frac{x^2}{9} & - & \frac{x^3}{27} & + & \frac{5x^4}{243} & - & \frac{10x^5}{729} & + & \text{\&c} \\
 & & - & \frac{x^2}{9} & - & \frac{x^3}{27} & + & \frac{x^4}{81} & - & \frac{5x^5}{729} & + & \text{\&c} \\
 & & & & + & \frac{5x^3}{81} & + & \frac{5x^4}{243} & - & \frac{5x^5}{729} & + & \text{\&c} \\
 & & & & & & - & \frac{10x^4}{243} & - & \frac{10x^5}{729} & - & \text{\&c} \\
 & & & & & & & & + & \frac{22x^5}{729} & - & \text{\&c} \\
\hline
1 & + \frac{2x}{3} & - & \frac{x^2}{9} & + & \frac{4x^3}{81} & - & \frac{7x^4}{243} & + & \frac{14x^5}{729} & & \text{\&c.}
\end{array}
$$

$$
\begin{array}{rcccccccccccl}
1 & + & \frac{2x}{3} & - & \frac{x^2}{9} & + & \frac{4x^3}{81} & - & \frac{7x^4}{243} & + & \frac{14x^5}{729} & - & \&c \\
1 & + & \frac{x}{3} & - & \frac{x^2}{9} & + & \frac{5x^3}{81} & - & \frac{10x^4}{243} & + & \frac{22x^5}{729} & - & \&c \\
\hline
1 & + & \frac{2x}{3} & - & \frac{x^2}{9} & + & \frac{4x^3}{81} & - & \frac{7x^4}{243} & + & \frac{14x^5}{729} & - & \&c \\
 & + & \frac{x}{3} & + & \frac{2x^2}{9} & - & \frac{x^3}{27} & + & \frac{4x^4}{243} & - & \frac{7x^5}{729} & + & \&c \\
 & & & - & \frac{x^2}{9} & - & \frac{2x^3}{27} & + & \frac{x^4}{81} & - & \frac{4x^5}{729} & + & \&c \\
 & & & & & + & \frac{5x^3}{81} & + & \frac{10x^4}{243} & - & \frac{5x^5}{729} & + & \&c \\
 & & & & & & & - & \frac{10x^4}{243} & - & \frac{20x^5}{729} & + & \&c \\
 & & & & & & & & & + & \frac{22x^5}{729} & + & \&c \\
\hline
1 & + & x & & * & & * & & * & & * & & \&c.
\end{array}
$$

It appears therefore that the series $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} -$ &c is really and truly, as far as relates to the first six terms of it, the cube-root of the binomial quantity $1 + x$, and consequently that the binomial theorem is true in the case of cube-roots as well as in that of square-roots.

Examples of the Extraction of the Roots of some particular Powers of the Binomial Quantity $1 + x$ by means of the Series given above in Art. 3; in which the Numerator of the Fractional Index $\frac{m}{n}$ is supposed to be less than its Denominator.

12. In the next place we will investigate the value of $\overline{1 + x}|^{\frac{2}{3}}$, or of the cube-root of the square of the binomial quantity $1 + x$, by means of the series $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 -$ &c given in Art. 3 as the value of $\overline{1 + x}|^{\frac{m}{n}}$.

Now in this case m is $= 2$, and n is $= 3$. And therefore we shall have $2n = 6$, $3n = 9$, $4n = 12$, and $5n = 15$,

and $n - m \; (= \; 3 - 2) = 1$,
$2n - m \; (= \; 6 - 2) = 4$,
$3n - m \; (= \; 9 - 2) = 7$,
and $4n - m \; (= 12 - 2) = 10$,

and consequently $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 -$ &c $= 1 + \frac{2}{3} A x - \frac{1}{6} B x^2 + \frac{4}{9} C x^3 - \frac{7}{12} D x^4 + \frac{10}{15} E x^5 -$ &c $= 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} -$ &c. Therefore, if the binomial theorem is true in the case of the powers of roots, or the roots of powers,

$\overline{1+x}|^{\frac{2}{3}}$, or the cube-root of the ſquare of the binomial quantity $1+x$, will be equal to the ſeries $1+\frac{2x}{3}-\frac{x^2}{9}+\frac{4x^3}{81}-\frac{7x^4}{243}+\frac{14x^5}{729}-$ &c.

13. Now that this ſeries is really equal to $\overline{1+x}|^{\frac{2}{3}}$, or to the cube-root of the ſquare of the binomial quantity $1+x$, or to the cube-root of the trinomial quantity $1+2x+xx$, will appear by multiplying the ſaid ſeries twice into itſelf. For we ſhall find that the product of theſe multiplications will be the ſaid trinomial quantity. Theſe multiplications will be as follows:

$$\begin{array}{rrrrrrl}
1 & +\frac{2x}{3} & -\frac{x^2}{9} & +\frac{4x^3}{81} & -\frac{7x^4}{243} & +\frac{14x^5}{729} & -\&c \\
1 & +\frac{2x}{3} & -\frac{x^2}{9} & +\frac{4x^3}{81} & -\frac{7x^4}{243} & +\frac{14x^5}{729} & -\&c \\
\hline
1 & +\frac{2x}{3} & -\frac{x^2}{9} & +\frac{4x^3}{81} & -\frac{7x^4}{243} & +\frac{14x^5}{729} & -\&c \\
 & +\frac{2x}{3} & +\frac{4x^2}{9} & -\frac{2x^3}{27} & +\frac{8x^4}{243} & -\frac{14x^5}{729} & +\&c \\
 & & -\frac{x^2}{9} & -\frac{2x^3}{27} & +\frac{x^4}{81} & -\frac{4x^5}{729} & +\&c \\
 & & & +\frac{4x^3}{81} & +\frac{8x^4}{243} & -\frac{4x^5}{729} & +\&c \\
 & & & & -\frac{7x^4}{243} & -\frac{14x^5}{729} & +\&c \\
 & & & & & +\frac{14x^5}{729} & +\&c \\
\hline
1 & +\frac{4x}{3} & +\frac{2x^2}{9} & -\frac{4x^3}{81} & +\frac{5x^4}{243} & -\frac{8x^5}{729} & \&c.
\end{array}$$

$$\begin{array}{rrrrrrl}
1 & +\frac{4x}{3} & +\frac{2x^2}{9} & -\frac{4x^3}{81} & +\frac{5x^4}{243} & -\frac{8x^5}{729} & +\&c \\
1 & +\frac{2x}{3} & -\frac{x^2}{9} & +\frac{4x^3}{81} & -\frac{7x^4}{243} & +\frac{14x^5}{729} & -\&c \\
\hline
1 & +\frac{4x}{3} & +\frac{2x^2}{9} & -\frac{4x^3}{81} & +\frac{5x^4}{243} & -\frac{8x^5}{729} & +\&c \\
 & +\frac{2x}{3} & +\frac{8x^2}{9} & +\frac{4x^3}{27} & -\frac{8x^4}{243} & +\frac{10x^5}{729} & -\&c \\
 & & -\frac{x^2}{9} & -\frac{4x^3}{27} & -\frac{2x^4}{81} & +\frac{4x^5}{729} & -\&c \\
 & & & +\frac{4x^3}{81} & +\frac{16x^4}{243} & +\frac{8x^5}{729} & -\&c \\
 & & & & -\frac{7x^4}{243} & -\frac{28x^5}{729} & -\&c \\
 & & & & & +\frac{14x^5}{729} & +\&c \\
\hline
1 & +2x & +xx & * & * & * & \&c.
\end{array}$$

It appears therefore that the ſeries $1+\frac{2x}{3}-\frac{x^2}{9}+\frac{4x^3}{81}-\frac{7x^4}{243}+\frac{14x^5}{729}-$ &c is really and truly the cube-root of the trinomial quantity $1+2x+xx$, or of the ſquare of the binomia quantity $1+x$, and conſequently that the binomial theorem

theorem is true in the case of the cube-root of the square of a binomial quantity, or when m, the numerator of the fraction $\frac{m}{n}$, which is the index of the power of $1 + x$, is $= 2$, and n, the denominator of the said fraction, is $= 3$.

14. As another example of the investigation of the value of $\overline{1+x}|^{\frac{m}{n}}$ by means of the foregoing series $1 + \frac{m}{n}Ax - \overline{\frac{n-m}{2n}}\,Bx^2 + \frac{2n-m}{3n}Cx^3 - \overline{\frac{3n-m}{4n}}\,Dx^4 + \frac{4n-m}{5n}Ex^5 - \&c$, we will suppose m to be $= 3$, and n to be $= 5$, or $\overline{1+x}|^{\frac{m}{n}}$ to be equal to $\overline{1+x}|^{\frac{3}{5}}$, or to the fifth root of the cube of the binomial quantity $1 + x$, or to the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$.

In this case we shall have $2n = 10$, $3n = 15$, $4n = 20$, and $5n = 25$, and consequently

$$\begin{aligned} n - m\ (&= 5 - 3) = 2, \\ \text{and } 2n - m\ (&= 10 - 3) = 7, \\ \text{and } 3n - m\ (&= 15 - 3) = 12, \\ \text{and } 4n - m\ (&= 20 - 3) = 17; \end{aligned}$$

and $1 + \frac{m}{n}Ax - \overline{\frac{n-m}{2n}}\,Bx^2 + \frac{2n-m}{3n}Cx^3 - \overline{\frac{3n-m}{4n}}\,Dx^4 + \frac{4n-m}{5n}Ex^5 - \&c$ $= 1 + \frac{3}{5}Ax - \frac{2}{10}Bx^2 + \frac{7}{15}Cx^3 - \frac{12}{20}Dx^4 + \frac{17}{25}Ex^5 - \&c = 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c$. Therefore, if the binomial theorem is true in the case of the powers of roots, or the roots of powers, the quantity $\overline{1+x}|^{\frac{3}{5}}$, or the fifth root of the cube of $1 + x$, or the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will be equal to the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c$.

15. Now that this series is really equal to $\overline{1+x}|^{\frac{3}{5}}$, or to the fifth root of the cube of the binomial quantity $1 + x$, or to the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will appear by raising the said series to the fifth power by multiplying it first into itself, whereby we shall obtain its square, and afterwards multiplying the said square into itself, whereby we shall obtain its fourth power, and, lastly, multiplying the said fourth power of it into the series itself, whereby we shall obtain its fifth power. For we shall find that the product of these three multiplications will be the said quadrinomial quantity $1 + 3x + 3x^2 + x^3$. These multiplications will be as follows:

$$\begin{array}{l} 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c \\ 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c \\ \hline \end{array}$$

$$\begin{array}{rrrrrrl}
1 & +\frac{3x}{5} & -\frac{3x^2}{25} & +\frac{7x^3}{125} & -\frac{21x^4}{625} & +\frac{357x^5}{15625} & -\&c \\
 & +\frac{3x}{5} & +\frac{9x^2}{25} & -\frac{9x^3}{125} & +\frac{21x^4}{625} & -\frac{63x^5}{3125} & +\&c \\
 & & -\frac{3x^2}{25} & -\frac{9x^3}{125} & +\frac{9x^4}{625} & -\frac{21x^5}{3125} & +\&c \\
 & & & +\frac{7x^3}{125} & +\frac{21x^4}{625} & -\frac{21x^5}{3125} & +\&c \\
 & & & & -\frac{21x^4}{625} & -\frac{63x^5}{3125} & +\&c \\
 & & & & & +\frac{357x^5}{15625} & +\&c \\
\hline
1 & +\frac{6x}{5} & +\frac{3x^2}{25} & -\frac{4x^3}{125} & +\frac{9x^4}{625} & -\frac{126x^5}{15625} & +\&c.
\end{array}$$

This is the square of the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c.$

$$\begin{array}{rrrrrrl}
1 & +\frac{6x}{5} & +\frac{3x^2}{25} & -\frac{4x^3}{125} & +\frac{9x^4}{625} & -\frac{126x^5}{15625} & +\&c \\
1 & +\frac{6x}{5} & +\frac{3x^2}{25} & -\frac{4x^3}{125} & +\frac{9x^4}{625} & -\frac{126x^5}{15625} & +\&c \\
\hline
1 & +\frac{6x}{5} & +\frac{3x^2}{25} & -\frac{4x^3}{125} & +\frac{9x^4}{625} & -\frac{126x^5}{15625} & +\&c \\
 & +\frac{6x}{5} & +\frac{36x^2}{25} & +\frac{18x^3}{125} & -\frac{24x^4}{625} & +\frac{54x^5}{3125} & +\&c \\
 & & +\frac{3x^2}{25} & +\frac{18x^3}{125} & +\frac{9x^4}{625} & -\frac{12x^5}{3125} & +\&c \\
 & & & -\frac{4x^3}{125} & -\frac{24x^4}{625} & -\frac{12x^5}{3125} & +\&c \\
 & & & & +\frac{9x^4}{625} & +\frac{54x^5}{3125} & +\&c \\
 & & & & & -\frac{126x^5}{15625} & -\&c \\
\hline
1 & +\frac{12x}{5} & +\frac{42x^2}{25} & +\frac{28x^3}{125} & -\frac{21x^4}{625} & +\frac{168x^5}{15625} & +\&c.
\end{array}$$

This is the fourth power of the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c.$

$$\begin{array}{rrrrrrl}
1 & +\frac{12x}{5} & +\frac{42x^2}{25} & +\frac{28x^3}{125} & -\frac{21x^4}{625} & +\frac{168x^5}{15625} & -\&c \\
1 & +\frac{3x}{5} & -\frac{3x^2}{25} & +\frac{7x^3}{125} & -\frac{21x^4}{625} & +\frac{357x^5}{15625} & -\&c \\
\hline
1 & +\frac{12x}{5} & +\frac{42x^2}{25} & +\frac{28x^3}{125} & -\frac{21x^4}{625} & +\frac{168x^5}{15625} & -\&c \\
 & +\frac{3x}{5} & +\frac{36x^2}{25} & +\frac{126x^3}{125} & +\frac{84x^4}{625} & -\frac{63x^5}{3125} & +\&c \\
 & & -\frac{3x^2}{25} & -\frac{36x^3}{125} & -\frac{126x^4}{625} & -\frac{84x^5}{3125} & +\&c \\
 & & & +\frac{7x^3}{125} & +\frac{84x^4}{625} & +\frac{294x^5}{3125} & +\&c \\
 & & & & -\frac{21x^4}{625} & -\frac{252x^5}{3125} & -\&c \\
 & & & & & +\frac{357x^5}{15625} & -\&c \\
\hline
1 & +3x & +3x^2 & +x^3 & * & * & \&c.
\end{array}$$

It appears therefore that the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} -$ &c, is really and truly the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, or of the cube of the binomial quantity $1 + x$, and consequently that the binomial theorem is true in the case of the fifth root of the cube of a binomial quantity, or when m, the numerator of the fraction $\frac{m}{n}$, which is the index of the power of $1 + x$, is $= 3$, and n, the denominator of the said fraction, is $= 5$.

Examples of the Extraction of the Roots of some particular Powers of the Binomial Quantity $1 + x$ by means of the Series given above in Art. 4 and 5, in which the Numerator of the Fractional Index $\frac{m}{n}$ is supposed to be greater than its Denominator.

16. In the next place we will investigate the value of $\overline{1 + x}|^{\frac{3}{2}}$, or of the square-root of the cube of the binomial quantity $1 + x$ by means of the series set down in Art. 4 and 5. Now in this case m is $= 3$, and n is $= 2$, and consequently m is greater than n, but less than $2n$. Therefore, by Art. 6, the series that is equal to $\overline{1 + x}|^{\frac{m}{n}}$ will, in this case, be $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \left[\frac{2n-m}{3n}\right. Cx^3 + \frac{3n-m}{4n}Dx^4 - \left[\frac{4n-m}{5n}\right. Ex^5 +$ &c.

Now, since m is $= 3$, and n is $= 2$, we shall have $m - n$ $(= 3 - 2) = 1$, and $2n = 4$, $3n = 6$, $4n = 8$, and $5n = 10$, and

$$2n - m\ (= 4 - 3) = 1,$$
$$3n - m\ (= 6 - 3) = 3,$$
$$4n - m\ (= 8 - 3) = 5,$$

and the series $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \left[\frac{2n-m}{3n}\right. Cx^3 + \frac{3n-m}{4n}Dx^4 - \left[\frac{4n-m}{5n}\right. Ex^5 +$ &c $= 1 + \frac{3}{2}Ax + \frac{1}{4}Bx^2 - \frac{1}{6}Cx^3 + \frac{3}{8}Dx^4 - \frac{5}{10}Ex^5 +$ &c $=$ $1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} +$ &c. Therefore, if the binomial theorem is true in the case of the roots of powers, the quantity $\overline{1 + x}|^{\frac{3}{2}}$, or the square-root of the cube of the binomial quantity $1 + x$, or the square-root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will be equal to the series $1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} +$ &c.

17. Now that this series is really equal to the square-root of the cube of $1 + x$, or to the square-root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will appear by multiplying the said series into itself. For we shall find that the product of the said multiplication will be the said quadrinomial quantity. This multiplication will be as follows:

$$
\begin{array}{r}
1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+\frac{3x^4}{128}-\frac{3x^5}{256}+\&c\\
1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+\frac{3x^4}{128}-\frac{3x^5}{256}+\&c\\
\hline
1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+\frac{3x^4}{128}-\frac{3x^5}{256}+\&c\\
+\frac{3x}{2}+\frac{9x^2}{4}+\frac{9x^3}{16}-\frac{3x^4}{32}+\frac{9x^5}{256}-\&c\\
+\frac{3x^2}{8}+\frac{9x^3}{16}+\frac{9x^4}{64}-\frac{3x^5}{128}+\&c\\
-\frac{x^3}{16}-\frac{3x^4}{32}-\frac{3x^5}{128}+\&c\\
+\frac{3x^4}{128}+\frac{9x^5}{256}+\&c\\
-\frac{3x^5}{256}-\&c\\
\hline
1+3x+3x^2+x^3 \quad * \quad *
\end{array}
$$

It appears therefore that the series $1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+\frac{3x^4}{128}-\frac{3x^5}{256}+\&c$, is really and truly the square root of the quadrinomial quantity $1+3x+3x^2+x^3$, or of the cube of the binomial quantity $1+x$, and consequently that the binomial theorem is true in the case of the square-root of the cube of a binomial quantity, or when m is $=3$, and n is $=2$.

18. I shall add one more example of the series given in Art. 4 and 5, in which it is supposed that the numerator m is greater than the denominator n.

Let it be required to find, by means of the said series, the value of $\overline{1+x}|^{\frac{5}{3}}$, or of the cube-root of the fifth power of the binomial quantity $1+x$, or of the cube-root of the sextinomial quantity $1+5x+10x^2+10x^3+5x^4+x^5$.

In this case m is $=5$, and n is $=3$. Therefore m (though greater than n,) is less than $2n$, and consequently, (by Art. 6,) $\overline{1+x}|^{\frac{m}{n}}$ will be equal to the series $1+\frac{m}{n}Ax+\frac{m-n}{2n}Bx^2-\frac{2n-m}{3n}Cx^3+\frac{3n-m}{4n}Dx^4-\frac{4n-m}{5n}Ex^5+$ &c.

Now, since m is $=5$, and n is $=3$, we shall have have $m-n$ $(=5-3)$ $=2$, and $2n=6$, $3n=9$, $4n=12$, $5n=15$, and

$$2n-m\ (=\ 6-5)=1,$$
$$3n-m\ (=\ 9-5)=4,$$
$$\text{and } 4n-m\ (=12-5)=7,$$

and consequently the series $1+\frac{m}{n}Ax+\frac{m-n}{2n}Bx^2-\frac{2n-m}{3n}Cx^3+\frac{3n-m}{4n}Dx^4-\frac{4n-m}{5n}Ex^5+\&c=1+\frac{5}{3}Ax+\frac{2}{6}Bx^2-\frac{1}{9}Cx^3+\frac{4}{12}Dx^4-\frac{7}{15}Ex^5+\&c=1+\frac{5x}{3}+\frac{5x^2}{9}-\frac{5x^3}{81}+\frac{5x^4}{243}-\frac{7x^5}{729}+\&c$. Therefore, if the binomial theorem is true in the case of the roots of powers, the quantity $\overline{1+x}|^{\frac{5}{3}}$, or the cube-root of the fifth power of the binomial quantity $1+x$, or the

the cube-root of the sextinomial quantity $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, will be equal to the series $1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} +$ &c.

19. Now that this series is really equal to the cube-root of the fifth power of $1 + x$, or to the cube-root of the sextinomial quantity $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be the said sextinomial quantity. These multiplications will be as follows:

$$\begin{array}{rrrrrrl}
1 & + \frac{5x}{3} & + \frac{5x^2}{9} & - \frac{5x^3}{81} & + \frac{5x^4}{243} & - \frac{7x^5}{729} & + \&c \\
1 & + \frac{5x}{3} & + \frac{5x^2}{9} & - \frac{5x^3}{81} & + \frac{5x^4}{243} & - \frac{7x^5}{729} & + \&c \\
\hline
1 & + \frac{5x}{3} & + \frac{5x^2}{9} & - \frac{5x^3}{81} & + \frac{5x^4}{243} & - \frac{7x^5}{729} & + \&c \\
 & + \frac{5x}{3} & + \frac{25x^2}{9} & + \frac{25x^3}{27} & - \frac{25x^4}{243} & + \frac{25x^5}{729} & - \&c \\
 & & + \frac{5x^2}{9} & + \frac{25x^3}{27} & + \frac{25x^4}{81} & - \frac{25x^5}{729} & + \&c \\
 & & & - \frac{5x^3}{81} & - \frac{25x^4}{243} & - \frac{25x^5}{729} & + \&c \\
 & & & & + \frac{5x^4}{243} & + \frac{25x^5}{729} & + \&c \\
 & & & & & - \frac{7x^5}{729} & - \&c \\
\hline
1 & + \frac{10x}{3} & + \frac{35x^2}{9} & + \frac{140x^3}{81} & + \frac{35x^4}{243} & - \frac{14x^5}{729} & \&c.
\end{array}$$

$$\begin{array}{rrrrrrl}
1 & + \frac{10x}{3} & + \frac{35x^2}{9} & + \frac{140x^3}{81} & + \frac{35x^4}{243} & - \frac{14x^5}{729} & \&c \\
1 & + \frac{5x}{3} & + \frac{5x^2}{9} & - \frac{5x^3}{81} & + \frac{5x^4}{243} & - \frac{7x^5}{729} & + \&c \\
\hline
1 & + \frac{10x}{3} & + \frac{35x^2}{9} & + \frac{140x^3}{81} & + \frac{35x^4}{243} & - \frac{14x^5}{729} & \&c \\
 & + \frac{5x}{3} & + \frac{50x^2}{9} & + \frac{175x^3}{27} & + \frac{700x^4}{243} & + \frac{175x^5}{729} & - \&c \\
 & & + \frac{5x^2}{9} & + \frac{50x^3}{27} & + \frac{175x^4}{81} & + \frac{700x^5}{729} & - \&c \\
 & & & - \frac{5x^3}{81} & - \frac{50x^4}{243} & - \frac{175x^5}{729} & + \&c \\
 & & & & + \frac{5x^4}{243} & + \frac{50x^5}{729} & + \&c \\
 & & & & & - \frac{7x^5}{729} & - \&c \\
\hline
1 & + 5x & + 10x^2 & + 10x^3 & + 5x^4 & + x^5 & \&c.
\end{array}$$

It appears therefore that the series $1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} +$ &c is really and truly the cube-root of the fifth power of the binomial quantity $1 + x$, or the cube-root of the sextinomial quantity $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, and consequently that the binomial theorem is true in the case of the cube-root of the fifth power of a binomial quantity, or when m is $= 5$, and n is $= 3$.

20. These

20. These six examples of the application of the binomial theorem to the investigation of the square and cube-roots of the binomial quantity $1 + x$, and of the cube-root of the square of $1 + x$, and of the fifth root of the cube of $1 + x$, and of the square-root of its cube, and the cube-root of its fifth power, may suffice for the purpose of illustrating the meaning of that celebrated theorem, and the manner of applying it to particular cases, when the index of the power of $1 + x$ is either the fraction $\frac{1}{n}$ or the fraction $\frac{m}{n}$. And the proofs that have been given of the truth of the serieses obtained by means of the binomial theorem in these examples, by raising the serieses so obtained to the proper powers, and shewing that the powers thereby produced are exactly what they ought to be, afford a very strong presumption, "that, since the binomial theo- "rem is true in these particular values of m and n, it must also be true in all "other values of them whatsoever." But in the mathematical sciences we ought to aim at something more than this high degree of probability, and to endeavour, if possible, to demonstrate the propositions we advance. And this is what I shall now proceed to do with respect to this theorem in the cases mentioned in Art. 2, 3, 4, and 5. And, that we may not fall into confusion by making our inquiries too extensive, and embracing too many objects at the same time, I shall first consider the case of $\overline{1 + x}|^{\frac{1}{n}}$, or the nth root of the binomial quantity $1 + x$, and endeavour to demonstrate, that, whatever be the whole number denoted by n, the value of $\overline{1 + x}|^{\frac{1}{n}}$ will be equal to the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5 -$ &c, or $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c, set forth above in Art. 2, and afterwards shall proceed to give a like demonstration of the truth of the other serieses set forth in Art. 3, 4, and 5, for the values of $\overline{1 + x}|^{\frac{m}{n}}$.

Before we enter on the investigation of this series, it will be necessary to make the following observations.

Observations preparatory to the Investigation of the Series that is equal to $\overline{1 + x}|^{\frac{1}{n}}$, or the n*th root of the binomial quantity $1 + x$.*

21. In the first place, we must observe that the first term of the series that is equal to $\overline{1 + x}|^{\frac{1}{n}}$ or $\sqrt[n]{1 + x}$, must always be 1.

This follows from the common manner of extracting the roots of a binomial, or other more compound quantity, of which 1 is the first term. For, if we were required to extract the square-root, or the cube-root, or any other root, of

of ſuch compound quantity, according to the common rules laid down in books of arithmetick, or algebra, for ſuch extractions, the firſt ſtep of ſuch an extraction would be to extract the ſquare-root, or the cube-root, or other higher root, of its firſt member 1; which ſquare-root, or cube-root, or other higher root, of 1, would be 1; ſo that 1 would be the firſt term of the ſaid ſquare-root, or cube-root, or other higher root, of the ſaid binomial, or other more compound, quantity $1 + x$, or $1 + x +$ &c. Q. E. D.

The truth of this obſervation may alſo be deduced from the following conſideration, to wit, "that the ſeries that is equal to $\overline{1 + x}|^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$, muſt "be equal to it in all magnitudes of x, or while x is of any magnitude leſs than "1, which is ſuppoſed to be the firſt and greater member of the binomial quan- "tity $1 + x$." For, it follows from hence that the ſaid ſeries muſt be equal to $\overline{1 + x}|^{\frac{1}{n}}$, or to $\sqrt[n]{1 + x}$, when x is $= 0$. But, when x is $= 0$, $\overline{1 + x}|^{\frac{1}{n}}$ is $= \overline{1 + 0}|^{\frac{1}{n}} = \overline{1}|^{\frac{1}{n}} = 1$. Therefore, the ſaid ſeries muſt, in that caſe, be $= 1$. Therefore, the firſt term of the ſaid ſeries cannot be involved with any of the powers of x, (by which it would be made to become equal to 0 when x was equal to 0,) but muſt be $= 1$. Q. E. D.

22. In the next place we muſt obſerve, that the ſecond and third, and other following terms, of the ſeries that is equal to $\overline{1 + x}|^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$, will contain the ſeveral powers of x, to wit, x, x^2, x^3, x^4, x^5, &c, in their natural order, without interruption, and conſequently that the ſaid quantity will be equal to a ſeries of the following form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c; in which the ſecond and third, and other following terms, Bx, Cx^2, Dx^3, Ex^4, Fx^5, are to be either added to the firſt term 1, and in that caſe to have the ſign + prefixed to them, or to be ſubtracted from the ſaid firſt term, and in that caſe to have the ſign − prefixed to them. But, as it is not yet certain to which of the ſaid terms the ſign + is to be prefixed, and which of them are to be marked with the ſign −, or which of them are to be added to the ſaid firſt term 1, and which of them are to be ſubtracted from it, I have not, on this occaſion, (where it was only neceſſary to mention the form of the ſeries which is equal to $\overline{1 + x}|^{\frac{1}{n}}$,) prefixed either of theſe ſigns to any of them, but have only ſeparated the ſeveral terms of the ſaid ſeries one from another by placing a comma after each term.

23. Now, "that this obſervation is true, or that $\overline{1 + x}|^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$, "will be equal to a ſeries of the ſaid form, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c," has already been ſhewn in two inſtances, to wit, when n is $= 2$, and when n is $= 3$, or in the caſes of the ſquare-root and the cube-root of $1 + x$. For it has been ſhewn by Art. 9, that the ſeries $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} -$ &c is equal

to the ſquare root of $1 + x$; and it has been ſhewn in Art. 11, that the ſeries $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} -$ &c is equal to the cube-root of $1 + x$. And both theſe ſerieſes are of the ſame form as the general ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c above-mentioned. And, "that all other roots of the bino-"mial quantity $1 + x$ muſt likewiſe be equal to ſerieſes of the ſame form," will appear from conſidering the manner of extracting any root of ſuch a binomial quantity. For the firſt ſtep of ſuch an extraction gives us the firſt term of the root ſought; which firſt term (as we have already obſerved) is always 1: and the ſecond ſtep of ſuch an extraction is to ſquare the ſaid firſt term, 1, of the root ſought, if we are performing the extraction of the ſquare-root of $1 + x$; or to cube the ſaid firſt term, 1, of the root ſought, if we are performing the extraction of the cube-root of $1 + x$; or, in general, to raiſe the ſaid firſt term, 1, of the root ſought to the nth power, if we are performing the extraction of the nth root of $1 + x$; and then to ſubtract ſuch ſquare, cube, or nth power of the ſaid firſt term, 1, of the root ſought (which will always be 1, becauſe all the powers of 1 are equal to 1), from $1 + x$, or the original quantity of which the root is ſought: in conſequence of which ſubtraction there will remain the quantity x (which is the ſecond and leſſer member of the binomial quantity $1 + x$,) for the ground-work of the gradual evolution of the ſecond and third and other following terms of the ſeries, or root, ſought. And the third ſtep, or proceſs, of ſuch extraction (whereby we ſhall obtain the ſecond term of the ſeries that is equal to the root ſought) is to double the firſt term of it already found, to wit, the term 1, in the caſe of the extraction of the ſquare root; and to treble the ſaid firſt term 1, in the caſe of the extraction of the cube-root; and, in general, to multiply the ſaid firſt term, 1, by n in the caſe of the extraction of the nth root; and then to divide the ſaid remainder x by the ſaid product; to wit, by 2×1, or 2, in the caſe of the ſquare-root; and by 3×1, or 3, in the caſe of the cube-root; and by $n \times 1$, or n, in the caſe of the nth root: which will give us $\frac{x}{2}$ in the caſe of the ſquare-root, and $\frac{x}{3}$ in the caſe of the cube-root, and $\frac{x}{n}$ in the caſe of the nth root, for the ſecond term of the ſeries that is equal to $\overline{1 + x}|^{\frac{1}{2}}$, $\overline{1 + x}|^{\frac{1}{3}}$, or $\overline{1 + x}|^{\frac{1}{n}}$: ſo that the two firſt terms of the ſeries that is equal to $\overline{1 + x}|^{\frac{1}{2}}$ will be $1 + \frac{x}{2}$, and the two firſt terms of the ſeries that is equal to $\overline{1 + x}|^{\frac{1}{3}}$ will be $1 + \frac{x}{3}$, and in general, the two firſt terms of the ſeries that is equal to $\overline{1 + x}|^{\frac{1}{n}}$ will be $1 + \frac{x}{n}$.

And the next operations of theſe extractions will be to raiſe $1 + \frac{x}{2}$ to the ſecond power in the extraction of the ſquare-root, and to raiſe $1 + \frac{x}{3}$ to the third power in the extraction of the cube-root, and, in general, to raiſe $1 + \frac{x}{n}$ to the nth power in the extraction of the nth root; and then to ſubtract theſe powers from the original binomial quantity $1 + x$, of which we are ſeeking the root; or,

to ſpeak more correctly, (becauſe theſe powers of $1 + \frac{x}{2}$, $1 + \frac{x}{3}$, and $1 + \frac{x}{n}$ will be greater than $1 + x$,) to ſubtract $1 + x$ from the ſaid powers: after which the third term of the ſeries that is equal to $\overline{1+x}|^{\frac{1}{2}}$, or $\overline{1+x}|^{\frac{1}{3}}$, or $\overline{1+x}|^{\frac{1}{n}}$, will be obtained by dividing the remainder of the laſt ſubtraction by $2 \times \overline{1 + \frac{x}{2}}$, or $2 + x$, in the extraction of the ſquare-root; or by $3 \times \overline{1 + \frac{x}{3}}$, or $3 + x$, in the extraction of the cube-root; or, in general, by $n \times \overline{1 + \frac{x}{n}}$, or $n + x$, in the extraction of the nth root. Now, it it is evident, that, by raiſing the aforeſaid powers of the binomial quantities $1 + \frac{x}{2}$, $1 + \frac{x}{3}$, and $1 + \frac{x}{n}$, (in the ſecond member of each of which the ſimple power of x occurs,) we ſhall obtain quantities in which the powers of x will occur in their natural order, to wit, x, x^2, x^3, x^4, x^5, &c, without any interruption. Thus $\overline{1 + \frac{x}{2}}|^2$ is $= 1 + x + \frac{xx}{4}$, and $\overline{1 + \frac{x}{3}}|^3$ is $(= 1 + 3 \times 1 \times \frac{x}{3} + 3 \times 1 \times \frac{x^2}{9} + \frac{x^3}{27}) = 1 + x + \frac{xx}{3} + \frac{x^3}{27}$, and $\overline{1 + \frac{x}{n}}|^n$ is (by the binomial theorem in the caſe of integral powers, which has been already demonſtrated above, in the laſt tract but one, from page 153 to page 169,) $= 1 + \frac{n}{1} \times \frac{x}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \overline{\frac{x}{n}}|^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \overline{\frac{x}{n}}|^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \overline{\frac{x}{n}}|^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \overline{\frac{x}{n}}|^5 +$ &c $= 1 + \frac{n}{1} \times \frac{x}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{x^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{x^3}{n^3} + \frac{n}{1} + \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{x^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{x^5}{n^5} +$ &c $= 1 + x + \frac{n-1}{2n} x^2 + \frac{n-1}{2n} \times \frac{n-2}{3n} x^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} x^4 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} x^5 +$ &c; in all which powers of the ſaid binomial quantities $1 + \frac{x}{2}$, $1 + \frac{x}{3}$, and $1 + \frac{x}{n}$ the powers of x occur in their natural order, to wit, x, x^2, x^3, x^4, x^5, &c, without any interruption. Therefore, when $1 + x$ has been ſubtracted from each of theſe powers, the remainders, (which will be $\frac{xx}{4}$ in the caſe of the ſquare-root, and $\frac{xx}{3} + \frac{x^3}{27}$ in the caſe of the cube-root, and $\frac{n-1}{2n} \times x^2 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times x^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times x^4 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} \times x^5 +$ &c in the caſe of the nth root,) will conſiſt of the regular powers of x, beginning with xx, combined with certain numeral co-efficients; and conſequently the quotients of the diviſions of theſe remainders by $2 + x$, and $3 + x$, and $n + x$, reſpectively, will be quantities that involve in them the ſquare of x; to wit, in the caſe of the ſquare-root, the ſaid quotient will

will be $\frac{\frac{xx}{4}}{2+x} = \frac{\frac{xx}{4}}{2} - \&c = \frac{xx}{8} - \&c$; and in the case of the cube-root the said quotient will be $\frac{\frac{xx}{3} + \frac{x^3}{27}}{3+x} = \frac{\frac{xx}{3}}{3}\ \&c = \frac{xx}{9}\ \&c$; and, in the case of the nth root the said quotient will be $\frac{\frac{n-1}{2n} xx + \frac{n-1}{2n} \times \frac{n-2}{3n} \times x^3 + \&c}{n+x} = \frac{\frac{n-1}{2n} xx}{n}$ $\&c = \frac{n-1}{2nn} xx\ \&c$; so that the third term of the series that is equal to $\overline{1+x}|^{\frac{1}{2}}$ will be $\frac{xx}{8}$ and the third term of the series that is equal to $\overline{1+x}|^{\frac{1}{3}}$ will be $\frac{xx}{9}$ and, in general, the third term of the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$ will be $\frac{n-1}{2nn}$ xx; all which third terms involve in them the square of x. And thus we see that the three first terms of the series that is equal to $\overline{1+x}|^{\frac{1}{2}}$ will be $1 + \frac{x}{2} - \frac{xx}{8}$, and the three first terms of the series that is equal to $\overline{1+x}|^{\frac{1}{3}}$ will be $1 + \frac{x}{3} - \frac{xx}{9}$, and the three first terms of the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$ will be $1 + \frac{x}{n} - \frac{n-1}{2nn} xx$; in all which serieses the powers of x ascend regularly.

And as the following terms of these serieses are derived from the three first terms of them by the repetition of the same processes of multiplication, subtraction, and division, by which the third term is derived from the first and second terms, and the second term is derived from the first, it follows that the fourth and fifth and sixth and other following terms must involve in them the cube and fourth power and fifth power, and other following powers, of x, in their natural order, without interruption, and consequently that the form of the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$ will be 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or, rather, (since we now know that the second term of the series is to be added to the first term, 1, and that the third term of it is to be subtracted from them,) $1 + Bx - Cx^2$, Dx^3, Ex^4, Fx^5, &c. Q. E. D.

24. The foregoing reasonings may, perhaps, seem rather abstract and difficult, as they are expressed in words in the last article. But the force of them will become more apparent by actually exhibiting to the reader's view an example of the extraction of a root of the binomial quantity $1 + x$ in the manner alluded to in them. Now the extraction of the square-root of $1 + x$ is performed in the following manner.

The

The Extraction of the square-root of the binomial quantity $1 + x$.

$$
\begin{array}{lllllll}
 & & & 1 + x\,(1 + \frac{x}{2} - \frac{xx}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \&c \\
 & & & \frac{1}{*} + x \\
2 + \frac{x}{2}) & & & \\
 & & & + x + \frac{xx}{4} \\
\hline
2 + x - \frac{xx}{8}) & & & * - \frac{xx}{4} \\
 & & & - \frac{xx}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
\hline
2 + x - \frac{xx}{4} + \frac{x^3}{16}) & & & * + \frac{x^3}{8} - \frac{x^4}{64} \\
 & & & + \frac{x^3}{8} + \frac{x^4}{16} - \frac{x^5}{64} + \frac{x^6}{256} \\
\hline
2 + x - \frac{xx}{4} + \frac{x^3}{8} - \&c) & & & * - \frac{5x^4}{64} + \frac{x^5}{64} - \frac{x^6}{256} \\
 & & & - \frac{5x^4}{64} - \frac{5x^5}{128} + \frac{5x^6}{512} + \&c \\
\hline
2 + x - \frac{xx}{4} + \frac{x^3}{8} - \&c \quad) & & & * + \frac{7x^5}{128} - \frac{7x^6}{512}\ \&c \\
 & & & + \frac{7x^5}{128} + \frac{7x^6}{256} - \&c \\
\hline
2 + x - \frac{xx}{4} + \frac{x^3}{8} - \&c \quad) & & & * - \frac{21x^6}{512}\ \&c \\
 & & & - \frac{21x^6}{512}\ \&c \\
\hline
 & & & *
\end{array}
$$

25. It appears from the foregoing extraction of the square-root of $1 + x$, that the said square-root is equal to a series of quantities of which the first seven terms are $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} \times \frac{7x^5}{256} - \frac{21x^6}{1024}$ in which the powers of x increase regularly from x to x^6. And it is plain from the manner in which he second of those terms, to wit, $\frac{x}{2}$, and all the terms that follow it, are gradually derived from x, or the remainder of the first subtraction, that the following terms of the series, to whatever number they should be continued, would involve in them the next following powers of x, to wit, x^7, x^8, x^9, x^{10}, x^{11}, &c in their natural order. And the same thing would appear in the extraction of the cube root of $1 + x$, or of its fourth root, or its fifth root, or any higher root of it whatsoever; though the operations necessary to these extractions would be much more complicated and laborious than those that occur in the foregoing extraction of the square-root. And therefore we may conclude in general from the nature of these extractions, that the nth root of the binomial quantity $1 + x$ will in all cases be equal to a series of the before-mentioned form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, in which the powers of x occur in their natural order, or increase gradually by the continual multiplication of x. Q. E. D.

26. In

26. In the third place we must observe, that the second term Bx, of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, must be added to the first term 1, and consequently have the sign + prefixed to it.

This observation has been already shewn to be true in the reasonings used in Art: 23 to demonstrate the second observation. But, as it will be referred to in the following investigation as a necessary preliminary proposition, it will not be amiss to give the following additional demonstration of it before we begin that investigation.

Now, since $1 + x$ is greater than 1, it follows that $\overline{1+x}|^{\frac{1}{n}}$, or the nth root of $1 + x$, or the first of $n - 1$ geometrical mean proportionals between 1 and $1 + x$, must also be greater than 1. Consequently the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, must also be greater than 1. And this must always be true, of how small a magnitude soever we suppose x to be taken, so long as it has any magnitude at all. But, in order that the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, may be always greater than 1, however small the magnitude of x be taken, it is necessary that Bx should be added to the first term 1, and not subtracted from it. For, if Bx were subtracted from 1, it would be possible, by diminishing the magnitude of x, to make all the terms Cx^2, Dx^3, Ex^4, Fx^5, &c that follow Bx, put together become less than Bx, however great we may suppose the magnitudes of the co-efficients C, D, E, F, &c, to be; in which case it is evident that the whole series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c would (even though all the terms after Bx were to be added to the first term 1, and the series were to be $1 - Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c,) be less than the first term 1: which is contrary to the supposition. Therefore the second term, Bx, cannot be subtracted from the first term 1, but must be added to it, and consequently must be marked with the sign +; and therefore the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be of this form, $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c. Q. E. D.

27. By the help of these three preliminary observations we may now proceed to the investigation of the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$, or the nth root of the binomial quantity $1 + x$, when n is supposed to stand for any whole number whatsoever. This investigation may be made as follows.

An

AN ADDITION

TO THE

Difcourfe on the Binomial Theorem.

[*The Binder is defired to place this and the other fheets with ftarred fignatures and folios after page* 216 *in the fecond volume.*]

ARTICLE 1. The foregoing conclufion contained in articles 22, 23, 24, and 25, to wit, "that $\overline{1 + x}|^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$, or the nth root of the binomial quantity $1 + x$, is equal to a feries of the before-mentioned form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c (in which the powers of x follow each other in their natural order without any interruption, or increafe gradually by the continual multiplication of x) or that a feries of this form may always exift (to whatever whole number we fuppofe the letter n to be equal) which fhall be equal to the faid quantity $\overline{1 + x}|^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$;" may be alfo fhewn in the following manner.

2. Whatever whole number may be fuppofed to be denoted by n, it is evident that a feries of the before-mentioned form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, will be equal to it, if the feveral numeral co-efficients B, C, D, E, F, &c, of x, x^2, x^3, x^4, x^5, &c, in the fecond and other following terms of fuch feries, are of fuch magnitudes, and the faid fecond and other following terms, to wit, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are fo connected with the firft term 1, by the figns + and −, or by addition and fubtraction, that, when the faid feries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c is raifed to the nth power, or multiplied $n - 1$ times into itfelf, the compound feries which, it is evident, will be produced by fuch multiplication, fhall be equal to the binomial quantity $1 + x$, or that the two firft terms of the faid compound feries fhall be 1 and x, and that the fign prefixed to the faid fecond term x fhall be +, and that the third, and fourth, and fifth, and fixth, and other following terms of the faid compound feries fhall, each of them, be equal to nothing, or that the members of each of the faid terms of the faid compound feries, after the fecond term, fhall be marked with both the figns + and −, that is, fome of them with the fign +, and others of them with the fign −, and that the fum of thofe of the faid members of each term which are marked with the fign − fhall be equal to the fum of the other members of the fame term which are marked with the fign +, fo as thereby to counterbalance, or deftroy, them, and make the whole of the faid compound term be equal to nothing.

3. Thus, for example, if n is $= 2$, the series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, will be equal to $\overline{1+x}|^{\frac{1}{2}}$, or $\sqrt[2]{1+x}$, or the square-root of the binomial quantity $1+x$, if the numeral co-efficients B, C, D, E, F, &c, of x, x^2, x^3, x^4, x^5, &c, in the second and other following terms of the said series, to wit, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are of such magnitudes, and the said second and other following terms of the said series are so connected with the first term 1 by the signs $+$ and $-$, or by addition and subtraction, that, when the said series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, is raised to the square, or second power, or multiplied $2-1$ times, or once, into itself, the compound series which, it is evident, will be produced by such multiplication, shall be equal to the binomial quantity $1+x$, or that the two first terms of the said compound series shall be 1 and x, and that the sign prefixed to the said second term x shall be $+$, and that the third, and fourth, and fifth, and sixth, and other following terms of the said compound series shall, each of them, be equal to nothing, or that the members of each of the said terms of the said compound series, after the said second term, shall be marked with both the signs $+$ and $-$, that is, some of them with the sign $+$, and the others with the sign $-$, and that the sum of those members of each term which are marked with the sign $-$ shall be equal to the sum of the other members of the same term which are marked with the sign $+$, so as thereby to counterbalance, or destroy, them, and to make the whole of the said term be equal to nothing.

4. And, if n is $= 3$, the series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, will be equal to $\overline{1+x}|^{\frac{1}{3}}$, or $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $1+x$, if the numeral co-efficients B, C, D, E, F, &c of x, x^2, x^3, x^4, x^5, &c, in the second and other following terms of the said series, are of such magnitudes, and the said second and other following terms of the said series, to wit, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are so connected with the first term 1 by the signs $+$ and $-$, or by addition and subtraction, that when the said series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, is raised to the cube, or third power, or multiplied $3-1$ times, or twice, into itself, the compound series which, it is evident, will be produced by the said two multiplications, shall be equal to the binomial quantity $1+x$, or that the two first terms of the said compound series shall be 1 and x, and that the sign prefixed to the said second term x shall be $+$, and that the third, and fourth, and fifth, and sixth, and other following terms of the said compound series shall, each of them, be equal to nothing, or that the members of each of the said terms of the said compound series, after the said second term, shall be marked with both the signs $+$ and $-$, that is, some of them with the sign $+$, and the others with the sign $-$, and that the sum of those members of each term which are marked with the sign $-$ shall be equal to the sum of the other members of the same term which are marked with the sign $+$, so as thereby to counterbalance, or destroy, them, and to make the whole of the said term be equal to nothing.

5. We

5. We muſt therefore endeavour to ſhew that, if n be equal to 2, or to 3, or to any other whole number whatſoever, it will be poſſible for a ſeries of the before-mentioned form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to exiſt, in which the magnitudes of the numeral co-efficients B, C, D, E, F, &c, of x, x^2, x^3, x^4, x^5, and the following powers of x, in the ſecond and other following terms of the ſaid ſeries, to wit, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are ſuch, and the ſaid ſecond and other following terms of the ſaid ſeries are ſo connected with the firſt term 1 by the ſigns + and −, or by addition and ſubtraction, that, if the ſaid ſeries be raiſed to the ſquare or the cube, or the nth power, or be multiplied into itſelf $2 - 1$, or $3 - 1$ times (that is, once or twice) or $n - 1$ times, the compound ſeries that will ariſe from ſuch multiplications ſhall be equal to the binomial quantity $1 + x$, or that the two firſt terms of the ſaid compound ſeries ſhall be 1 and x, and that the ſaid ſecond term x ſhall have the ſign + prefixed to it, or ſhall be added to the firſt term 1, and that the third, and fourth, and fifth, and ſixth, and all the following terms of the ſaid compound ſeries, ſhall, each of them, be equal to nothing, or that ſome of the members of each of the ſaid terms of the ſaid compound ſeries, after the ſecond term, ſhall be marked with the ſign +, and the others with the ſign −, and that the ſum of the latter members which are marked with the ſign −, ſhall be equal to the ſum of the former members, which are marked with the ſign +, ſo as thereby to counterbalance, or deſtroy, them, and to make the whole of the ſaid term be equal to nothing.

Of the Square-root of $1 + x$.

6. Now that it is poſſible for ſuch a ſeries of the before-mentioned form, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to exiſt in the caſe of the ſquare-root of $1 + x$, or when n is equal to 2, may be ſhewn in the following manner.

Let the ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, be multiplied into itſelf; which may be done ~~as follows~~.

1,	Bx,	Cx^2,	Dx^3,	Ex^4,	Fx^5,	&c
1,	Bx,	Cx^2,	Dx^3,	Ex^4,	Fx^5,	&c
1,	Bx,	Cx^2,	Dx^3,	Ex^4,	Fx^5,	&c
	Bx,	B^2x^2,	BCx^3,	BDx^4,	BEx^5,	&c
		Cx^2,	BCx^3,	C^2x^4,	CDx^5,	&c
			Dx^3,	BDx^4,	CDx^5,	&c
				Ex^4,	BEx^5,	&c
					Fx^5,	&c
1,	$2Bx$,	$2Cx^2$,	$2Dx^3$,	$2Ex^4$,	$2Fx^5$,	&c
		B^2x^2,	$2BCx^3$,	$2BDx^4$,	$2BEx^5$,	&c
				C^2x^4,	$2CDx^5$,	&c

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This compound series is the square of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c. Therefore, if we can prove that it is possible that the co-efficients B, C, D, E, F, &c, may be taken of such magnitudes, and that the second and other following terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, may be so connected with its first term 1 by the signs + and —, or by addition and subtraction, as to make the said compound series become equal to $1 + x$, it will follow that the said simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, will be equal to the square-root of $1 + x$. We must therefore endeavour to shew that it is possible for the numeral co-efficients B, C, D, E, F, &c, to be of such magnitudes, and for the terms Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to be so connected with the first term 1 of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, by the signs + and —, or by addition and subtraction, that the whole compound series above-mentioned shall be equal to $1 + x$. This may be done in the manner following.

7. In the first place it is evident that the first term of the said compound series is 1. This is too plain to admit of any proof.

In the next place it is plain that, if we prefix the sign + to the second term Bx of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, so as to convert the said series into the series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, the sign + will likewise be prefixed to $2Bx$, or the second term of the compound series $\left\{\begin{array}{lll} 1, & 2Bx, & 2Cx^2, \&c \\ & & B^2x^2, \&c \end{array}\right\}$ above-mentioned, so as to convert it into the series $\left\{\begin{array}{lll} 1 + & 2Bx, & 2Cx^2, \&c \\ & & B^2x^2, \&c \end{array}\right\}$.

And, in the 3d place, it is evident that, if we suppose $2B$ to be equal to 1, or B to be equal to $\frac{1}{2}$, the second term, $2Bx$, of the said compound series will be equal to $2 \times \frac{1}{2} \times x$, or $\frac{2}{2} \times x$, or x, and consequently the two first terms of the said compound series $\left\{\begin{array}{lll} 1, & 2Bx, & 2Cx^2, \&c \\ & & B^2x^2, \&c \end{array}\right\}$, or $\left\{\begin{array}{lll} 1 + & 2Bx, & 2Cx^2, \&c \\ & & B^2x^2, \&c \end{array}\right\}$, will be $1 + x$.

If, therefore, we can prove that it is possible to take the following numeral co-efficients C, D, E, F, &c, of such magnitudes, and to connect the third and fourth, and other following terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to wit, the terms Cx^2, Dx^3, Ex^4, Fx^5, &c, with the two first terms 1, Bx, or $1 + Bx$, or $1 + \frac{x}{2}$, by the signs + and —, or by addition and subtraction, in such a manner, that every following term of the compound series

$$\left\{\begin{array}{llllll} 1 + 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & & C^2x^4, & 2CDx^5, & \&c \end{array}\right\}$$

or

$$\text{or}\left.\begin{array}{llllll} 1+x, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & & C^2x^4, & 2CDx^5, & \&c \end{array}\right\}$$

ſhall be equal to nothing, or that ſome of their members (for each of theſe terms is evidently a compound quantity, or a quantity conſiſting of more than one member) ſhall be marked with the ſign +, and others of them ſhall be marked with the ſign —, and that the ſum of the latter members in each term, which are marked with the ſign —, ſhall be equal to the ſum of the former members in the ſame term, which are marked with the ſign +, ſo as to exactly counterbalance, or deſtroy, the ſaid former members, and make the whole term become equal to 0, it will then be evident that the whole compound ſeries above mentioned will be equal to its two firſt terms $1+x$, and conſequently that the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1+Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1+\frac{x}{2}$, Cx^2, Dx^3, Ex^4, Fx^5, &c, (of which the ſaid compound ſeries is the ſquare) will be equal to the ſquare-root of $1+x$. This therefore is what we muſt now endeavour to prove.

8. In order to this, we muſt obſerve, in the firſt place, that in the firſt, or upper, horizontal row of terms in the ſaid compound ſeries, to wit, 1, $2Bx$, $2Cx^2$, $2Dx^3$, $2Ex^4$, $2Fx^5$, &c, or $1+x$, $2Cx^2$, $2Dx^3$, $2Ex^4$, $2Fx^5$, &c, the ſign + or —, that is to be prefixed to every term of it, muſt be the ſame which is prefixed to the correſponding term of the original ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1+\frac{x}{2}$, Cx^2, Dx^3, Ex^4, Fx^5, &c, or term which involves the ſame capital letter, C, or D, or E, or F, &c. Thus, if the ſign — is to be prefixed to the term Cx^2 of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, the ſame ſign — muſt likewiſe be prefixed to the term $2Cx^2$, in the upper horizontal row of terms in the compound ſeries $\left.\begin{array}{llll} 1, 2Bx, & 2Cx^2, & 2Dx^3, & \&c \\ & B^2x^2, & 2BCx^3, & \&c \end{array}\right\}$, or $\left.\begin{array}{llll} 1+x, & 2Cx^2, & 2Dx^3, & \&c \\ & B^2x^2, & 2BCx^3, & \&c \end{array}\right\}$; becauſe the ſaid term, $2Cx^2$, in the ſaid compound ſeries, ariſes from the addition of the two like terms Cx^2 and Cx^2 to each other, of which the firſt Cx^2 is the product of the multiplication of Cx^2 by 1, and the other Cx^2 is the product of the multiplication of 1 by Cx^2, and each of theſe products muſt evidently have the ſame ſign + or — prefixed to it as is prefixed to its factor Cx^2, or the third term of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, its other factor being 1. And for the ſame reaſon each of the following terms $2Dx^3$, $2Ex^4$, $2Fx^5$, &c, in the ſaid firſt, or upper, horizontal row of terms muſt have the ſame ſign + or — prefixed to it as is prefixed to the correſponding term, or term involving the ſame capital letter D, or E, or F, &c, in the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1+\frac{x}{2}$, Cx^2, Dx^3, Ex^4, Fx^5, &c.

In the ſecond place, it is evident from the laſt obſervation that, if we can de-

termine the ſign + or —, which ought to be prefixed to any term in the ſaid upper horizontal row of terms 1, $2Bx$, $2Cx^2$, $2Dx^3$, $2Ex^4$, $2Fx^5$, &c, or $1+x$, $2Cx^2$, $2Dx^3$, $2Ex^4$, $2Fx^5$, &c, we ſhall at the ſame time determine the ſign + or —, which ought to be prefixed to the correſponding term (or term involving the ſame capital letter, D, or E, or F, &c) in the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1+\frac{x}{2}$, Cx^2, Dx^3, Ex^4, Fx^5, &c, the ſigns to be prefixed to the ſaid two correſponding terms being always the ſame.

In the third place it is evident that in the firſt, or upper, horizontal row of terms in the above-mentioned compound ſeries

$$\left.\begin{array}{llllll} 1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5,\ \&c \\ & & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5,\ \&c \\ & & & & C^2x^4, & 2CDx^5,\ \&c \end{array}\right\}$$

(which is equal to the ſquare of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c) the co efficients of x, x^2, x^3, x^4, x^5, &c, are 2 B, 2 C, 2 D, 2 E, 2 F, &c, or exactly the doubles of the co-efficients of the ſame powers of x in the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, ſo that a new capital letter C, or D, or E, or F, &c, enters into every new term of the ſaid firſt, or upper horizontal row of terms.

For each of theſe terms in the ſaid upper horizontal row is the ſum of two equal terms, of which one is placed in the higheſt horizontal row of terms of the original ſet of ſeparate products ariſing from the multiplication of the ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, into itſelf, before the ſimilar terms in the ſaid ſeparate products are added together at the bottom, and the other is placed at the bottom of the ſame vertical column of terms in which the former is the upper term. Thus, in the firſt vertical column of terms in the ſaid original ſet of products ariſing from the multiplication of the ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, into itſelf, there are two equal terms, to wit, Bx and Bx, whoſe ſum forms the term $2Bx$ in the upper horizontal row of terms of the compound ſeries at the bottom, to wit, the ſeries

$$\begin{array}{llllll} 1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5,\ \&c \\ & & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5,\ \&c \\ & & & & C^2x^4, & 2CDx^5,\ \&c, \end{array}$$

which is equal to the ſquare of the ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c; and in the ſecond vertical column of the ſame original ſet of products, there are the terms Cx^2, B^2x^2, and Cx^2, of which the higheſt and loweſt are equal to each other, and their ſum is conſequently equal to $2Cx^2$; and in the third vertical column there are the terms Dx^3, BCx^3, BCx^3, and Dx^3, of which the higheſt and the loweſt are equal to each other, and their ſum is conſequently equal to $2Dx^3$; and in the fourth vertical column there are the terms Ex^4, BDx^4, C^2x^4, BDx^4, and Ex^4, of which the higheſt and the loweſt are equal to each other, and their ſum is conſequently equal to $2Ex^4$; and in the fifth vertical column there are the terms Fx^5, BEx^5, CDx^5, CDx^5, BEx^5, and Fx^5, of which the higheſt and the loweſt are equal to each other, and their ſum is conſequently equal to $2Fx^5$. And it is evident that the ſame thing would take place in all the following vertical columns of terms, to whatever number of terms

terms the said series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, should be continued: So that the following terms of the upper horizontal row of terms after the term $2Fx^5$ must be $2Gx^6$, $2Hx^7$, $2Ix^8$, $2Kx^9$, $2Lx^{10}$, $2Mx^{11}$, &c, *ad infinitum*.

And in the fourth place it must be observed, that the capital letter C, or D, or E, or F, &c, which enters into the highest term of any of the said vertical columns of terms (which highest terms are $2Cx^2$, $2Dx^3$, $2Ex^4$, $2Fx^5$, $2Gx^6$, &c) will not enter into any of the other terms of the same vertical column, but the said other, or lower, terms of the said vertical column will involve in them only such capital letters as had appeared in the upper terms of the preceeding vertical columns. Thus for example, in the vertical column $2Cx^2$, B^2x^2 (which is the first vertical column of the compound series in which the similar terms have been added to each other) the letter C enters only in the upper term $2Cx^2$, and the other term B^2x^2 involves only the preceeding letter B; and in the second vertical column of terms in the same compound series, to wit, the vertical column $2Dx^3$, $2BCx^3$, the letter D enters only in the upper term $2Dx^3$, and the other term $2BCx^3$ involves only the preceeding letters B and C; and in the third vertical column $2Ex^4$, $2BDx^4$, C^2x^4, the letter E enters only in the upper term $2Ex^4$, and the two other terms $2BDx^4$, and $C\ x^4$, involve only the preceeding letters B, C, and D; and in the fourth vertical column $2Fx^5$, $2BEx^5$, $2CDx^5$, the letter F enters only in the upper term $2Fx^5$, and the other two terms $2BEx^5$ and $2CDx^5$ involve only the preceeding letters B, C, D, and E.

And it is easy to perceive, from the nature of the multiplication by which the said compound series is obtained, that the same thing would take place in all the following vertical columns of terms in the said compound series, to whatever number of terms the said series should be continued. We may therefore conclude that there will be one, and but one, new capital letter B, or C, or D, or E, or F, &c, contained in every new vertical column of terms in the abovementioned compound series, which is equal to the square of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, and that this new letter will appear only in the upper term of such vertical column.

This is a most important remark, and ought to be well understood and remembered.

In the 5th place it is evident, that when the signs + and −, which are to be prefixed to Bx and Cx^2 and Dx^3, or any greater number of the terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, after the first term 1, have been determined, the signs that are to be prefixed to all the several terms in the next following vertical columns, except the highest term, will be thereby determined likewise. Thus, if the signs + and −, which are to be prefixed to the second, third, and fourth terms, Bx, Cx^2, and Dx^3, of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are determined, the signs of all the terms, except the highest, in the next following vertical column of terms, which involves in it the next co-efficient E, to wit, the column containing the terms $2Ex^4$, $2BDx^4$, C^2x^4, will be determined likewise; because they will involve in them only the old letters B and C and D, which are the co-efficients of the powers of

x in

x in the terms Bx, Cx^2, and Dx^3, of which the ſigns are already ſuppoſed to be determined. For example, if the firſt four terms of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are $1 + Bx - Cx^2 + Dx^3$ (as in truth they are found to be) the ſaid following vertical column of terms, which involves in it the letter E, to wit, the column containing the terms $2Ex^4$, $2BDx^4$, and C^2x^4, will be $2Ex^4 + 2BDx^4 + C^2x^4$. For $2BDx^4$ and C^2x^4 being produced by the multiplications of Bx into Dx^3 and of Cx^2 into Cx^2, it is evident that, when it is aſcertained that the ſign + is to be prefixed to Bx and to Dx^3, and that the ſign — is to be prefixed to Cx^2, it will neceſſarily follow that BDx^4 will be $(= + Bx \times + Dx^3) = + BDx^4$, and conſequently that $2BDx^4$ will be $= + 2BDx^4$, and that $Cx^2 \times Cx^2$ will be $(= - Cx^2 \times - Cx^2) = + C^2x^4$, or that the ſign + muſt be prefixed both to the term $2BDx^4$ and to the term C^2x^4. This is a neceſſary conſequence of the common rules of multiplication in Algebra, according to which when the ſigns of the factors of any product are known or determined, the ſign to be prefixed to the product may be thence determined likewiſe.

These five obſervations being well underſtood and aſſented to as ſufficiently demonſtrated, we may now deduce from them a proof of the propoſition that was ſtated in the latter part of art. 7, to wit, that it is poſſible for a ſeries of the form 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to exiſt, in which the co-efficients C, D, E, F, &c, of the third, fourth, fifth, ſixth, and other following terms of it, to wit, Cx^2, Dx^3, Ex^4, Fx^5, &c, ſhall be of ſuch magnitudes, and the ſaid terms themſelves ſhall be connected with the two firſt terms $1 + Bx$, or $1 + \frac{x}{2}$, by the ſigns + and —, or by addition and ſubtraction, in ſuch a manner, that every following term of the compound ſeries

$$\left.\begin{array}{llllll} 1 + 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & & C^2x^4, & 2CDx^5, & \&c \end{array}\right\},$$

$$\left.\begin{array}{llllll} \text{or } 1 + x & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & & C^2x^4, & 2CDx^5, & \&c \end{array}\right\},$$

ſhall be equal to nothing, or that ſome of its members (for each of theſe terms is evidently a compound quantity, or quantity conſiſting of more than one member) ſhall be marked with the ſign +, and others of them ſhall be marked with the ſign —, and that the ſum of the latter members in each term, which are marked with the ſign —, ſhall be equal to the ſum of the former members in the ſame term, which are marked with the ſign +, ſo as to exactly counterbalance, or deſtroy, the ſaid former members, and make the whole term be equal to nothing; in conſequence of which the whole of the ſaid compound ſeries will be equal to its two firſt terms $1 + 2Bx$, or $1 + 2 \times \frac{1}{2}x$, or $1 + x$. This propoſition may be proved in the manner following.

9. Since the ſecond term Bx of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, is to have the ſign + prefixed to it, ſo that the ſaid ſeries will be 1 + Bx,

$+ Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, it follows (by art. 8, obf. 5) that the quantity B^2x^2, which forms the fecond member of the third term, $2Cx^2$, B^2x^2, of the foregoing compound feries

$$\left.\begin{array}{llllll} 1, \; 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \\ & & & C^2x^4, & 2CDx^5, & \end{array}\right\}$$

muft alfo have the fign $+$ prefixed to it. For it is the product of the multiplication of $+ Bx$ into $+ Bx$, which is $+ B^2x^2$.

And, fince the faid fecond member B^2x^2 of the third term $2Cx^2$, B^2x^2 of the faid compound feries is to have the fign $+$ prefixed to it, it is evident that, in order to make the whole of the faid term $2Cx^2$, B^2x^2, or $2Cx^2 + B^2x^2$, be equal to 0, we muft prefix the contrary fign $-$ to its firft member $2Cx^2$, and at the fame time muft take C of fuch a magnitude that the faid firft member $2Cx^2$ fhall be equal to the fecond member B^2x^2, that is, we muft take C equal to $\frac{B^2}{2}$, or (becaufe B has been already found to be equal to $\frac{1}{2}$) we muft take C equal to $\frac{1}{2} \times \frac{1}{4}$, or $\frac{1}{8}$. And then we fhall have the whole third term $2Cx^2$, $B^2x^2 = -2Cx^2 + B^2x^2 = -2 \times \frac{1}{8}x^2 + \frac{1}{2} \times \frac{1}{2}x^2 = -\frac{1}{4}x^2 + \frac{1}{4}x^2 = 0$.

And, becaufe the fign $-$ is to be prefixed to $2Cx^2$ in the aforefaid compound feries, which is equal to the fquare of the fimple feries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, it muft likewife (by art. 8, obf. 1) be prefixed to the correfponding, or third, term Cx^2 of the faid fimple feries; and confequently the three firft terms of the faid fimple feries will be $1 + Bx - Cx^2$, or $1 + \frac{1}{2}x - \frac{1}{8}xx$, or $1 + \frac{x}{2} - \frac{xx}{8}$.

And therefore, if the three firft terms of the fimple feries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are $1 + \frac{x}{2} - \frac{xx}{8}$, the three firft terms of the foregoing compound feries

$$\left.\begin{array}{llllll} 1, \; 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & & C^2x^4, & 2CDx^5, & \&c \end{array}\right\}$$

(which is equal to the fquare of the faid fimple feries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c) will be

$$\left.\begin{array}{l} 1 + 2Bx - Cx^2 \\ \qquad\quad\;\; + B^2x^2 \end{array}\right\},$$

or $1 + 2 \times \frac{1}{2}x - 0$, or $1 + x - 0$; which are equal to $1 + x$.

 10. The

10. The fourth term of the foregoing compound series is $2Dx^3$, $2BCx^3$, or (because the three first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, have been shewn to be $1 + Bx - Cx^2$) $2Dx^3 - 2BCx^3$.

Now, since the second member, $2BCx^3$, of this fourth term, $2Dx^3 - 2BCx^3$, has the sign $-$ prefixed to it, it is evident that, in order to make the whole term be equal to 0, we must prefix the sign $+$ to its first member $2Dx^3$, and we must at the same time take D of such a magnitude as shall make $2Dx^3$ be exactly equal to $2BCx^3$, that is, we must take $D = BC$, or (because it has been shewn that B is $= \frac{1}{2}$, and that C is $= \frac{1}{8}$) we must take $D = \frac{1}{2} \times \frac{1}{8}$, or $\frac{1}{16}$.

And, because the sign $+$ is to be prefixed to $2Dx^3$ in the aforesaid compound series, which is equal to the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, or $1 + Bx - Cx^2, Dx^3, Ex^4, Fx^5$, &c, it must likewise (by art. 8, obs. 1) be prefixed to the corresponding, or fourth, term, Dx^3, of the said simple series, and consequently the four first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3$, or $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$, or $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$.

And therefore, if the four first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, are $1 + \frac{x}{2} - \frac{xx}{8} + \frac{x^3}{16}$, the four first terms of the foregoing compound series (which is equal to the square of the said simple series) will be

$$\left\{\begin{array}{llll} 1 + 2Bx & - Cx^2 & + 2Dx^3 \\ & + B^2x^2 & - 2BCx^3 \end{array}\right\},$$

or $1 + 2 \times \frac{1}{2}x - 0 + 0$, or $1 + x - 0 + 0$; which are equal to $1 + x$.

11. The fifth term of the foregoing compound series

$$\left\{\begin{array}{lllllll} 1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\ & & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & & & C^2x^4, & 2CDx^5, & \&c \end{array}\right\}$$

or (prefixing the proper signs to the several terms that involve only the letters B, C, and D, which have been already investigated) of the compound series

$$\left\{\begin{array}{lllllll} 1 + 2Bx & - 2Cx^2 & + 2Dx^3, & & 2Ex^4, & 2Fx^5, & \&c \\ & + B^2x^2 & - 2BCx^3 & + & 2BDx^4, & 2BEx^5, & \&c \\ & & & + & C^2x^4 & - 2CDx^5, & \&c \end{array}\right\}$$

is $2Ex^4 + 2BDx^4 + C^2x^4$.

Now, since the sign $+$ is prefixed to the second and third members, $2BDx^4$ and C^2x^4, of this fourth term, it is evident that, in order to make the whole

whole term be equal to o, we must prefix the contrary sign — to its first member $2\,Ex^4$, and must likewise take E of such a magnitude that $2\,Ex^4$ shall be equal to the sum of the other two members $2\,BDx^4$ and C^2x^4; that is, we must take $E = BD + \frac{C^2}{2} = \frac{1}{2} \times \frac{1}{16} + \frac{1}{2} \times \frac{1}{8} \times \frac{1}{8} = \frac{1}{32} + \frac{1}{128} = \frac{4}{128} + \frac{1}{128} = \frac{5}{128}$.

And, because the sign — is to be prefixed to the quantity $2\,Ex^4$ in the foregoing compound series which is equal to the square of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + Bx - Cx^2 + Dx^3$, Ex^4, Fx^5, &c, it must likewise (by art. 8, obs. 1) be prefixed to the corresponding, or fifth, term, Ex^4, of the said simple series; and consequently the five first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3 - Ex^4$, or $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$, or $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$.

And therefore, if the five first terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are $1 + \frac{x}{2} - \frac{xx}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$, the five first terms of the foregoing compound series (which is equal to the square of the said simple series) will be

$$\begin{array}{lllll} 1 + 2\,Bx & - 2\,Cx^2 & + 2\,Dx^3 & - 2\,Ex^4 \\ & + \;B^2x^2 & - 2\,BCx^3 & + 2\,BDx^4 \\ & & & + \;C^2x^4, \end{array}$$

or $1 + 2 \times \frac{1}{2}x - 0 + 0 - 0$, or $1 + x - 0 + 0 - 0$; which are equal to $1 + x$.

12. The sixth term of the foregoing compound series

$$\begin{array}{llllll} 1, \; 2\,Bx, & 2\,Cx^2, & 2\,Dx^3, & 2\,Ex^4, & 2\,Fx^5, & \&c \\ & B^2x^2, & 2\,BCx^3, & 2\,BDx^4, & 2\,BEx^5, & \&c \\ & & & C^2x^4, & 2\,CDx^5, & \&c \end{array}$$

or (prefixing the proper signs to the several terms that involve only the letters B, C, D, and E, which have been already investigated) of the compound series

$$\begin{array}{llllll} 1 + 2\,Bx & - 2\,Cx^2 & + 2\,Dx^3 & - 2\,Ex^4 & 2\,Fx^5 & \&c \\ & + \;B^2x^2 & - 2\,BCx^3 & + 2\,BDx^4 & - 2\,BEx^5 & \&c \\ & & & + \;C^2x^4 & - 2\,CDx^5, & \&c \end{array},$$

is $2\,Fx^5 - 2\,BEx^5 - 2\,CDx^5$.

Now, since the sign — is prefixed to the second and third members, $2\,BEx^5$ and $2\,CDx^5$, of this sixth term, it is evident, that, in order to make the whole term be equal to nothing, we must prefix the contrary sign + to its first member $2\,Fx^5$, and must likewise take F of such a magnitude that the said first

member $2Fx^5$ shall be equal to the sum of the other two members, $2BEx^5$ and $2CDx^5$; that is, we must take F equal to BE + CD, or (because it has been shewn that B is $= \frac{1}{2}$, and that C is $= \frac{1}{8}$, and that D is $= \frac{1}{16}$, and that E is $= \frac{5}{128}$) we must take F equal to $\frac{1}{2} \times \frac{5}{128} + \frac{1}{8} \times \frac{1}{16}$, or to $\frac{5}{256} + \frac{1}{128}$, or to $\frac{5}{256} + \frac{2}{256}$, or to $\frac{7}{256}$.

And, because the sign + is to be prefixed to the quantity $2Fx^5$ in the foregoing compound series which is equal to the square of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + Bx - Cx^2 + Dx^3 - Ex^4$, Fx^5, &c, it must likewise (by art. 8, obs. 1) be prefixed to the corresponding, or sixth, term, Fx^5, of the said simple series; and consequently the six first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5$, or $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5$, or $1 + \frac{x}{2} - \frac{xx}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}$.

And therefore, if the six first terms of the said simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c are $1 + \frac{x}{2} - \frac{xx}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}$, the six first terms of the foregoing compound series (which is equal to the square of the said simple series) will be

$$\left.\begin{array}{llllll} 1 + 2Bx & - 2Cx^2 & + 2Dx^3 & - 2Ex^4 & + 2Fx^5 \\ & + B^2x^2 & - 2BCx^3 & + 2BDx^4 & - 2BEx^5 \\ & & & + C^2x^4 & - 2CDx^5 \end{array}\right\},$$

or $1 + 2 \times \frac{1}{2}x - 0 + 0 - 0 + 0$, or $1 + x - 0 + 0 - 0 + 0$; which are equal to $1 + x$.

13. And in like manner it will evidently be possible to take the following numeral co-efficients G, H, I, K, L, M, &c of x^6, x^7, x^8, x^9, x^{10}, x^{11}, and of the other following powers of x, in the seventh, eighth, ninth, tenth, eleventh, and twelfth, and other following terms of the series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, of such magnitudes, and so to connect the said seventh, eighth, ninth, tenth, eleventh, twelfth, and other following terms of the said series with its first term 1 by the signs + and −, or by addition and subtraction, as to make the whole of the seventh, and of the eighth, and of the ninth, and of the tenth, and of the eleventh, and of the twelfth, and of every following term of the compound series which is the square of the said simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, be equal to 0, and consequently to make the whole of the said compound series be equal to its two first terms $1 + x$. And therefore we may conclude that there is a certain infinite series of the aforesaid form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, which is equal to $\overline{1 + x}|^{\frac{1}{2}}$, or to $\sqrt[2]{1 + x}$, or the square-root of the binomial quantity $1 + x$. Q. E. D.

14. Having

14. Having thus proved the truth of this proposition in the case of the square-root of the binomial quantity $1 + x$, or when the number n in the general expression $\overline{1 + x}|^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$, is equal to 2, we will now proceed to shew that it is also true in the case of the cube-root of $1 + x$, or when n is equal to 3; after which we will endeavour to shew that it will also be true in the case of any other root of $1 + x$, or when n is equal to any other whole number whatsoever.

Of the cube root of the binomial quantity $1 + x$.

15. Now it is evident (as has been already observed in art. 4) that $\overline{1 + x}|^{\frac{1}{3}}$, or $\sqrt[3]{1 + x}$, or the cube-root of the binomial quantity $1 + x$, will be equal to an infinite series of this form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, if it is possible to assign such magnitudes to the numeral co-efficients B, C, D, E, F, &c, of the powers of x in the second and other following terms of this series, and so to connect the said second and other following terms of the said series with the first term 1, by the signs + and −, or by addition and subtraction, that the second term of the compound series which is the cube of the said series, or is the product that arises by multiplying it twice into itself, shall be equal to x and shall have the sign + prefixed to it, or shall be added to 1 (which must evidently be the first term of the said cube or compound series), and that the third term of the said compound series shall be equal to nothing, and that the fourth term of it shall also be equal to nothing, and that every following term of it shall in like manner be equal to nothing, or that each of the said terms, after the said second term, shall consist of two, or more, members, and that some of the said members of each term shall be marked with the sign +, and the other members of the same term shall be marked with the sign −, and that the sum of the latter members of each term, which are marked with the sign −, shall be equal to the sum of the former members of the same term, which are marked with the sign +, so as to counterbalance, or destroy, them, and to make the whole of the said term be equal to nothing. We must therefore shew that it is possible to assign such magnitudes to the numeral co-efficients B, C, D, E, F, &c, and so to connect the terms Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, with its first term 1 by the signs + and −, as to produce the effects just now described.

16. It has been shewn above in art. 6, that, if the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, be multiplied once into itself, the product thence arising, or the square of the said simple series, will be the following compound series, to wit,

$$\left.\begin{array}{llllll} 1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, \&c \\ & & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, \&c \\ & & & & C^2x^4, & 2CDx^5, \&c \end{array}\right\}.$$

Therefore,

Therefore, in order to obtain the cube of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, we must multiply this compound series (which is equal to its square) into the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, itself. This may be done in the manner following.

$$
\begin{array}{lllllll}
1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\
 & & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\
 & & & & C^2x^4, & 2CDx^5, & \&c \\
\hline
1, & Bx, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\
\hline
1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, & \&c \\
 & & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\
 & & & & C^2x^4, & 2CDx^5, & \&c \\
 & Bx, & 2B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\
 & & & B^3x^3, & 2B^2Cx^4, & 2B^2Dx^5 & \&c \\
 & & & & & BC^2x^5, & \&c \\
 & & Cx^2, & 2BCx^3, & 2C^2x^4, & 2CDx^5, & \&c \\
 & & & & B^2Cx^4, & 2BC^2x^5, & \&c \\
 & & & Dx^3, & 2BDx^4, & 2CDx^5, & \&c \\
 & & & & & B^2Dx^5, & \&c \\
 & & & & Ex^4, & 2BEx^5, & \&c \\
 & & & & & Fx^5, & \&c \\
\hline
1, & 3Bx, & 3Cx^2, & 3Dx^3, & 3Ex^4, & 3Fx^5, & \&c \\
 & & 3B^2x^2, & 6BCx^3, & 6BDx^4, & 6BEx^5, & \&c \\
 & & & B^3x^3, & 3C^2x^4, & 6CDx^5, & \&c \\
 & & & & 3B^2Cx^4, & 3B^2Dx^5, & \&c \\
 & & & & & 3BC^2x^5, & \&c.
\end{array}
$$

Therefore this last compound series

$$
\left.\begin{array}{lllllll}
1, & 3Bx, & 3Cx^2, & 3Dx^3, & 3Ex^4, & 3Fx^5, & \&c \\
 & & 3B^2x^2, & 6BCx^3, & 6BDx^4, & 6BEx^5, & \&c \\
 & & & B^3x^3, & 3C^2x^4, & 6CDx^5, & \&c \\
 & & & & 3B^2Cx^4, & 3B^2Dx^5, & \&c \\
 & & & & & 3BC^2x^5, & \&c
\end{array}\right\}
$$

will

will be the cube of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, *ad infinitum*. Let this compound series (for the sake of brevity) be denoted by the Greek capital letter Γ.

We are therefore now to prove that it is possible to assign such magnitudes to the numeral co-efficients B, C, D, E, F, &c, of the powers of x in the second and other following terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, and so to connect the said second and other following terms of the said series with its first term 1 by the signs + and −, that the second term $3Bx$ of the said compound series Γ shall be equal to x and shall have the sign + prefixed to it, or shall be added to the first term 1 of the said series, and that the whole compound third term of the said series, to wit, $3Cx^2$, $3B^2x^2$, shall be equal to 0, and that the whole compound fourth term of the said series, to wit, $3Dx^3$, $6BCx^3$, B^3x^3, shall also be equal to 0, and that the whole compound fifth term, to wit, $3Ex^4$, $6BDx^4$, $3C^2x^4$, $3B^2Cx^4$, shall also be equal to 0, and that the whole compound sixth term, to wit, $3Fx^5$, $6BEx^5$, $6CDx^5$, $3B^2Dx^5$, $3BC^2x^5$, shall also be equal to 0, and, in like manner, that every following compound term shall also be equal to 0, and consequently that the whole of the said compound series Γ shall be equal to its two first terms, or to the binomial quantity $1 + x$.

17. Now, to the end that the second term $3Bx$ of the compound series Γ may have the sign + prefixed to it, we need only prefix the same sign + to the second term Bx of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c; it being evident from the rules of multiplication that, if the sign + be prefixed to the term Bx in the series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, it must likewise be prefixed to the term $3Bx$ in the series Γ, which arises by multiplying the said series twice into itself.

And to the end that $3Bx$, or the second term of the compound series Γ, may be equal to x, we need only suppose that $3B$ is equal to 1, or that B is equal to $\frac{1}{3}$. For then we shall have $3Bx = 3 \times \frac{1}{3} \times x = \frac{3}{3} \times x = x$.

Therefore, if the two first terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, be $1 + \frac{1}{3}x$, or $1 + \frac{x}{3}$, the two first terms of the compound series Γ will be $1 + x$.

18. In the next place we must observe that in the compound series Γ (which is equal to the cube of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c) as well as in the former compound series (which was equal to the square of the said series) one new capital letter C, or D, or E, or F, &c, and but one, will enter into every new compound term, or vertical column of terms; and this new letter will appear in the highest term of every such vertical column, and only in the said highest term, and the other terms of such vertical column, or those which are placed immediately under such highest term, will involve in them only such of the capital letters B, C, D, E, F, &c, as had occurred in the preceeding compound

 pound

pound terms, or vertical columns of terms. And, further, when the magnitudes of any number of the ſaid capital letters, B, C, D, E, F, &c (or co-efficients of the powers of x in the terms of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c) have been determined, and the ſigns + and —, which are to be prefixed to thoſe of the terms Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c of the ſaid ſimple ſeries, in which the ſaid letters are involved, have been likewiſe determined, the magnitudes of all the quantities in the compound ſeries Γ which involve the ſaid letters of which the magnitudes have been determined, and the ſigns + and —, that are to be prefixed to ſuch quantities, will all be determined likewiſe.

Theſe obſervations are obvious conſequences from the rules of algebraïck multiplication and the manner in which the compound ſeries Γ has been derived from the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to wit, by multiplying it twice into itſelf. And they are true likewiſe, with reſpect to all higher powers of the ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, whatſoever, as well as with reſpect to its cube.

19. Since, by art. 17, it appears that Bx, or the ſecond term of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, is to have the ſign + prefixed to it, it follows that the ſame ſign + muſt alſo be prefixed to the quantity $3B^2x^2$, which is the ſecond member of the third term, $3Cx^2$, $3B^2x^2$, of the compound ſeries Γ, ſo that the ſaid third term will be $3Cx^2 + 3B^2x^2$. Therefore, to the end that the whole of the ſaid third term $3Cx^2 + 3B^2x^2$ may be equal to nothing, we muſt prefix the contrary ſign — to the firſt member of it, to wit, $3Cx^2$, and muſt likewiſe take C of ſuch a magnitude that $3Cx^2$ ſhall be equal to $3B^2x^2$, or that $3C$ ſhall be equal to $3B^2$, that is, we muſt take C equal to B^2, or to $\frac{1}{3} \times \frac{1}{3}$ or $\frac{1}{9}$. And then we ſhall have the ſaid third term $3Cx^2 + 3B^2x^2 = -3Cx^2 + 3B^2x^2 = -3 \times \frac{1}{9}x^2 + 3 \times \frac{1}{9}x^2 = 0$.

And, becauſe $3Cx^2$ in the compound ſeries Γ has the ſign — prefixed to it, the third term Cx^2 in the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, which correſponds to it, muſt alſo have the ſign — prefixed to it, and conſequently the three firſt terms of the ſaid ſimple ſeries will be $1 + Bx - Cx^2$, or $1 + \frac{1}{3}x - \frac{1}{9}xx$, or $1 + \frac{x}{3} - \frac{xx}{9}$.

And, therefore, if the three firſt terms of the ſaid ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are $1 + \frac{x}{3} - \frac{xx}{9}$, the three firſt terms of the compound ſeries Γ (which is equal to the cube of the ſaid ſimple ſeries) will be

$$\begin{array}{l} 1 + 3Bx - 3Cx^2 \\ + 3B^2x^2, \end{array}$$

or $1 + 3 \times \frac{1}{3}x - 0$, or $1 + x - 0$; which are equal to $1 + x$.

20. The

20. The fourth term of the compound series Γ is $3\,Dx^3$, $6\,BCx^3$, B^3x^3, or (because the three first terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, have been found to be $1 + Bx - Cx^2$) $3\,Dx^3 - 6\,BCx^3 + B^3x^3$, or (because B has been found to be equal to $\frac{1}{3}$, and C has been found to be equal to $\frac{1}{9}$) $3\,Dx^3 - 6 \times \frac{1}{3} \times \frac{1}{9}x^3 + \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}x^3$, or $3\,Dx^3 - \frac{6}{27}x^3 + \frac{1}{27}x^3$, or $3\,Dx^3 - \frac{5}{27}x^3$. Now, since the sign $-$ is prefixed to the second member of this term, to wit, $\frac{5}{27}x^3$, it is evident that, in order to make the whole of the said term be equal to o, we must prefix the sign $+$ to the first member of it, $3\,Dx^3$, and we must also take D of such a magnitude as will make the said first member of it $3\,Dx^3$ be equal to the second member $\frac{5}{27}x^3$, that is, we must take D equal to $\frac{1}{3} \times \frac{5}{27}$, or $\frac{5}{81}$. And, because the sign $+$ is to be prefixed to the quantity $3\,Dx^3$ in the compound series Γ, we must prefix the same sign $+$ to the correspondent term Dx^3 of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, and consequently the four first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3$, or $1 + \frac{1}{3}x - \frac{1}{9}xx + \frac{5}{81}x^3$, or $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81}$.

And therefore, if the four first terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81}$, the four first terms of the compound series Γ (which is equal to the cube of the said simple series) will be $1 + 3\,Bx - o + o$, or $1 + 3 \times \frac{1}{3}x - o + o$, or $1 + x - o + o$; which are equal to $1 + x$.

21. The fifth term of the said compound series Γ is $3\,Ex^4$, $6\,BDx^4$, $3\,C^2x^4$, $3\,B^2Cx^4$, or (prefixing the proper signs $+$ and $-$ to the three last members of this term, to wit, $6\,BDx^4$, $3\,C^2x^4$, and $3\,B^2Cx^4$, which involve only the co-efficients B, C, and D, which have already been investigated) $3\,Ex^4 + 6\,BDx^4 + 3\,C^2x^4 - 3\,B^2Cx^4$, or (because B has been found to be $= \frac{1}{3}$, and C has been found to be $= \frac{1}{9}$, and D has been found to be $= \frac{5}{81}$) $3\,Ex^4 + 6 \times \frac{1}{3} \times \frac{5}{81}x^4 + 3 \times \frac{1}{9} \times \frac{1}{9}x^4 - 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{9}x^4$, or $3\,Ex^4 + \frac{10}{81}x^4 + \frac{3}{81}x^4 - \frac{3}{81}x^4$, or $3\,Ex^4 + \frac{10}{81}x^4$. Now, in order to make this whole term, $3\,Ex^4 + \frac{10}{81}x^4$ (of which the second member $\frac{10}{81}x^4$ is marked with the sign $+$) be equal to o, we must prefix the contrary sign $-$ to its first member $3\,Ex^4$, and we must likewise take E of such a magnitude that $3\,Ex^4$ shall be equal to $\frac{10}{81}x^4$, that is, we must take $E = \frac{1}{3} \times \frac{10}{81} = \frac{10}{243}$. And, because the

the ſign $-$ is to be prefixed to $3Ex^4$ in the compound ſeries Γ, it muſt alſo be prefixed to the correſponding term, Ex^4, in the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c; and conſequently the five firſt terms of the ſaid ſimple ſeries will be $1 + Bx - Cx^2 + Dx^3 - Ex^4$, or $1 + \frac{1}{3}x - \frac{1}{9}xx + \frac{5}{81}x^3 - \frac{10}{243}x^4$, or $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$.

And therefore, if the five firſt terms of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, are $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$, the firſt five terms of the compound ſeries Γ (which is equal to the cube of the ſaid ſeries) will be $1 + 3Bx - 0 + 0 - 0$, or $1 + 3 \times \frac{1}{3}x - 0 + 0 - 0$, or $1 + x - 0 + 0 - 0$, which are equal to $1 + x$.

22. And the ſixth term of the ſaid compound ſeries Γ is $3Fx^5$, $6BEx^5$, $6CDx^5$, $3B^2Dx^5$, $3BC^2x^5$, or (prefixing the proper ſigns $+$ and $-$ to the ſecond, and third, and fourth, and fifth members of this term, to wit, $6BEx^5$, $6CDx^5$, $3B^2Dx^5$, and $3BC^2x^5$, which involve only the co-efficients B, C, D, and E, which have already been inveſtigated) $3Fx^5 - 6BEx^5 - 6CDx^5 + 3B^2Dx^5 + 3BC^2x^5$, or (becauſe B has been found to be $= \frac{1}{3}$, and C to be $= \frac{1}{9}$, and D to be $= \frac{5}{81}$, and E to be $= \frac{10}{243}$) $3Fx^5 - 6 \times \frac{1}{3} \times \frac{10}{243}x^5 - 6 \times \frac{1}{9} \times \frac{5}{81}x^5 + 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{5}{81}x^5 + 3 \times \frac{1}{3} \times \frac{1}{9} \times \frac{1}{9}x^5$, or $3Fx^5 - \frac{20}{243}x^5 - \frac{10}{243}x^5 + \frac{5}{243}x^5 + \frac{3}{243}x^5$, or $3Fx^5 - \frac{30}{243}x^5 + \frac{8}{243}x^5$, or $3Fx^5 - \frac{22}{243}x^5$. Now, in order to make this whole term $3Fx^5 - \frac{22}{243}x^5$ (of which the ſecond member $\frac{22}{243}x^5$ is marked with the ſign $-$) be equal to 0, we muſt prefix the contrary ſign $+$ to its firſt member $3Fx^5$, and we muſt likewiſe take F of ſuch a magnitude that $3Fx^5$ ſhall be equal to $\frac{22}{243}x^5$, that is, we muſt take $F = \frac{1}{3} \times \frac{22}{243}$, or $\frac{22}{729}$. And, becauſe the ſign $+$ is to be prefixed to $3Fx^5$ in the compound ſeries Γ, it muſt alſo be prefixed to the correſponding term Fx^5 of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c; and conſequently the firſt ſix terms of the ſaid ſeries will be $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5$, or $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \frac{22}{729}x^5$, or $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729}$.

And therefore, if the ſix firſt terms of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c, are $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729}$, the firſt ſix terms of the compound ſeries Γ (which is equal to the cube of the ſaid ſeries) will

will be $1 + 3Bx - 0 + 0 - 0 + 0$, or $1 + 3 \times \frac{1}{3}x - 0 + 0 - 0 + 0$, or $1 + x - 0 + 0 - 0 + 0$; which are equal to $1 + x$.

23. And in the ſame manner it will evidently be poſſible to take the following numeral co-efficients, G, H, I, K, L, M, &c, of x^6, x^7, x^8, x^9, x^{10}, x^{11}, &c, in the 7th, 8th, 9th, 10th, 11th, 12th, and other following terms of the ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, of ſuch magnitudes, and ſo to connect the ſaid 7th, 8th, 9th, 10th, 11th, 12th, and other following terms of the ſaid ſeries with its firſt term 1 by the ſigns + and −, or by addition and ſubtraction, as to make the whole of the ſeventh term, and the whole of the eighth term, and the whole of the ninth term, and of the tenth term, and of the eleventh term, and of the twelfth term, and of every following term, of the compound ſeries Γ (which is equal to the cube of the ſaid ſimple ſeries) be equal to nothing, and conſequently to make the whole of the ſaid compound ſeries be equal to its two firſt terms $1 + x$. And therefore we may conclude that there is a certain infinite ſeries of the aforeſaid form, to wit, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, which is equal to $\overline{1+x}|^{\frac{1}{3}}$, or to $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $1 + x$.

Q. E. D.

24. We have now ſhewn that an infinite ſeries of the foregoing form, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, may exiſt, that ſhall be equal to $\overline{1+x}|^{\frac{1}{2}}$, or $\sqrt[2]{1+x}$, or the ſquare-root of the binomial quantity $1 + x$; and that another infinite ſeries of the ſame form, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, but with different values of the co-efficients B, C, D, E, F, &c, may exiſt, that ſhall be equal to $\overline{1+x}|^{\frac{1}{3}}$, or $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $1 + x$. It remains that we prove that, if n be any other whole number whatſoever greater than 3, it will always be poſſible for an infinite ſeries of the ſame form 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to exiſt, that ſhall be equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, or the nth root of the binomial quantity $1 + x$.

Of the n*th root of the binomial quantity* $1 + x$, *when* n *is any whole number whatſoever.*

25. Now the poſſibility of the exiſtence of ſuch a ſeries in all values of the index n may be deduced from an attentive conſideration of the two foregoing compound ſerieſes (which are equal to the ſquare and the cube of the ſaid ſimple ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c) and of the ſeveral properties

of

of their terms described above in art. 8, and resulting from the nature of multiplication. For, it will appear, upon such a consideration of those serieses, that, if we were to raise the said simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, to the fourth power, and to the fifth power, and to the sixth power, and to the seventh power, and to any greater number of its following powers whatsoever, by continual multiplications of the next preceeding powers of it into the said series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, itself, the observations contained in art. 8 would always be true of all these powers, or products, as well as of the two former and less complicated compound serieses which are equal to its square and cube.

26. And hence it follows that, if the said simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, were to be raised to the nth power (n being any whole number whatsoever) the first, or upper, horizontal row of terms in the compound series that would be equal to such nth power of it, would be 1, nBx, nCx^2, nDx^3, nEx^4, nFx^5, &c *ad infinitum*; just as, when n was $= 2$, we have seen that it was 1, $2Bx$, $2Cx^2$, $2Dx^3$, $2Ex^4$, $2Fx^5$, &c; and, when n was $= 3$, we have seen that it was 1, $3Bx$, $3Cx^2$, $3Dx^3$, $3Ex^4$, $3Fx^5$, &c.

And it will likewise be evident that the second term, or term involving x, in the said compound series will be only a single quantity, to wit, nBx, but that every following term of the said compound series involving any of the following powers of x, to wit, xx, x^3, x^4, x^5, x^6, x^7, &c, will be a compound term, or will consist of two, or more, single quantities.

And, 3dly, it will be evident, that in the third term, or term involving xx (which will be the first compound term, or vertical column of terms) the capital letter C, which enters in the highest term of the said vertical column, to wit, in the term nCx^2, will not be contained in the other, or lower, terms of the same vertical column; but the said lower terms will only involve the preceeding capital letter B. And, in like manner, the capital letter D, which enters in the highest term of the next vertical column of terms, involving x^3, to wit, in the term nDx^3, will not be contained in the other, or lower, terms of the same vertical column; but the said lower terms will only involve the preceeding capital letters B and C. And, in like manner, the capital letter E, or F, or G, or H, &c, that will enter in the highest term of any subsequent vertical column of terms, will not be contained in the other, or lower, terms of the same vertical column; but the said lower terms will involve only the capital letters preceeding the said capital letter that enters in the highest term of the said vertical column.

And therefore, 4thly, if the magnitudes of the capital letters contained in any given number of terms in the first, or upper, horizontal row of terms of the said compound series, to wit, 1, nBx, nCx^2, nDx^3, nEx^4, nFx^5, &c, have been determined, and the signs + and −, which are to be prefixed to the said terms in

in the said horizontal row, have also been determined, and consequently the signs which are to be prefixed to the corresponding terms of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c (which are always the same with the former signs, by art. 8, obs. 1.) have also been determined, the magnitudes of all the terms of the next following vertical column of terms, except the highest term, will be thereby determined; and likewise the signs + and − which are to be prefixed to all the said terms of the said next following vertical column of terms, except the highest term, will also be thereby determined.

And therefore, 5thly, since both the magnitudes of all the terms in the said next vertical column of terms, except the highest term, and the signs + and −, that are to be prefixed to the said terms, are determined, in the case here supposed, or when the preceeding terms of the first, or upper horizontal row of terms 1, nBx, nCx^2, nDx^3, nEx^4, nFx^5, &c, and the signs + and −, that are to be prefixed to them, have been determined, it follows that in the same case *the resulting value* of all the terms of the said next vertical column of terms, except the highest term, or the value resulting from the computation of their separate values, and the addition of those separate values to each other, or the subtraction of some of them from the others, or from the first term 1, of the said compound series, according as the signs + or − are prefixed to them, will likewise be determined.

27. If these conclusions are allowed to be just, they will enable us to prove that the said compound series (which is equal to the nth power of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c) may be made to be equal to the binomial quantity $1 + x$, and consequently that the said simple series may be made to be equal to the nth root of the said binomial quantity, by taking the co-efficients B, C, D, E, F, &c, of certain proper magnitudes, and by connecting the second, and third, and other following, terms of the said simple series, to wit, the terms Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, with its first term 1 by the signs + and −, or by addition and subtraction, in a certain proper manner;—I say, the foregoing conclusions, if admitted to be just, will enable us to prove this proposition in the manner following.

In the first place, let the sign + be prefixed to the second term Bx of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c. And it will follow that the second term nBx of the compound series which is equal to the nth power of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, must likewise have the same sign + prefixed to it; and consequently the two first terms of the said compound series will be $1 + nBx$.

Secondly, let us suppose B to be equal to $\frac{1}{n}$. Then will nBx be $= n \times \frac{1}{n}x = x$, and consequently the two first terms $1 + nBx$ of the compound series which

which is equal to the nth power of the simple series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, will be $1 + n \times \frac{1}{n}x$, or $1 + x$.

Thirdly, since the sign + is prefixed to the second term Bx of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, and B has been taken = to $\frac{1}{n}$, let the result of the value of the lower term of the first vertical column of terms in the said compound series (in which lower term only the capital letter B will enter) be computed, and be marked with its proper sign + or −; and then let the contrary sign be prefixed to nCx^2, or the upper term of the said vertical column; and let C be taken of such a magnitude as to make nCx^2 equal to the said result. And, lastly, let the same sign + or −, which is prefixed to nCx^2, be also prefixed to the corresponding term, Cx^2, of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c. And the consequence will be that the whole of the said vertical column, or third term of the said compound series, will be equal to 0, and consequently that the three first terms of the said compound series will be $1 + x$, 0; which are equal to $1 + x$.

Fourthly, since the values of B and C are now determined, and likewise the signs + and −, which are to be prefixed to the terms Bx and Cx^2 in the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, let the values of the several lower terms of the vertical column of terms which involves x^3, and of which the highest term is nDx^3, be computed; which will evidently be possible, because all the said lower terms will involve only the two capital letters B and C, which have been already determined: and let the values of the said several lower terms be added to each other, or some of them be subtracted from the others, or from the first term 1 of the said compound series, according as the sign + or the sign − is prefixed to them; and let the result of such additions and subtractions be marked with its proper sign + or −. And then let the contrary sign be prefixed to the highest term, nDx^3, of the said vertical column, which involves the new capital letter D; and let the said capital letter D be taken of such a magnitude that the said highest term nDx^3 shall be equal to the said result of the values of all the lower terms placed under it in the same vertical column. And, lastly, let the same sign which is prefixed to nDx^3, be also prefixed to the corresponding term Dx^3 of the simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c. And it is evident that the whole of the said vertical column of terms of which nDx^3 is the highest term, or the whole of the fourth term of the said compound series will be equal to 0, and consequently that the four first terms of the said compound series will be $1 + x$, 0, 0; which are equal to $1 + x$.

And, in like manner, by computing the several lower terms of the next vertical column of terms, of which nEx^4 is the highest term, and finding the result of them, and prefixing the proper sign + or − to such result, and then prefixing the contrary sign to the said highest term nEx^4, and supposing E to be of such a magnitude as to make nEx^4 be equal to the said result, and, lastly, prefixing to the corresponding term, Ex^4, of the said simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, the same sign as was prefixed to the said highest term, nEx^4,

$n\,E\,x^4$, of the said vertical column involving x^4, we shall make the whole of the said vertical column, or the whole fifth term of the said compound series, to be equal to 0, and consequently the five first terms of it to be $1 + x$, 0, 0, 0; which are equal to $1 + x$.

And in the same manner it will be possible to assign certain proper magnitudes, or values, to the following capital letters F, G, H, I, K, L, M, &c, which are involved in the following terms Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, of the said simple series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c (to whatever number of terms the said series may be continued) and to connect the said terms with the first term 1 of the said series by the signs + and —, or by addition and subtraction, in such a manner, that the whole of the sixth term, and of every following term, of the compound series that is equal to the nth power of the said simple series, or that arises by the multiplication of the said series $n - 1$ times into itself, shall also be equal to 0, and consequently that the whole of the said compound series shall be $1 + x$, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, &c *ad infinitum*; which are equal to $1 + x$. And therefore an infinite series of the foregoing form, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, may exist, which shall be equal to $\overline{1 + x}|^{\frac{1}{n}}$, or to $\sqrt[n]{1 + x}$, or to the nth root of the binomial quantity $1 + x$.

Q. E. D.

An Investigation of the Series

$$1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \&c,$$

which is set down above in Art. 2, as being equal to

$$\overline{1+x}\big|^{\frac{1}{n}} \text{ or } \sqrt[n]{1+x}.$$

28. Let 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, be the series that is equal to $\overline{1+x}\big|^{\frac{1}{n}}$, or $\sqrt[n]{\overline{1+x}}$; in which series the capital letters B, C, D, E, F, &c, stand for certain numbers, or numeral co-efficients of the several powers of x, which are hitherto unknown, and which it is the object of this investigation to discover. And further, since we have seen that the second term, Bx, of this series, is always to be added to the first term 1, let us prefix the sign + to the said second term; and then we shall have $\overline{1+x}\big|^{\frac{1}{n}}$ = the series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c.

29. Further, let all the terms of the said series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, except the first, be represented by the letter y; so that $\overline{1+x}\big|^{\frac{1}{n}}$ shall be equal $1 + y$.

Then will the nth power of $\overline{1+x}\big|^{\frac{1}{n}}$ be equal to the nth power of $1+y$; that is, $1 + x$ will be $= \overline{1+y}\big|^{n} =$ (by the binomial theorem in the case of integral powers, which has been already demonstrated above in the last tract but one, in pages 153, 154, &c, ... 169,) to the series $1 + \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c$; and consequently, (subtracting 1 from both sides,) x will be equal to the series $\frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c.$

For the sake of brevity, let the small letters c, d, e, and f, &c, be substituted instead of $\frac{n}{1} \times \frac{n-1}{2}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c, respectively, in the last equation. And we shall then have $x = ny + cy^2 + dy^3 + ey^4 + fy^5 + \&c.$

30. In the next place let the series Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to y, be raised to the second, third, fourth, fifth, and other following

 powers,

powers, in order to have the values of y^2, y^3, y^4, y^5, &c, expressed in the powers of x. This may be done in the manner following.

$$\begin{array}{rlllll}
y = & Bx, & Cx^2, & Dx^3, & Ex^4, & Fx^5, \&c \\
y = & Bx, & Cx^2, & Dx^3, & Ex^4, & Fx^5, \&c \\
\hline
 & B^2x^2, & BCx^3, & BDx^4, & BEx^5, & \&c \\
 & & BCx^3, & C^2x^4, & CDx^5, & \&c \\
 & & & BDx^4, & CDx^5, & \&c \\
 & & & & BEx^5, & \&c \\
\hline
y^2 = \Big\{ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\
 & & & C^2x^4, & 2CDx^5, & \&c
\end{array}$$

$$\begin{array}{rllll}
y = & Bx, & Cx^2, & Dx^3, & \&c \\
\hline
 & B^3x^3, & 2B^2Cx^4, & 2B^2Dx^5, & \&c \\
 & & B^2Cx^4, & BC^2x^5, & \&c \\
 & & & 2BC^2x^5, & \&c \\
 & & & B^2Dx^5, & \&c \\
\hline
y^3 = \Big\{ & B^3x^3, & 3B^2Cx^4, & 3B^2Dx^5, & \&c \\
 & & & 3BC^2x^5, & \&c
\end{array}$$

$$\begin{array}{rlll}
y = & Bx, & Cx^2, & \&c \\
\hline
 & B^4x^4, & 3B^3Cx^5, & \&c \\
 & & B^3Cx^5, & \&c \\
\hline
y^4 = & B^4x^4, & 4B^3Cx^5, & \&c \\
y = & Bx, & \&c & \\
\hline
y^5 = & B^5x^5, & \&c &
\end{array}$$

Now let these values of y, y^2, y^3, y^4, y^5, &c, be substituted instead of y, y^2, y^3, y^4, y^5, &c, respectively, in the equation $x = ny + cy^2 + dy^3 + ey^4 + fy^5 + \&c$. And we shall have

$$x = \left\{\begin{array}{llllll}
nBx, & nCx^2, & nDx^3, & nEx^4, & nFx^5, & \&c \\
 & +cB^2x^2, & 2cBCx^3, & 2cBDx^4, & 2cBEx^5, & \&c \\
 & & & 2cC^2x^4, & 2cCDx^5, & \&c \\
 & & +dB^3x^3, & 3dB^2Cx^4, & 3dB^2Dx^5, & \&c \\
 & & & & 3dBC^2x^5, & \&c \\
 & & & +eB^4x^4, & 4eB^3Cx^5, & \&c \\
 & & & & +fB^5x^5, & \&c.
\end{array}\right.$$

By means of this equation we may determine the values of as many of the co-efficients B, C, D, E, F, &c, as we think proper, by proceeding in the manner following.

31. To

31. To find the value of B, we muſt proceed as follows. Divide all the terms of this equation by x; and we ſhall have

$$1 = \left\{ \begin{array}{llllll} nB, & nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\ & +cB^2x, & 2cBCx^2, & 2cBDx^3, & 2cBEx^4, & \&c \\ & & & 2cC^2x^3, & 2cCDx^4, & \&c \\ & & +dB^3x^2, & 3dB^2Cx^3, & 3dB^2Dx^4, & \&c \\ & & & & 3dBC^2x^4, & \&c \\ & & & +eB^4x^3, & 4eB^3Cx^4, & \&c \\ & & & & +fB^5x^4, & \&c \end{array} \right.$$

This equation is always true, of how ſmall a magnitude ſoever we ſuppoſe x to be taken. And therefore it will be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x in them will be equal to 0 likewiſe, and conſequently the equation will be $1 = nB$. Therefore B will be $= \frac{1}{n}$; and conſequently the two firſt terms of the ſeries $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$ will be $1 + \frac{1}{n}x$, or $1 + \frac{x}{n}$. Q. E. I.

This value of the co-efficient B agrees with that which we before found for it in Art. 23, in which it appeared that the three firſt terms of the ſeries that is equal to $\overline{1+x}|^{\frac{1}{n}}$ would be $1 + \frac{x}{n} - \left|\frac{n-1}{2nn}\right. xx$, or $1 + \frac{x}{n} - \left|\frac{n-1}{2nn}\right. xx$.

32. To find the value of C, the co-efficient of the third term, Cx^2, of the ſaid ſeries $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, and likewiſe to determine, which of the two ſigns + and − is to be prefixed to the ſaid term, or whether the ſaid term is to be added to the two foregoing terms $1 + Bx$, or $1 + \frac{x}{n}$, or to be ſubtracted from them, we muſt proceed as follows.

Since 1, which forms the left-hand ſide of the equation obtained in Art. 31, is equal to nB, which is the firſt term of the upper line of terms on the right-hand ſide of that equation, it follows that all the other terms on the right-hand ſide of that equation, taken together, muſt be equal to 0; that is, the compound ſeries

$$\begin{array}{llllll} nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\ +cB^2x, & 2cBCx^2, & 2cBDx^3, & 2cBEx^4, & \&c \\ & & 2cC^2x^3, & 2cCDx^4, & \&c \\ & +dB^3x^2, & 3dB^2Cx^3, & 3dB^2Dx^4, & \&c \\ & & & 3dBC^2x^4, & \&c \\ & & +eB^4x^3, & 4eB^3Cx^4, & \&c \\ & & & +fB^5x^4, & \&c \end{array}$$

will be $= 0$; that is, ſome of the terms of this ſeries muſt be ſubtracted from the others, and the ſum of thoſe that are ſo ſubtracted muſt be equal to the ſum of the other terms from which they are ſubtracted.

Now let all the terms of this compound series be divided by x. And it is evident that the terms that were subtracted from the others, and which were equal to them before such division, will still be equal to them after it. Therefore the new series that will result from such division, will still be $= 0$; that is, the compound series

$$\begin{array}{rrrrl} nC, & nDx, & nEx^2, & nFx^3, & \&c \\ +cB^2, & 2cBCx, & 2cBDx^2, & 2cBEx^3, & \&c \\ & & 2cC^2x^2, & 2cCDx^3, & \&c \\ & +dB^3x, & 3dB^2Cx^2, & 3dB^2Dx^3, & \&c \\ & & & 3dBC^2x^3, & \&c \\ & & +eB^4x^2, & 4eB^3Cx^3, & \&c \\ & & & +fB^5x^3, & \&c \end{array}$$

will be $= 0$. And this equation will be true in all the possible magnitudes of x; and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms of the said series that involve x in them will be equal to 0 likewise, and the whole series will be reduced to the two terms $nC + cB^2$. Therefore these two terms $nC + cB^2$ will be $= 0$; and consequently nC must be subtracted from cB^2, and marked with the sign $-$. For, if it were to be added to cB^2, it would not be possible that $nC + cB^2$, (which on that supposition would mean the sum of the two quantities nC and cB^2) could be equal to 0. Therefore nC must have the sign $-$ prefixed to it; and consequently the third term, Cx^2, of the assumed series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is $= \overline{1+x}|^{\frac{1}{n}}$, (from which third term the said quantity nC has been derived by multiplication and division in the course of this investigation,) must likewise have the sign $-$ prefixed to it, and must be subtracted from the two former terms $1 + Bx$, or $1 + \frac{x}{n}$; so that the three first terms of the said assumed series, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be $1 + Bx - Cx^2$, or $1 + \frac{x}{n} - Cx^2$.

And, for determining the magnitude of the co-efficient C, we shall have the equation $-nC + cB^2 = 0$; whence cB^2 will be $= nC$, and C will be $= \frac{1}{n} \times cB^2 = \frac{1}{n} \times \frac{n}{1} \times \frac{n-1}{2} \times B^2 = \frac{n-1}{2} \times B^2 = \frac{n-1}{2} \times \frac{1}{nn} = \frac{n-1}{2n} \times \frac{1}{n}$. Therefore the three first terms of the assumed series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be $1 + \frac{x}{n} - \frac{1}{n} \times \frac{n-1}{2n}xx$, or $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bxx$.

Q. E. I.

33. To find the value of D, the co-efficient of Dx^3, the fourth term of the assumed series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal $\overline{1+x}|^{\frac{1}{n}}$, and

and to determine whether the sign + or the sign − is to be prefixed to the said fourth term, we must proceed as follows.

In the course of the last article we obtained the following equation; to wit,

$$\left.\begin{array}{rrrrl}
nC, & nDx, & nEx^2, & nFx^3, & \&c \\
+\,cB^2, & 2cBCx, & 2cBDx^2, & 2cBEx^3, & \&c \\
 & & 2cC^2x^2, & 2cCDx^3, & \&c \\
 & +\,dB^3x, & 3dB^2Cx^2, & 3dB^2Dx^3, & \&c \\
 & & & 3dBC^2x^3, & \&c \\
 & & +\,eB^4x^2, & 4eB^3Cx^3, & \&c \\
 & & & +\,fB^5x^3, & \&c
\end{array}\right\} = 0.$$

And it was also found in the last article, that the term nC in this equation was to have the sign − prefixed to it. Therefore the terms $2cBCx$ and $3dB^2Cx^2$, and $4eB^3Cx^3$, in the same equation, (which contain the simple power of C, or are produced by multiplications into C,) must likewise have the sign − prefixed to them; but the terms $2cC^2x^2$ and $3dBC^2x^3$, (which contain the square of C,) must have the sign + prefixed to them, because $-C \times -C$ is $= +C^2$. Therefore, if these several terms have their proper signs prefixed to them, the said equation will be as follows; to wit,

$$\left.\begin{array}{rrrrl}
-\,nC, & nDx, & nEx^2, & nFx^3, & \&c \\
+\,cB^2 & -\,2cBCx, & 2cBDx^2, & 2cBEx^3, & \&c \\
 & & +\,2cC^2x^2, & 2cCDx^3, & \&c \\
 & +\,dB^3x & -\,3dB^2Cx^2, & 3dB^2Dx^3, & \&c \\
 & & & +\,3dBC^2x^3, & \&c \\
 & & +\,eB^4x^2 & -\,4eB^3Cx^3, & \&c \\
 & & & +\,fB^5x^3, & \&c
\end{array}\right\} = 0.$$

But it has been also shown in the last article, that $-nC + cB^2$ is $= 0$. Therefore, if we drop these two terms, the remaining terms on the left-hand side of the equation will still be equal to 0; that is,

$$\left.\begin{array}{rrrl}
nDx, & nEx^2, & nFx^3, & \&c \\
-\,2cBCx, & 2cBDx^2, & 2cBEx^3, & \&c \\
 & +\,2cC^2x^2, & 2cCDx^3, & \&c \\
+\,dB^3x & -\,3dB^2Cx^2, & 3dB^2Dx^3, & \&c \\
 & & +\,3dBC^2x^3, & \&c \\
 & +\,eB^4x^2 & -\,4eB^3Cx^3, & \&c \\
 & & +\,fB^5x^3, & \&c
\end{array}\right\} \text{will be} = 0.$$

And, if we divide all the terms on the left-hand side of this equation by x, the equation will still be true, or the terms on the left-hand side of it will still be equal to 0, or those of the said terms which are marked, or to be marked, with the sign −, and are to be subtracted from the other terms, will still be equal to the terms from which they are subtracted; and therefore the equation will be as follows, to wit,

$$\left.\begin{array}{llll} nD, & nEx, & nFx^2, & \&c \\ -2cBC, & 2cBDx, & 2cBEx^2, & \&c \\ & +2cC^2x, & 2cCDx^2, & \&c \\ +dB^3 & -3dB^2Cx, & 3dB^2Dx^2, & \&c \\ & & +3dBC^2x^2, & \&c \\ & +eB^4x & -4eB^3Cx^2, & \&c \\ & & +fB^5x^2, & \&c \end{array}\right\} = 0.$$

And this equation will be true in all the poſſible magnitudes of x: and therefore it will alſo be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewiſe, and conſequently the equation will be $nD - 2cBC + dB^3 = 0$. Therefore, by adding $2cBC$ to both ſides, we ſhall have $nD + dB^3 = 2cBC$, that is, (becauſe d is $= \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = c \times \frac{n-2}{3}$) $nD + \frac{n-2}{3} \times cB^3 = 2cBC$, or (becauſe B is $= \frac{1}{n}$) $nD + \frac{n-2}{3} \times c \times \frac{1}{n^3} = 2c \times \frac{1}{n} \times C$, or (becauſe C is $= \frac{1}{n} \times \frac{n-1}{2n}$) $nD + \frac{n-2}{3} \times c \times \frac{1}{n^3} = 2c \times \frac{1}{n} \times \frac{1}{n} \times \frac{n-1}{2n}$, or $nD + \frac{n-2}{3n} \times \frac{1}{nn} \times c = \frac{n-1}{n} \times \frac{1}{nn} \times c$, or $nD + \frac{n-2}{3n^3} \times c = \frac{n-1}{n^3} \times c = \frac{3n-3}{3n^3} \times c$, and (adding $\frac{3}{3n^3} \times c$ to both ſides,) $nD + \frac{n+1}{3n^3} \times c = \frac{3n}{3n^3} \times c$. Now, becauſe n is a whole number, and conſequently greater than 1, $n + n$ muſt be greater than $n + 1$, that is, $2n$ muſt be greater than $n + 1$, and, *à fortiori*, $3n$ (which is greater than $2n$,) will be greater than $n + 1$. Therefore $\frac{3n}{3n^3} \times c$ will be greater than $\frac{n+1}{3n^3} \times c$; and conſequently, to the end that $nD + \frac{n+1}{3n^3} \times c$ may be equal to $\frac{3n}{3n^3} \times c$, it is neceſſary that nD ſhould be added to $\frac{n+1}{3n^3} \times c$, and not ſubtracted from it. We muſt therefore prefix the ſign + to nD, and conſequently to the fourth term, Dx^3, of the aſſumed ſeries $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, (which is equal to $\overline{1+x}|^{\frac{1}{n}}$,) from which fourth term the term nD was derived by the operations of multiplication and diviſion in the courſe of this inveſtigation. Therefore the four firſt terms of the ſaid ſeries, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be $1 + Bx - Cx^2 + Dx^3$, or $1 + \frac{1}{n}x - \frac{1}{n} + \frac{n-1}{2n}xx + Dx^3$; which was one of the points we were to determine.

And to determine the magnitude of the co-efficient D, we ſhall have the ſaid equation $+ nD + \frac{n+1}{3n^3} \times c = \frac{3n}{3n^3} \times c$; whence nD will be $= \frac{3n}{3n^3} \times c - \frac{n+1}{3n^3} \times c = \frac{2n-1}{3n^3} \times c$, or (becauſe c is $= \frac{n}{1} \times \frac{n-1}{2}$) nD will be $= \frac{2n-1}{3n^3}$

$\times n$

$\times \frac{n}{1} \times \frac{n-1}{2} = \frac{2n-1}{3n^2} \times \frac{n-1}{2} = \frac{2n-1}{3n} \times \frac{n-1}{2n}$; and conſequently D will be $= \frac{2n-1}{3n} \times \frac{n-1}{2n} \times \frac{1}{n} = \frac{2n-1}{3n} \times C$. Therefore Dx^3, or the fourth term of the ſeries $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c. which is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3$, or $\frac{2n-1}{3n} Cx^3$; and conſequently the four firſt terms of the ſaid ſeries will be $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3$, or $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3$. Q. E. I.

34. By reaſoning in the ſame manner as in the foregoing articles on the equation obtained in the laſt article, to wit, the equation

$$\left.\begin{array}{rrrl}
nD, & nEx, & nFx^2, & \&c \\
-2cBC, & 2cBDx, & 2cBEx^2, & \&c \\
 & +2cC^2x, & 2cCDx^2, & \&c \\
+dB^3 & -3dB^2Cx, & 3dB^2Dx^2, & \&c \\
 & & +3dBC^2x^2, & \&c \\
 & +eB^4x & -4eB^3Cx^2, & \&c \\
 & & +fB^5x^2, & \&c
\end{array}\right\} = 0,$$

or (if we prefix the ſign + to the terms nD, $2cBDx$, and $3dB^2Dx^2$, and the ſign − to the term $2cCDx^2$,) the equation

$$\left.\begin{array}{rrrl}
+nD, & nEx, & nFx^2, & \&c \\
-2cBC & +2cBDx, & 2cBEx^2, & \&c \\
 & +2cC^2x & -2cCDx^2, & \&c \\
+dB^3 & -3dB^2Cx & +3dB^2Dx^2, & \&c \\
 & & +3dBC^2x^2, & \&c \\
 & +eB^4x & -4eB^3Cx^2, & \&c \\
 & & +fB^5x^2, & \&c
\end{array}\right\} = 0,$$

we may obtain the two following equations for the determination of the values of E and F, and of the ſigns that are to be prefixed to them; to wit, the equation

$$nE + 2cBD + 2cC^2 - 3dB^2C + eB^4 = 0,$$

$$\text{or } nE + 2cBD + 2cC^2 + eB^4 = 3dB^2C,$$

for the determination of E; and the equation

$$nF - 2cBE - 2cCD + 3dB^2D + 3dBC^2$$
$$- 4eB^3C + fB^5 = 0, \text{ or}$$
$$nF + 3dB^2D + 3dBC^2 + fB^5 =$$

$2cBE + 2cCD + 4eB^3C$, for the determination of F. But the labour of reſolving theſe equations by ſubſtituting in them, inſtead of the ſmall letters c, d, e, f, their reſpective values, to wit, $\frac{n}{1} \times \frac{n-1}{2}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, $\frac{n}{1} \times \frac{n-1}{2}$

$\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, and, instead of the capital letters B, C, D and E, their respective values, to wit, $\frac{1}{n}$, $\frac{1}{n} \times \frac{n-1}{2n}$, $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}$ (for this last quantity will be found to be the value of E,) will be found to be very great; more especially in the latter equation, by which the value of F is to be determined. And in the investigation of the following co-efficients of the powers of x, to wit, the co-efficients G, H, I, K, L, &c, the intricacy of the calculations becomes so excessive as to make the discovery of these co-efficients in this method become absolutely impracticable.

35. And further, if this method of investigating the values of the co-efficients of the terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, and the signs that are to be prefixed to them, was not, after the first four or five terms, so exceedingly troublesome as soon to become impracticable, it would still be liable to another objection. For, to whatever number of terms we had carried the investigation,—as, for example, if we had discovered twenty terms of the said series,—it would still be impossible to see, from this method of obtaining these terms, that the next, or twenty-first, term (which we had not actually investigated by resolving the simple equation that belongs to it,) would observe the same law of generation, or derivation from the preceeding terms, which had been found to take place amongst the twenty terms which had been investigated: so that we should not be able to discover with certainty any more terms of the said series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, (which is equal to $\overline{1 + x}|^{\frac{1}{n}}$,) than we should actually have investigated. And therefore I shall dwell no longer on the foregoing method of investigating the values of these co-efficients, but shall proceed to explain another method of investigating them, which will be both much easier to practise in the few co-efficients we may think it necessary to investigate, and will, from the simplicity and regularity of the several simple equations involving the co-efficients C, D, E, F, &c, by the resolution of which the values of those co-efficients are to be determined, enable us to perceive that the law of the generation, or derivation, of the said co-efficients one from another, which takes place in the first four or five terms which we shall have actually investigated, must likewise take place in all the remaining terms of the series, to whatever number they may be supposed to be continued.

Another

Another Investigation of the Series

$$1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5 - \&c,$$

$$\text{or, } 1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \&c,$$

which is equal to $\overline{1+x}|^{\frac{1}{n}}$, *or* $\sqrt[n]{1+x}$; *by which the law of the continuation of the co-efficients* A, B, C, D, E, F, G, H, I, K, L, *&c, will be apparent.*

36. In the course of the foregoing investigation of the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, of which we have found the four first terms to be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3$, or $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3$, the principal difficulties that we had to encounter related to the discovery of the third and fourth terms $\frac{n-1}{2n}Bxx$, and $\frac{2n-1}{3n}Cx^3$; of which the former is to be marked with the sign $-$, or to be subtracted from the two first terms $1 + \frac{1}{n}Ax$, or $1 + \frac{1}{n}x$; and the latter, to wit, $\frac{2n-1}{3n}Cx^3$, is to be marked with the sign $+$, or to be added to the said two first terms. But it was clear beyond a doubt, both from the said foregoing investigation and from the three preliminary observations in art. 21, 22, 23, — — 26, that the two first terms of the said series would be 1 and $\frac{1}{n}x$, or $\frac{x}{n}$, and that the second of these terms is to be added to the first; and also that the form of the said series (which is equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$) would be 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, in which the several powers of x follow each other in their natural order, without any interruption. We may therefore assume it as a truth sufficiently ascertained, and a legitimate ground-

work of the investigation we are now going to explain, "that $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt{n}$ "$\overline{1+x}$, will be equal to the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c." This being premised, the said investigation will be as follows.

37. It having been proved in art. 21, 22, 23, — — 26, that, if x is of any magnitude less than 1, the quantity $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt{n}\ \overline{1+x}$, will be equal to the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c; it follows that, if y be any quantity greater than x, but less than 1, the quantity $\overline{1+y}|^{\frac{1}{n}}$, or $\sqrt{n}\ \overline{1+y}$, will, in like manner, be equal to the series $1 + \frac{1}{n}y, Cy^2, Dy^3, Ey^4, Fy^5$, &c. Now let d be equal to the difference by which y exceeds x, so that $x + d$ shall be $= y$; and let $x + d$ be substituted instead of y in the last equation $\overline{1+y}|^{\frac{1}{n}} = 1 + \frac{1}{n}y, Cy^2, Dy^3, Ey^4, Fy^5$, &c. And we shall then have $\overline{1+x+d}|^{\frac{1}{n}}$ = the series $1 + \frac{1}{n} \times \overline{x+d}$, $C \times \overline{x+d}|^2$, $D \times \overline{x+d}|^3$, $E \times \overline{x+d}|^4$, $F \times \overline{x+d}|^5$, &c = the series $1 + \frac{1}{n} \times \overline{x+d}$, $C \times \overline{xx + 2xd + dd}$, $D \times \overline{x^3 + 3x^2d + 3xd^2 + d^3}$, $E \times \overline{x^4 + 4x^3d + 6x^2d^2 + 4xd^3 + d^4}$, $F \times \overline{x^5 + 5x^4d + 10x^3d^2 + 10x^2d^3 + 5xd^4 + d^5}$, &c = the compound series

$$\begin{array}{llllll}
1 + \frac{1}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\
+ \frac{1}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c.
\end{array}$$

38. Let f be $= 1 + x$; and we shall have $f + d = 1 + x + d$, and $\overline{f+d}|^{\frac{1}{n}} = \overline{1+x+d}|^{\frac{1}{n}}$. But, because $f + d$ is $= f \times \overline{1 + \frac{d}{f}}$, it follows that $\overline{f+d}|^{\frac{1}{n}}$ will be $= f^{\frac{1}{n}} \times \overline{1 + \frac{d}{f}}|^{\frac{1}{n}}$. But, because $\overline{1+x}|^{\frac{1}{n}}$ is equal to the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$ &c; it follows that $\overline{1 + \frac{d}{f}}|^{\frac{1}{n}}$ will, in like manner, be equal to the series $1 + \frac{1}{n} \times \frac{d}{f}$, $C \times \frac{d}{f}|^2$, $D \times \frac{d}{f}|^3$, $E \times \frac{d}{f}|^4$, $F \times \frac{d}{f}|^5$, &c $= 1 + \frac{1}{n} \times \frac{d}{f}$, $C \times \frac{d^2}{f^2}$, $D \times \frac{d^3}{f^3}$, $E \times \frac{d^4}{f^4}$, $F \times \frac{d^5}{f^5}$, &c.

&c. Therefore $f^{\frac{1}{n}} \times \overline{1+\frac{d}{f}}|^{\frac{1}{n}}$ will be $= f^{\frac{1}{n}} \times$ the ſeries $1 + \frac{1}{n} \times \frac{d}{f}$, $C \times \frac{d^2}{f^2}$, $D \times \frac{d^3}{f^3}$, $E \times \frac{d^4}{f^4}$, $F \times \frac{d^5}{f^5}$, &c $=$ the ſeries $f^{\frac{1}{n}} + \frac{1}{n} \times f^{\frac{1}{n}} \times \frac{d}{f}$, $C \times f^{\frac{1}{n}} \times \frac{d^2}{f^2}$, $D \times f^{\frac{1}{n}} \times \frac{d^3}{f^3}$, $E \times f^{\frac{1}{n}} \times \frac{d^4}{f^4}$, $F \times f^{\frac{1}{n}} \times \frac{d^5}{f^5}$, &c. Therefore $\overline{f+d}|^{\frac{1}{n}}$ (which is $= f^{\frac{1}{n}} \times \overline{1+\frac{d}{f}}|^{\frac{1}{n}}$) will alſo be equal to the ſeries $f^{\frac{1}{n}} + \frac{1}{n} \times f^{\frac{1}{n}} \times \frac{d}{f}$, $C \times f^{\frac{1}{n}} \times \frac{d^2}{f^2}$, $D \times f^{\frac{1}{n}} \times \frac{d^3}{f^3}$, $E \times f^{\frac{1}{n}} \times \frac{d^4}{f^4}$, $F \times f^{\frac{1}{n}} \times \frac{d^5}{f^5}$, &c. Therefore, if we ſubſtitute $1 + x$ in this laſt equation inſtead of f, to which it is equal, we ſhall have $\overline{1+x+d}|^{\frac{1}{n}} =$ the ſeries $\overline{1+x}|^{\frac{1}{n}} + \frac{1}{n} \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c.

39. But it has been ſhewn, in art. 37, that $\overline{1+x+d}|^{\frac{1}{n}}$ is equal to the compound ſeries

$$\begin{array}{llllll} 1 + \frac{1}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\ + \frac{1}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\ & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\ & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\ & & & Ed^4, & 5Fxd^4, & \&c \\ & & & & Fd^5, & \&c. \end{array}$$

Therefore the ſeries $\overline{1+x}|^{\frac{1}{n}} + \frac{1}{n} \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c will be equal to the ſaid compound ſeries

$$\begin{array}{llllll} 1 + \frac{1}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\ + \frac{1}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\ & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\ & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\ & & & Ed^4, & 5Fxd^4, & \&c \\ & & & & Fd^5, & \&c. \end{array}$$

Therefore,

Therefore, if we ſubtract $\overline{1+x}|^{\frac{1}{n}}$ from the left-hand ſide of this equation, and the ſeries $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{1}{n}}$,) from the right-hand ſide of it, we ſhall have the ſeries $\frac{1}{n} \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c = the compound ſeries

$$\begin{array}{llllll}
\frac{1}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c;
\end{array}$$

and, if we divide all the terms by d (which is involved in every term on both ſides of this equation), we ſhall have the ſeries

$\frac{1}{n} \times \overline{1+x}|^{\frac{1}{n}} \times \frac{1}{1+x}$, $C \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^2}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^3}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^4}{\overline{1+x}|^5}$, &c = the compound ſeries

$$\begin{array}{llllll}
\frac{1}{n}, & 2Cx, & 3Dx^2, & 4Ex^3, & 5Fx^4, & \&c \\
 & Cd, & 3Dxd, & 6Ex^2d, & 10Fx^3d, & \&c \\
 & & Dd^2, & 4Exd^2, & 10Fx^2d^2, & \&c \\
 & & & Ed^3, & 5Fxd^3, & \&c \\
 & & & & Fd^4, & \&c
\end{array}$$

40. This equation is always true, how ſmall ſoever we may ſuppoſe d to be. And therefore it will alſo be true when d is $= 0$. But, when d is $= 0$, all the terms in the equation that involve d will be equal to 0 likewiſe; and conſequently the equation will then be as follows, to wit, $\frac{1}{n} \times \overline{1+x}|^{\frac{1}{n}} \times \frac{1}{1+x} =$ the ſeries $\frac{1}{n}$, $2Cx$, $3Dx^2$, $4Ex^3$, $5Fx^4$, &c. Therefore, if we multiply all the terms of this equation by n, we ſhall have $\overline{1+x}|^{\frac{1}{n}} \times \frac{1}{1+x} =$ the ſeries 1, $2nCx$, $3nDx^2$, $4nEx^3$, $5nFx^4$, &c; and, if we multiply both ſides of this laſt equation by $1+x$, we ſhall have $\overline{1+x}|^{\frac{1}{n}} =$ the compound ſeries

$$\begin{array}{llllll}
1, & 2nCx, & 3nDx^2, & 4nEx^3, & 5nFx^4, & \&c \\
+ & x, & 2nCx^2, & 3nDx^3, & 4nEx^4, & \&c
\end{array}$$

41. But

41. But $\overline{1+x}|^{\frac{1}{n}}$ is equal to the ſeries $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c. Therefore the ſeries $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, will be equal to the compound ſeries

$$\begin{array}{l} 1,\ 2nCx,\ 3nDx^2,\ 4nEx^3,\ 5nFx^4,\ \&c \\ +\quad x,\ 2nCx^2,\ 3nDx^3,\ 4nEx^4,\ \&c; \end{array}$$

and conſequently (ſubtracting 1 from both ſides of the equation) we ſhall have the ſeries $\frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c = the compound ſeries

$$\begin{array}{l} 2nCx,\ 3nDx^2,\ 4nEx^3,\ 5nFx^4,\ \&c \\ +\ x,\ 2nCx^2,\ 3nDx^3,\ 4nEx^4,\ \&c; \end{array}$$ and,

laſtly, dividing all the terms by x, we ſhall have the ſimple ſeries $\frac{1}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c = the compound ſeries

$$\begin{array}{l} 2nC,\ 3nDx,\ 4nEx^2,\ 5nFx^3,\ \&c \\ +\ 1,\ 2nCx,\ 3nDx^2,\ 4nEx^3,\ \&c; \end{array}$$

by the help of which equation we may determine both the values of the ſeveral coefficients C, D, E, F, &c, of the terms Cx^2, Dx^3, Ex^4, Fx^5, &c, of the aſſumed ſeries $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, and the ſigns + and − that are to be prefixed to the ſaid terms reſpectively, by reaſonings ſimilar to thoſe uſed in the former inveſtigation, in deriving the values of the co-efficients B, C, and D from the much more complicated equation obtained above in art. 30. This may be done in the manner following:

42. In the firſt place we may determine both the ſign to be prefixed to the term Cx^2 in the ſeries $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{1}{n}}$), and the magnitude of the co-efficient C, by proceeding as follows.

The final equation obtained in the laſt article, to wit, the equation between the ſimple ſeries $\frac{1}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c, and the compound ſeries

$$\begin{array}{l} 2nC,\ 3nDx,\ 4nEx^2,\ 5nFx^3,\ \&c \\ +\ 1,\ 2nCx,\ 3nDx^2,\ 4nEx^3,\ \&c, \end{array}$$ is

always true, how ſmall ſoever we may ſuppoſe x to be. And therefore it will be true when x is $= 0$. But when x is $= 0$, all the terms in the foregoing equation that involve x in them will be equal to 0 likewiſe; and the ſaid equation will conſequently then be $\frac{1}{n} = 2nC, + 1$, or $\frac{1}{n} = 1, 2nC$; that is, $\frac{1}{n}$ will be equal to 1, together with $2nC$ either added to it, or ſubtracted from it, as may be neceſſary to produce the ſaid equality. But, becauſe n is a whole number, $\frac{1}{n}$ muſt be leſs than 1, and therefore cannot be equal to 1 together

with

with $2nC$ added to it; therefore it muſt be equal to 1 with $2nC$ ſubtracted from it. Therefore the term $2nC$, in the equation $\frac{1}{n} = 1,\ 2nC$, muſt have the ſign — prefixed to it: and conſequently the terms $2nCx$ and Cx, in the final equation obtained in art. 41, and Cx^2, the third term of the ſeries $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c (from which third term the term $2nC$ has been derived, by various multiplications and diviſions, in the courſe of the preceeding inveſtigation), muſt likewiſe have the ſign — prefixed to them. So that the ſaid third term Cx^2 of the ſeries $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, muſt be ſubtracted from the two firſt terms of the ſaid ſeries; and the three firſt terms of the ſaid ſeries will therefore be $1 + \frac{1}{n}x - Cxx$. Q. E. I.

And, to determine the magnitude of C, we ſhall have the equation $\frac{1}{n} = 1 - 2nC$; whence (adding $2nC$ to both ſides) we ſhall have $\frac{1}{n} + 2nC = 1$; and (ſubtracting $\frac{1}{n}$ from both ſides) $2nC = 1 - \frac{1}{n} = \frac{n-1}{n}$; and (dividing both ſides by $2n$) $C = \frac{n-1}{n} \times \frac{1}{2n} = \frac{n-1}{2n} \times \frac{1}{n}$, or $\frac{1}{n} \times \frac{n-1}{2n}$.

Therefore the three firſt terms of the ſeries $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c (which is equal to $\overline{1+x}|^{\frac{1}{n}}$) will be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx$, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bxx$. Q. E. I.

43. Secondly, to find the value of the co-efficient D, and the ſign that is to be prefixed to Dx^3, the fourth term of the ſeries $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, we muſt proceed as follows.

Since it has been ſhewn in the laſt article that the ſign — is to be prefixed to the terms $2nC$ and $2nCx$ and Cx, in the final equation obtained in art. 41, to wit, the equation $\frac{1}{n},\ Cx,\ Dx^2,\ Ex^3,\ Fx^4$, &c = the compound ſeries

$$\begin{array}{llllll} 2nC, & 3nDx, & 4nEx^2, & 5nFx^3, & \&c \\ +1, & 2nCx, & 3nDx^2, & 4nEx^3, & \&c, \end{array}$$ let it be accordingly prefixed to the ſaid terms; and then the ſaid equation will be as follows, to wit, $\frac{1}{n} - Cx,\ Dxx,\ Ex^3,\ Fx^4$, &c = the compound ſeries

$$\begin{array}{llllll} -2nC, & 3nDx, & 4nEx^2, & 5nFx^3, & \&c \\ +1 - 2nCx, & 3nDx^2, & 4nEx^3, & \&c. \end{array}$$

Now let $2Cx$ be added to both ſides of this equation; and we ſhall then have

$\frac{1}{n} +$

$\frac{1}{n} + Cx, Dx^2, Ex^3, Fx^4,$ &c = the compound series

$$\begin{array}{l} -2nC,\ 3nDx,\ 4nEx^2,\ 5nFx^3,\ \&c \\ +1-2nCx,\ 3nDx^2,\ 4nEx^3,\ \&c. \\ +2Cx \end{array}$$

But it was shewn in the last article that $\frac{1}{n}$ is $= 1 - 2nC$. Therefore, if we subtract $\frac{1}{n}$ and $1 - 2nC$ from the opposite sides of this equation, the remainders will be equal; that is, the series $+ Cx, Dx^2, Ex^3, Fx^4,$ &c, will be = the compound series

$$\begin{array}{l} 3nDx,\ 4nEx^2,\ 5nFx^3,\ \&c \\ -2nCx,\ 3nDx^2,\ 4nEx^3,\ \&c \\ +\ 2Cx \end{array}$$

Therefore (dividing all the terms by x) we shall have $C, Dx, Ex^2, Fx^3,$ &c = the compound series

$$\begin{array}{l} 3nD,\ 4nEx,\ 5nFx^2,\ \&c \\ -2nC,\ 3nDx,\ 4nEx^2,\ \&c \\ +2C \end{array}$$

This equation will be true, how small soever we may suppose x to be. And therefore it will be true also when x is $= 0$. But, when x is $= 0$, all the terms which involve x will be equal to 0 likewise; and consequently the equation will be $C = 3nD - 2nC + 2C$. Therefore, if we add $2nC$ to both sides, we shall have $C + 2nC = 3nD + 2C$; and (subtracting C from both sides) $2nC = 3nD + C$, or $2nC = C, 3nD$; that is, $2nC$ will be equal to C, together with $3nD$ either added to it, or subtracted from it, as may be necessary to produce the said equality. But, since n is a whole number, and consequently greater than 1, $2nC$ must be greater than C, and therefore cannot be equal to C with $3nD$ subtracted from it, but must be equal to C with $3nD$ added to it. Therefore the sign, + must be prefixed to $3nD$, and consequently to the terms $3nDx$ and $3nDx^2$ in the last equation, and to the term Dx^2 in the final equation obtained in art. 41; and likewise to the fourth term, Dx^3, of the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5,$ &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$; from which fourth term Dx^3 all the other terms that involve D were derived, by the operations of multiplication and division, in the course of the foregoing investigation. Therefore the four first terms of the said series $1 \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5,$ &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + Dx^3$, or $1 + \frac{1}{n}x - Cxx + Dx^3$, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}|Bxx + Dx^3$.

Q. E. I.

And, to determine the magnitude of D, we shall have the equation $2nC = C + 3nD$; whence $3nD$ will be $= 2nC - C = \overline{2n-1}| \times C$, and D will

will be $= \frac{2n-1}{3n} \times C = \frac{2n-1}{3n} \times \frac{n-1}{2n} \times \frac{1}{n}$, or $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$. Therefore the four first terms of the series $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3$, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3$. Q. E. I.

44. To determine the sign that is to be prefixed to Ex^4, or the fifth term of the series $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, and to find the magnitude of the co-efficient E, we must proceed as follows.

In the last article we obtained the equation C, Dx, Ex^2, Fx^3, &c, = the compound series

$$\begin{array}{llll} 3nD, & 4nEx, & 5nFx^2, & \&c \\ -2nC, & 3nDx, & 4nEx^2, & \&c \\ +2C; & & & \end{array}$$

which, if we prefix the sign + to the terms that involve D (agreeably to what was found in the last article to be necessary), will become

$C + Dx$, Ex^2, Fx^3, &c = the compound series

$$\begin{array}{llll} +3nD, & 4nEx, & 5nFx^2, & \&c \\ -2nC & +3nDx, & 4nEx^2, & \&c \\ +2C. & & & \end{array}$$

Now it has been seen, in the last article, that C is $= +3nD - 2nC + 2C$. Therefore, if we subtract C and $3nD - 2nC + 2C$ from the opposite sides of this last equation, we shall have $+Dx$, Ex^2, Fx^3, &c = the compound series

$$\begin{array}{lll} 4nEx, & 5nFx^2, & \&c \\ +3nDx, & 4nEx^2, & \&c; \end{array}$$

and consequently (dividing all the terms by x) $+D$, Ex, Fx^2, &c = the compound series

$$\begin{array}{lll} 4nE, & 5nFx, & \&c \\ +3nD, & 4nEx, & \&c. \end{array}$$

And this equation will be true, how small soever we suppose x to be: and therefore it will also be true when x is = o. But, when x is = o, all the terms that involve x will be equal to o likewise; and the equation will be as follows, to wit, $+D = 4nE + 3nD$, or $+D = 3nD$, $4nE$; that is, D will be equal to $3nD$, together with $4nE$ either added to it, or subtracted from it, as may be necessary to produce the said equality. But because n is a whole number, nD must be greater than D; and therefore $3nD$ must, *à fortiori,* be greater than D. Therefore, in order to make $3nD$ be equal to D, the quantity $4nE$ must not be added to $3nD$, but must be subtracted from it; and consequently the sign − must be prefixed to $4nE$; and the last equation $+D = 3nD$, $4nE$ will be $+D = 3nD - 4nE$. Therefore the same sign − must likewise

wise be prefixed to Ex^4, the fifth term of the assumed series $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{1}{n}}$); from which fifth term the said term $4nE$ is derived, by the operations of multiplication and division, in the course of the foregoing investigations. Therefore the said fifth term, Ex^4, of the said series, must be subtracted from the first term 1; and consequently the five first terms of the said series will be $1 + \frac{1}{n}x - Cxx + Dx^3 - Ex^4$, or $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - Ex^4$. Q. E. I.

And, to determine the magnitude of the co-efficient E, we shall have the equation $+D = 3nD - 4nE$; whence $D + 4nE$ will be $= 3nD$, and $4nE$ will be $= 3nD - D = \overline{3n-1} \times D$, and E will be $= \frac{3n-1}{4n} \times D = \frac{3n-1}{4n} \times \frac{2n-1}{3n} \times \frac{n-1}{2n} \times \frac{1}{n}$, or $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}$. Therefore the four first terms of the series $1 + \frac{1}{n}x$, Cxx, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, will be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4$, or $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4$. Q. E. I.

45. To determine the sign that is to be prefixed to Fx^5, the sixth term of the series $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, and to find the magnitude of the co-efficient F, we must proceed as follows.

In the last article we obtained the equation $+D$, Ex, Fx^2, Gx^3, Hx^4, &c = the compound series

$$\begin{array}{lllll} 4nE, & 5nFx, & 6nGx^2, & 7nHx^3, & \&c \\ +3nD, & 4nEx, & 5nFx^2, & 6nGx^3, & \&c \end{array}$$

or (if we prefix the sign − to the terms $4nE$, $4nEx$, and Ex, agreeably to what was found in the last article to be necessary)

$$+D - Ex,\ Fx^2,\ Gx^3,\ Hx^4,\ \&c = \text{the compound series}$$

$$\begin{array}{lllll} -4nE, & 5nFx, & 6nGx^2, & 7nHx^3, & \&c \\ +3nD - & 4nEx, & 5nFx^2, & 6nGx^3, & \&c. \end{array}$$

Therefore, if we add $2Ex$ to both sides of this equation, we shall have

$$+D + Ex,\ Fx^2,\ Gx^3,\ Hx^4,\ \&c = \text{the compound series}$$

$$\begin{array}{lllll} -4nE, & 5nFx, & 6nGx^2, & 7nHx^3, & \&c \\ +3nD - & 4nEx, & 5nFx^2, & 6nGx^3, & \&c \\ & +2Ex. & & & \end{array}$$

And it has also been shewn in the last article, that D is $= 3nD - 4nE$. Therefore, if we subtract D and $3nD - 4nE$ from the opposite sides of the last

last equation, the remainders will be equal; that is, the simple series Ex, Fx^2, Gx^3, Hx^4, &c will be equal to the compound series

$$\begin{array}{llll} & 5nFx, & 6nGx^2, & 7nHx^3, \ \&c \\ - & 4nEx, & 5nFx^2, & 6nGx^3, \ \&c \\ + & 2\,Ex & & \end{array}$$

; and consequently (dividing all the terms by x) the simple series E, Fx, Gx^2, Hx^3, &c will be equal to the compound series

$$\begin{array}{llll} & 5nF, & 6nGx, & 7nHx^2, \ \&c \\ - & 4nE, & 5nFx, & 6nGx^2, \ \&c \\ + & 2E. & & \end{array}$$

And this equation is true, of how small a magnitude soever we suppose x to be taken: and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will then be $E = 5nF - 4nE + 2E$. Therefore (adding $4nE$ to both sides) we shall have $E + 4nE = 5nF + 2E$, and (subtracting E from both sides) we shall have $4nE = 5nF + E$, or $4nE = E$, $5nF$; that is $4nE$ will be equal to E, together with $5nF$ either added to it or subtracted from it, as may be necessary to produce such equality. But, because $4n$ is greater than 1, $4nE$ must be greater than E. Therefore, in order to make E be equal to $4nE$, it will be necessary to add $5nF$ to it; and consequently the sign $+$ must be prefixed to $5nF$, and the last equation $4nE = E$, $5nF$ will be $4nE = E + 5nF$. Therefore the sign $+$ must also be prefixed to the sixth term, Fx^5, of the assumed series $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c, which is equal to $\overline{1 + x}\big|^{\frac{1}{n}}$; from which sixth term the term $5nF$ has been derived by the operations of multiplication and division in the course of this investigation. Therefore the said sixth term, Fx^5, of the said series is to be added to the first term 1, and consequently the six first terms of the said series will be $1 + \frac{1}{n}x - Cx^2 + Dx^3 - Ex^4 + Fx^5$, or $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5$, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + Fx^5$. Q. E. I.

And to determine the magnitude of F, we shall have the aforesaid equation $4nE = E + 5nF$; whence $5nF$ will be $= 4nE - E = \overline{4n - 1}\big| \times E$, and F will be $= \frac{4n-1}{5n} \times E = \frac{4n-1}{5n} \times \frac{3n-1}{4n} \times \frac{2n-1}{3n} \times \frac{n-1}{2n} \times \frac{1}{n}$, or $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}$. Therefore the first six terms of the series $1 + \frac{1}{n}x$, Cxx, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c, which is equal to $\overline{1 + x}\big|^{\frac{1}{n}}$, will be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n}$

$\times \frac{n-1}{2n}$

$\times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5$, or

$$1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5.$$

Q. E. I.

46. The reader will obſerve, that in this way of inveſtigating the values of C, D, E, F, &c, there is no more difficulty in determining the values of E and F, than in determining the values of C and D; becauſe the ſimple equations, by the reſolution of which the ſeveral new co-efficients are to be determined, conſiſt always of only three terms, to wit, of one term derived from a term of the ſimple ſeries $\frac{1}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c, (which forms the left-hand ſide of the final equation obtained in art. 41,) by dividing it by the power of x with which it involved, and of two terms derived in like manner from two terms of the compound ſeries

$$\begin{array}{llllll} 2nC, & 3nDx, & 4nEx^2, & 5nFx^3, & \&c \\ +1, & 2nCx, & 3nDx^2, & 4nEx^3, & \&c \text{ (which} \end{array}$$

forms the right-hand ſide of the ſame final equation), by dividing them by the ſame power of x: whereas, in the former way of inveſtigating the values of thoſe co-efficients, ſet forth in art. 31, 32, and 33, the ſeveral ſimple equations, by the reſolution of which the ſaid co-efficients are to be determined, to wit, the equations $1 = nB$, and $cB^2 - nC = 0$, or $cB^2 = nC$, and $nD - 2cBC + dB^3 = 0$, or $nD + dB^3 = 2cBC$, and $nE + 2cBD + 2cC^2 - 3dB^2C + eB^4 = 0$, or $nE + 2cBD + 2cC^2 + eB^4 = 3dB^2C$, and $nF - 2cBE - 2cCD + 3dB^2D + 3dBC^2 - 4eB^3C + fB^5 = 0$, or $nF + 3dB^2D + 3dBC^2 + fB^5 = 2cBE + 2cCD + 4eB^3C$, increaſe continually in the number of their terms by the addition of two terms in every new equation; and likewiſe conſiſt of terms that are more and more complicated continually, ſo as ſoon to make the labour of reſolving them become intolerable. Therefore the preſent method of inveſtigating theſe co-efficients is, in a practical view, very much to be preferred to the former method; though the reaſonings uſed in that former method ſeem to be rather more direct and ſimple than thoſe which we were obliged to have recourſe to in art. 37, 38, 39, 40, and 41, in order to obtain the final equation ſet forth in the laſt of thoſe articles. Both the methods therefore have their ſeparate merits, and are worthy of our attention; and the former method ſerves as a proper introduction to the latter.

47. From the manner in which the grand final equation, $\frac{1}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c = the compound ſeries,

$$\begin{array}{llllll} 2nC, & 3nDx, & 4nEx^2, & 5nFx^3, & \&c \\ +1, & 2nCx, & 3nDx^2, & 4nEx^2, & \&c, \text{ in} \end{array}$$

art. 41, was obtained (which was by the multiplication of the ſeries 1, $2nCx$, $3nDx^2$, $4nEx^3$, $5nFx^4$, &c into $1 + x$ in art. 40), it is eaſy to ſee that the ſeveral equations for determining the values of the following co-efficients G,

2 H 2 H, I,

H, I, K, L, &c (which we have not yet investigated), and the signs + and −, that are to be prefixed to them, would be as follows.

The first equation would be $F = 6nG,\ 5nF$, or $F = 5nF,\ 6nG$, or (since we have already seen in art. 45, that $5nF$ and F must have the sign + prefixed to them) $+ F = + 5nF,\ 6nG$, and consequently (in order to make the said equation possible) $+ F = 5nF - 6nG$; whence it will follow that Gx^6, or the seventh term of the assumed series $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c (which is equal to $\overline{1 + x}|^{\frac{1}{n}}$), must be marked with the sign −, or be subtracted from the first term 1, and that G will be $= \frac{5n - 1}{6n} \times F$.

And the second equation would be $G = 7nH,\ 6nG$, or (because it has just now been shewn that the sign − is to be prefixed to $6nG$ and G) $- G = 7nH - 6nG$, and consequently (in order to make the said equation possible) $- G = + 7nH - 6nG$; whence it will follow that Hx^7, or the eighth term of the assumed series $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c (which is equal to $\overline{1 + x}|^{\frac{1}{n}}$), will be marked with the sign +, or added to the first term 1, and that H will be $= \frac{6n - 1}{7n} \times G$.

And the third equation would be $H = 8nI,\ 7nH$, or $H = 7nH,\ 8nI$, or (because it has been just now shewn that the terms $7nH$ and H must have the sign + prefixed to them), $+ H = + 7nH,\ 8nI$, and consequently (in order to make the said equation possible,) $+ H = 7nH - 8nI$; whence it will follow that Ix^8, or the ninth term of the series $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c (which is equal to $\overline{1 + x}|^{\frac{1}{n}}$), will be marked with the sign −, or subtracted from the first term 1, and that I will be $= \frac{7n - 1}{8n} \times H$.

And the fourth equation would be $I = 9nK,\ 8nI$, or (because it has been just now shewn that the terms $8nI$ and I must have the sign − prefixed to them) $- I = 9nK - 8nI$, and consequently (in order to make the said equation possible,) $- I = + 9nK - 8nI$; whence it will follow that Kx^9, or the tenth term of the series $1 + \frac{1}{n}x,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c (which is equal to $\overline{1 + x}|^{\frac{1}{n}}$), will be marked with the sign +, or must be added to the first term 1, and that K will be $= \frac{8n - 1}{9n} \times I$.

And the 5th equation would be $K = 10nL,\ 9nK$, or because it has been just now shewn that the terms $9nK$ and K must have the sign + prefixed to them) $+ K = 10nL + 9nK$, or $+ K = + 9nK,\ 10nL$, and consequently (in order to make the said equation possible) $+ K = + 9nK - 10nL$; whence it will follow that Lx^{10}, or the eleventh term of the series 1 +

6 $\frac{1}{n}x$,

$\frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{1}{n}}$), will be marked with the fign $-$, or fubtracted from the firft term 1, and that L will be $= \frac{9n-1}{10n} \times K$.

And the like fhort and eafy equations,

$$\begin{aligned} -L &= -10nL + 11nM, \\ +M &= +11nM - 12nN, \\ -N &= -12nN + 13nO, \\ +O &= +13nO - 14nP, \\ -P &= -14nP + 15nQ, \end{aligned}$$

&c, would be found for the determination of the values and the figns of the following co-efficients M, N, O, P, Q, &c, *ad infinitum*, or as far as we pleafed to continue the inveftigation. We may therefore now conclude with certainty, that the quantity $\overline{1+x}|^{\frac{1}{n}}$, or the nth root of the binomial quantity $1+x$, is equal to the feries $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5 -$ &c, or $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \frac{5n-1}{6n}Fx^6 + \frac{6n-1}{7n}Gx^7 - \frac{7n-1}{8n}Hx^8 + \frac{8n-1}{9n}Ix^9 - \frac{9n-1}{10n}Kx^{10}$ + &c *ad infinitum*, in which the figns of the terms that come after the fecond term $\frac{1}{n}x$ are alternately $-$ and $+$, and the law of the generation, or continuation of the terms one from another, is very manifeft, every new generating fraction of the co-efficients C, D, E, F, G, H, I, K, L, &c, being derived from the generating fraction which immediately preceeds it, by the addition of n to both its numerator and its denominator. Q. E. I.

48. This feries agrees exactly with that which was fet forth in art. 2, as being equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, and which had been derived from the binomial theorem in the cafe of integral powers.

A Review of the foregoing Inveftigation of the Series which is equal to $\overline{1+x}|^{\frac{1}{n}}$, *or* $\sqrt[n]{1+x}$.

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49. I have now completed the fecond inveftigation of the feries $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal

equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{\overline{1+x}}$. The eafe with which it enables us to find the values and the figns of the feveral co-efficients C, D, E, F, G, H, I, K, L, &c, gives it a great fuperiority over the former inveftigation explained above in art. 28, 29, 30, &c—34, and raifes it to an equality with any method of inveftigating this feries that I have yet feen. Yet, from the number of the fteps which it contains before we obtain the final equation in art. 41, it may, perhaps, be thought to be rather too abftrufe and difficult; though none of the fteps in it, taken feparately, feem to deferve that character. To explain it, therefore, as fully as I can to the reader, and make the connection of the reafonings ufed in it as apparent as poffible, I will here ftate them over again with fomewhat more brevity than at firft, in order to bring all the parts of the inveftigation into view at once.

Now the principal artifice of this inveftigation confifts in finding two different expreffions for the value of $\overline{1+x+d}|^{\frac{1}{n}}$. The firft of thefe expreffions is obtained by confidering the trinomial quantity $1+x+d$ as being a binomial quantity, of which the firft member is 1, and the fecond is $x+d$. By fuppofing y to be equal to $x+d$, and confequently $1+y$ to be equal to $1+x+d$, and affuming the feries $1+\frac{1}{n}y$, Cy^2, Dy^3, Ey^4, Fy^5, &c, for the value of $\overline{1+y}|^{\frac{1}{n}}$ (as we before affumed the feries $1+\frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c for the value of $\overline{1+x}|^{\frac{1}{n}}$), and then fubftituting in the faid feries $1+\frac{1}{n}y$, Cy^2, Dy^3, Ey^4, Fy^5, &c, inftead of y, y^2, y^3, y^4, y^5, &c, the like powers of $x+d$, to wit, $x+d$, $\overline{x+d}|^2$, $\overline{x+d}|^3$, $\overline{x+d}|^4$, $\overline{x+d}|^5$, &c, or $x+d$, $x^2+2xd+d^2$, $x^3+3x^2d+3xd^2+d^3$, $x^4+4x^3d+6x^2d^2+4xd^3+d^4$, $x^5+5x^4d+10x^3d^2+10x^2d^3+5xd^4+d^5$, &c, we obtained the equation, $\overline{1+x+d}|^{\frac{1}{n}}$, = the compound feries

$$\begin{array}{llllll}
1+\frac{1}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\
+\frac{1}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c.
\end{array}$$

The fecond expreffion of the value of $\overline{1+x+d}|^{\frac{1}{n}}$ was obtained by confidering the trinomial quantity $1+x+d$ as being a binomial quantity of which $1+x$ is the firft member, and d the fecond. By fuppofing f to be equal to $1+x$, and confequently $f+d$ to be equal to $1+x+d$, and therefore $f\times\overline{1+\frac{d}{f}}$ (which is $=f+d$) to be alfo equal to $1+x+d$, and affuming the

feries

series $1 + \frac{1}{n} \times \frac{d}{f}$, $C \times \overline{\frac{d}{f}}\big|^{2}$, $D \times \overline{\frac{d}{f}}\big|^{3}$, $E \times \overline{\frac{d}{f}}\big|^{4}$, $F \times \overline{\frac{d}{f}}\big|^{5}$, &c, for the value of $\overline{1 + \frac{d}{f}}\big|^{\frac{1}{n}}$ (as we before assumed the series $1 + \frac{1}{n} x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, for the value of $\overline{1+x}\big|^{\frac{1}{n}}$, and the series $1 + \frac{1}{n} y$, Cy^2, Dy^3, Ey^4, Fy^5, &c, for the value of $\overline{1+y}\big|^{\frac{1}{n}}$), we found $\overline{1+x+d}\big|^{\frac{1}{n}}$ to be equal to $f^{\frac{1}{n}} \times$ the series $1 + \frac{1}{n} \times \frac{d}{f}$, $C \times \overline{\frac{d}{f}}\big|^{2}$, $D \times \overline{\frac{d}{f}}\big|^{3}$, $E \times \overline{\frac{d}{f}}\big|^{4}$, $F \times \overline{\frac{d}{f}}\big|^{5}$, &c, and consequently to $f^{\frac{1}{n}} \times$ the series $1 + \frac{1}{n} \times \frac{d}{f}$, $C \times \frac{d^2}{f^2}$, $D \times \frac{d^3}{f^3}$, $E \times \frac{d^4}{f^4}$, $F \times \frac{d^5}{f^5}$, &c, and therefore to the series $f^{\frac{1}{n}} + \frac{1}{n} \times f^{\frac{1}{n}} \times \frac{d}{f}$, $C \times f^{\frac{1}{n}} \times \frac{d^2}{f^2}$, $D \times f^{\frac{1}{n}} \times \frac{d^3}{f^3}$, $E \times f^{\frac{1}{n}} \times \frac{d^4}{f^4}$, $F \times f^{\frac{1}{n}} \times \frac{d^5}{f^5}$, &c, that is (if we substitute $1 + x$ in the terms of this series instead of f, which is equal to it), to the series $\overline{1+x}\big|^{\frac{1}{n}} + \frac{1}{n} \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c.

We then equated this latter expression of the value of $\overline{1+x+d}\big|^{\frac{1}{n}}$ to the former, to wit, the compound series

$$\begin{array}{llllll}
1 + \frac{1}{n} x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\
+ \frac{1}{n} d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dx^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c;
\end{array}$$
and

then subtracted $\overline{1+x}\big|^{\frac{1}{n}}$ from the left-hand side of this last equation, and the series $1 + \frac{1}{n} x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}\big|^{\frac{1}{n}}$), from the right-hand side of the same equation; whereby we obtained the equation $\frac{1}{n} \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}\big|^{\frac{1}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c = the compound series

$\frac{1}{n} d$,

$$\begin{array}{llllll}
\frac{1}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c;
\end{array}$$

and (by dividing all the terms by d) the equation

$\frac{1}{n} \times \overline{1+x}|^{\frac{1}{n}} \times \frac{1}{1+x}$, $C \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d}{1+x}$, $D \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^2}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^3}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{1}{n}} \times \frac{d^4}{\overline{1+x}|^5}$, &c = the compound series

$$\begin{array}{llllll}
\frac{1}{n}, & 2Cx, & 3Dx^2, & 4Ex^3, & 5Fx^4, & \&c \\
 & Cd, & 3Dxd, & 6Ex^2d, & 10Fx^3d, & \&c \\
 & & Dd^2, & 4Exd^2, & 10Fx^2d^2, & \&c \\
 & & & Ed^3, & 5Fxd^3, & \&c \\
 & & & & Fd^4, & \&c;
\end{array}$$

and (by supposing d to become $= 0$, and consequently all the terms that involve d to become equal to 0 likewise) the equation $\frac{1}{n} \times \overline{1+x}|^{\frac{1}{n}} \times \frac{1}{1+x} =$ the series $\frac{1}{n}$, $2Cx$, $3Dx^2$, $4Ex^3$, $5Fx^4$, &c; and (by multiplying both sides into n) the equation $\overline{1+x}|^{\frac{1}{n}} \times \frac{1}{1+x} = 1$, $2nCx$, $3nDx^2$, $4nEx^3$, $5nFx^4$, &c; and, lastly (by multiplying both sides into $1+x$), the equation $\overline{1+x}|^{\frac{1}{n}}$

$$\begin{array}{lllllll}
= & 1, & 2nCx, & 3nDx^2, & 4nEx^3, & 5nFx^4, & \&c \\
+ & & x, & 2nCx^2, & 3nDx^3, & 4nEx^4, & \&c.
\end{array}$$

Having thus obtained a second expression of the value of $\overline{1+x}|^{\frac{1}{n}}$, involving the several powers of x, to wit, x, x^2, x^3, x^4, x^5, &c, in their natural order, without interruption, and involving likewise the co-efficients C, D, E, F, &c, of the powers of x in the series $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, that was at first assumed for the value of $\overline{1+x}|^{\frac{1}{n}}$, we equated these two expressions of the value of $\overline{1+x}|^{\frac{1}{n}}$ to each other, and thereby obtained the equation $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c = the compound series

$$\begin{array}{llllll}
1, & 2nCx, & 3nDx^2, & 4nEx^3, & 5nFx^4, & \&c \\
+ & x, & 2nCx^2, & 3nDx^3, & 4nEx^4, & \&c;
\end{array}$$

and (by subtracting 1 from both sides) the equation $\frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c = the compound series

$2nCx$,

$$\begin{array}{llllll} 2nCx, & 3nDx^2, & 4nEx^3, & 5nFx^4, & \&c \\ +x, & 2nCx^2, & 3nDx^3, & 4nEx^4, & \&c; \end{array}$$

and, lastly (by dividing all the terms by x), the equation $\frac{1}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c = the compound series

$$\begin{array}{llllll} 2nC, & 3nDx, & 4nEx^2, & 5nFx^3, & \&c \\ +1, & 2nCx, & 3nDx^2, & 4nEx^3, & \&c; \end{array}$$

which is the grand final equation from which we are to deduce the several particular equations (consisting of only three terms each), by means of which the signs of the terms Cx^2, Dx^3, Ex^4, Fx^5, &c, of the series $1 + \frac{1}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{1}{n}}$), and the magnitudes of the coefficients C, D, E, F, &c, are to be determined.

50. These were the several steps of the foregoing investigation, which, the reader will probably observe, bear some resemblance to Mr Landen's investigation of the more general series that is equal to $\overline{1+x}|^{\frac{m}{n}}$, which has been printed above in this collection, in pages 170, 171,——175, and of which an explanation has been given in pages 176, 177, 178, —— 193. And I readily acknowledge that it was after a very close and attentive perusal and consideration of that investigation of Mr Landen, and in consequence of the ideas which that perusal suggested to me, that I discovered the investigation that has been here explained. It seems, however, to be in many respects different from that of Mr Landen, though it had been originally suggested by it; and, in particular, I hope it will be found much less abstruse and difficult.

An Example to the foregoing Series.

51. Let it be required to find, by means of the foregoing series, the cube-root of the binomial quantity $1 + x$, or the value of $\overline{1+x}|^{\frac{1}{3}}$.

Here n is $= 3$. We shall therefore have

$$\begin{array}{lllll} B = \frac{1}{n}\ A = & & & & \frac{1}{3}A, \\ \text{and } C = \frac{n-1}{2n}B & = & \frac{3-1}{2\times 3}B & = & \frac{2}{6}B, \\ \text{and } D = \frac{2n-1}{3n}C & = & \frac{6-1}{9}C & = & \frac{5}{9}C, \\ \text{and } E = \frac{3n-1}{4n}D & = & \frac{9-1}{12}D & = & \frac{8}{12}D, \\ \text{and } F = \frac{4n-1}{5n}E & = & \frac{12-1}{15}E & = & \frac{11}{15}E, \\ \text{and } G = \frac{5n-1}{6n}F & = & \frac{15-1}{18}F & = & \frac{14}{18}F, \\ \text{and } H = \frac{6n-1}{7n}G & = & \frac{18-1}{21}G & = & \frac{17}{21}G, \\ \text{and } I = \frac{7n-1}{8n}H & = & \frac{21-1}{24}H & = & \frac{20}{24}H, \end{array}$$

$$\text{and } K = \frac{8n-1}{9n} I = \frac{24-1}{27} I = \frac{23}{27} I,$$
$$\text{and } L = \frac{9n-1}{10n} K = \frac{27-1}{30} K = \frac{26}{30} K,$$
$$\text{and } M = \frac{10n-1}{11n} L = \frac{30-1}{33} L = \frac{29}{33} L,$$
$$\text{and } N = \frac{11n-1}{12n} M = \frac{33-1}{36} M = \frac{32}{36} M,$$
$$\text{and } O = \frac{12n-1}{13n} N = \frac{36-1}{39} N = \frac{35}{39} N,$$
$$\text{and } P = \frac{13n-1}{14n} O = \frac{39-1}{42} O = \frac{38}{42} O,$$
$$\text{and } Q = \frac{14n-1}{15n} P = \frac{42-1}{45} P = \frac{41}{45} P,$$
$$\text{and } R = \frac{15n-1}{16n} Q = \frac{45-1}{48} Q = \frac{44}{48} Q,$$
$$\text{and } S = \frac{16n-1}{17n} R = \frac{48-1}{51} R = \frac{47}{51} R,$$
$$\text{and } T = \frac{17n-1}{18n} S = \frac{51-1}{54} S = \frac{50}{54} S.$$

Therefore the feries $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \frac{5n-1}{6n} Fx^6 + \frac{6n-1}{7n} Gx^7 - \frac{7n-1}{8n} Hx^8 + \frac{8n-1}{9n} Ix^9 - \frac{9n-1}{10n} Kx^{10} + \frac{10n-1}{11n} Lx^{11} - \frac{11n-1}{12n} Mx^{12} + \frac{12n-1}{13n} Nx^{13} - \frac{13n-1}{14n} Ox^{14} + \frac{14n-1}{15n} Px^{15} - \frac{15n-1}{16n} Qx^{16} + \frac{16n-1}{17n} Rx^{17} - \frac{17n-1}{18n} Sx^{18}$ + &c will, in this cafe, be $= 1 + \frac{1}{3} Ax - \frac{2}{6} Bx^2 + \frac{5}{9} Cx^3 - \frac{8}{12} Dx^4 + \frac{11}{15} Ex^5 - \frac{14}{18} Fx^6 + \frac{17}{21} Gx^7 - \frac{20}{24} Hx^8 + \frac{23}{27} Ix^9 - \frac{26}{30} Kx^{10} + \frac{29}{33} Lx^{11} - \frac{32}{36} Mx^{12} + \frac{35}{39} Nx^{13} - \frac{38}{42} Ox^{14} + \frac{41}{45} Px^{15} - \frac{44}{48} Qx^{16} + \frac{47}{51} Rx^{17} - \frac{50}{54} Sx^{18}$ + &c $= 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{154x^6}{6561} + \frac{374x^7}{19683} - \frac{935x^8}{59049} + \frac{21,505x^9}{1,594,323} - \frac{55,913x^{10}}{4,782,969} + \frac{147,407x^{11}}{14,348,907} - \frac{1,179,256x^{12}}{129,140,163} + \frac{3,174,920x^{13}}{387,420,489} - \frac{8,617,640x^{14}}{1,162,261,467} + \frac{70,664,648x^{15}}{10,460,353,203} - \frac{194,327,782x^{16}}{31,381,059,609} + \frac{537,259,162x^{17}}{94,143,178,827} - \frac{13,431,479,050x^{18}}{2,541,865,828,329}$ &c *; which therefore, is $= \overline{1 + x}|^{\frac{1}{3}}$, or the cube-root of the binomial quantity $1 + x$. Q. E. I.

52. Note. This feries for expreffing the value of $\overline{1 + x}|^{\frac{1}{3}}$, or the cube-root of the binomial quantity $1 + x$, and a fimilar feries for expreffing the value of $\overline{1 - x}|^{\frac{1}{3}}$, or the cube-root of the refidual quantity $1 - x$, will enable us to

* The co-efficients of x^{13}, x^{14}, x^{15}, x^{16}, x^{17}, x^{18} here given are equal to the co-efficients of the fame powers of x given above in page 192, but are reduced to fomewhat fmaller numbers by dividing both the numerators and denominators of the faid former co-efficients by 13 and 17.

extend

extend the well-known rules for resolving cubick equations of these two forms, to wit, $x^3 + qx = r$ and $x^3 - qx = r$, called *Cardan's Rules*, to that case of the second of these equations which they naturally are not fitted to solve, and which therefore has obtained the name of *the irreducible case*; which is the case in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$. This case must be subdivided into two branches, according as $\frac{rr}{4}$ is greater than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, (though less than $\frac{q^3}{27}$) and as it is less than $\frac{q^3}{54}$. When $\frac{rr}{4}$ is greater than $\frac{q^3}{54}$, though less than $\frac{q^3}{27}$, the equation $x^3 - qx = r$ may be resolved by extending to it, by a certain peculiar train of reasoning, the second of Cardan's rules, or that by which the same equation is to be resolved when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$; which extension may be performed by the help of the two infinite serieses that express the cube-roots of the binomial quantity $1 + x$ and the residual quantity $1 - x$, in a manner that is explained at considerable length in a Paper of mine published in the Philosophical Transactions for the year 1778, without any mention of impossible roots, or impossible quantities of any kind, or even of negative quantities. But when $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, the method explained in that Paper will not enable us to find the value of x in that equation, because the series obtained for the said value will not be a converging series. In this branch therefore of the irreducible case of the equation $x^3 - qx = r$, we are obliged to have recourse to another method of proceeding, and to determine the value of x by deriving it from that of the lesser of the two roots of the opposite equation $qx - x^3 = r$, which lesser root may be obtained by a similar extension of Cardan's first rule, or that by which he finds the root of the equation $x^3 + qx = r$, or $qx + x^3 = r$, to the equation $qx - x^3 = r$, by means of the said two serieses for expressing the cube-roots of $1 + x$ and $1 - x$; which extension may be made in a clear and intelligible manner, without any mention of impossible roots, or other impossible quantities, or even of negative quantities, any more than in the extension of Cardan's other rule. But this extension of Cardan's first rule to the resolution of the equation $qx - x^3 = r$ has not yet been published.

Of the Binomial Theorem in the case of $\overline{1 + x}\big|^{\frac{m}{n}}$, *or of the* nth *root of the* mth *power of the binomial quantity* $1 + x$, *or the* mth *power of its* nth *root, when* m *is any whole number whatsoever, and* n *any other whole number greater than* m.

53. Having now shewn in a pretty full, and, I hope, satisfactory manner, that the quantity $\overline{1 + x}\big|^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$, or the nth root of the binomial quantity

 $1 + x$,

$1 + x$, is equal to the feries $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5 -$ &c *ad infinitum*, or $1 + \frac{1}{n} A x - \overline{\left|\frac{n-1}{2n}\right.} B x^2 + \frac{2n-1}{3n} C x^3 - \overline{\left|\frac{3n-1}{4n}\right.} D x^4 + \frac{4n-1}{5n} E x^5 - \overline{\left|\frac{5n-1}{6n}\right.} F x^6 + \frac{6n-1}{7n} G x^7 - \overline{\left|\frac{7n-1}{8n}\right.} H x^8 + \frac{8n-1}{9n} I x^9 - \overline{\left|\frac{9n-1}{10n}\right.} K x^{10} +$ &c, *ad infinitum*, agreeably to what was afferted in art. 2; I fhall now proceed to confider the quantity $\overline{1+x}\big|^{\frac{m}{n}}$, or the nth root of the mth power of the binomial quantity $1 + x$, or (which comes to the fame thing) the mth power of its nth root, in the firft cafe of it, or when n, the index of the root, is greater than m the index of the power, and fhall endeavour to fhew that in this cafe, the faid quantity $\overline{1+x}\big|^{\frac{m}{n}}$ will be equal to the feries $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n} x^5 -$ &c *ad infinitum*, or (putting $A = 1$, and $B = \frac{m}{n}$, or the co-efficient of x, and $C = \frac{m}{n} \times \frac{n-m}{2n}$, or the co-efficient of x^2, and D, E, F, G, H, &c, for the co-efficients of x^3, x^4, x^5, x^6, x^7, and the following powers of x, in the fourth, fifth, fixth, feventh, eighth, and other following terms of the feries,) to the feries $1 + \frac{m}{n} A x - \overline{\left|\frac{n-m}{2n}\right.} B x^2 + \frac{2n-m}{3n} C x^3 - \overline{\left|\frac{3n-m}{4n}\right.} D x^4 + \frac{4n-m}{5n} E x^5 - \overline{\left|\frac{5n-m}{6n}\right.} F x^6 + \frac{6n-m}{7n} G x^7 - \overline{\left|\frac{7n-m}{8n}\right.} H x^8 + \frac{8n-m}{9n} I x^9 - \overline{\left|\frac{9n-m}{10n}\right.} K x^{10} +$ &c, *ad infinitum*, agreeably to what was afferted in art. 3. Now this may be done by methods of reafoning exactly fimilar to thofe which we have already made ufe of in finding the foregoing feries, which is equal to $\overline{1+x}\big|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$. Thefe methods may be explained as follows.

Obfervations preparatory to the Inveftigation of the Series fet forth in the foregoing article as being equal to the quantity $\overline{1+x}\big|^{\frac{m}{n}}$, *when* n *is greater than* m.

54. In the firft place, we muft obferve that the quantity $\overline{1+x}\big|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$, muft be equal to a feries

series of quantities of the same form with the series that is equal to $\overline{1 + x}|^{\frac{1}{n}}$, or the nth root itself of the same binomial quantity, that is, to a series of the form 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, of which the first term is 1, and the second and third, and other following terms, involve in them the several powers of x, to wit, x, x^2, x^3, x^4, x^5, &c, in their natural order without any interruption.

For, if a series of the said form 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, be multiplied into itself any number of times, or raised to any power denoted by the whole number m, it is evident that the product, or mth power of the said series, must always be a series of the same form, or a series of which 1 will be the first term, and of which the second and third, and other following terms, will involve the several powers of x in their natural order without any interruption. And therefore, since $\overline{1 + x}|^{\frac{1}{n}}$ has been shewn to be equal to a series of the said form 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, it follows that $\overline{1 + x}|^{\frac{m}{n}}$, or the mth power of $\overline{1 + x}|^{\frac{1}{n}}$, must also be equal to a series of the same form, 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c. Q. E. D.

55. In the second place, we must observe that the second term, Bx, of the series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1 + x}|^{\frac{m}{n}}$, must be added to the first term 1, and consequently marked with the sign +.

For, since $1 + x$ is greater than 1, the nth root of $1 + x$, or the first of $n - 1$ geometrical mean proportionals between 1 and $1 + x$, must also be greater than 1; and therefore, *à fortiori*, the mth power of the said nth root (which is greater than the said nth root itself) must also be greater than 1. Consequently the series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1 + x}|^{\frac{m}{n}}$, must also be greater than 1. And this must be true, of however small a magnitude we suppose x to be taken, so long as it has any magnitude at all. But, in order that the series 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, may be always greater than 1, however small the magnitude of x be taken, it is necessary that the second term Bx should be added to the first term 1. For, if Bx were subtracted from 1, it would be possible, by diminishing x, to make all the terms Cx^2, Dx^3, Ex^4, Fx^5, &c, that come after the second term Bx, become so much less than Bx, that the sum of them all put together should be less than Bx, whatever might be the magnitudes of the numeral co-efficients C, D, E, F, &c; in which case it is evident that the whole series $1 - Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, would (even though all the terms after Bx should be added to the first term 1, and the series should consequently be $1 - Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c) be less than the first term 1; which, we have shewn, it never can be. Therefore Bx cannot be subtracted from 1, but must be added to it; and the series that is equal to $\overline{1 + x}|^{\frac{m}{n}}$, must consequently be of this form, $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c. Q. E. D.

56. The

56. The truth of the foregoing obſervation, to wit, that the ſecond term, Bx, of the ſeries 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, is to be added to the firſt term 1, and not ſubtracted from it, will likewiſe appear by raiſing a few of the powers of the ſeries $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c$ (which is equal to $\overline{1+x}|^{\frac{1}{n}}$), by actual multiplication. For we ſhall find, in all the powers that we ſo raiſe, that the ſecond term is always added to the firſt; and we ſhall at the ſame time be eaſily able to perceive, from the manner of the operation, that the ſame thing muſt take place in all higher powers whatſoever of the ſame ſeries, and conſequently in the mth power of it, or in the ſeries that is equal to $\overline{1+x}|^{\frac{m}{n}}$. Now the ſecond, third, fourth, and fifth powers of the ſeries $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \&c$, carried only to three terms, may be raiſed in the manner following.

$$
\begin{array}{rcl}
\overline{1+x}|^{\frac{1}{n}} & = & 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c \\
\overline{1+x}|^{\frac{1}{n}} & = & 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c \\
\hline
 & & 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c \\
 & & \quad + \frac{1}{n}x + \frac{1}{n} \times \frac{1}{n}xx - \&c \\
 & & \qquad\qquad - \frac{1}{n} \times \frac{n-1}{2n}xx - \&c \\
\hline
\overline{1+x}|^{\frac{2}{n}} & = & 1 + \frac{2}{n}x - \frac{1}{n} \times \frac{2n-4}{2n}xx\ \&c \\
\overline{1+x}|^{\frac{1}{n}} & = & 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c \\
\hline
 & & 1 + \frac{2}{n}x - \frac{1}{n} \times \frac{2n-4}{2n}x^2\ \&c \\
 & & \quad + \frac{1}{n}x + \frac{1}{n} \times \frac{2}{n}x^2 - \&c \\
 & & \qquad\qquad - \frac{1}{n} \times \frac{n-1}{2n}x^2 - \&c \\
\hline
\overline{1+x}|^{\frac{3}{n}} & = & 1 + \frac{3}{n}x - \frac{1}{n} \times \frac{3n-9}{2n}x^2\ \&c \\
\overline{1+x}|^{\frac{1}{n}} & = & 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \&c \\
\hline
 & & 1 + \frac{3}{n}x - \frac{1}{n} \times \frac{3n-9}{2n}x^2\ \&c \\
 & & \quad + \frac{1}{n}x + \frac{1}{n} \times \frac{3}{n}x^2 - \&c \\
 & & \qquad\qquad - \frac{1}{n} \times \frac{n-1}{2n}x^2 - \&c
\end{array}
$$

$1 + x$

$$\overline{1+x}|^{\frac{4}{n}} = 1 + \frac{4}{n}x - \frac{1}{n} \times \frac{4n-16}{2n}x^2 \text{ \&c}$$

$$\overline{1+x}|^{\frac{1}{n}} = 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \text{\&c}$$

$$\begin{array}{l} 1 + \frac{4}{n}x - \frac{1}{n} \times \frac{4n-16}{2n}x^2 \text{ \&c} \\ \quad + \frac{1}{n}x + \frac{1}{n} \times \frac{4}{n}x^2 \text{ \&c} \\ \qquad\qquad - \frac{1}{n} \times \frac{n-1}{2n}x^2 - \text{\&c} \end{array}$$

$$\overline{1+x}|^{\frac{5}{n}} = 1 + \frac{5}{n}x - \frac{1}{n} \times \frac{5n-25}{2n}x^2 \text{ \&c.}$$

In all these powers of the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \text{\&c}$ (which is equal to $\overline{1+x}|^{\frac{1}{n}}$), the second term is added to the first term 1; and from the manner of raising these powers (which is done by multiplying the last series, or the series equal to the last power of $\overline{1+x}|^{\frac{1}{n}}$, by the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \text{\&c}$, and consequently by adding the product of the last series multiplied by $\frac{1}{n}x$ to the product of the same series multiplied by 1, or, properly speaking, not multiplied at all), it is evident that the same thing will always take place, or the second term will be always added to the first term, in all higher powers of the same series whatsoever, and consequently in the mth power of it, or in the series which is equal to $\overline{1+x}|^{\frac{m}{n}}$. Q. E. D.

57. And, in the 3d place, it is evident likewise from the foregoing operations (whereby we have obtained the three first terms of the serieses that are equal to $\overline{1+x}|^{\frac{2}{n}}$, $\overline{1+x}|^{\frac{3}{n}}$, $\overline{1+x}|^{\frac{4}{n}}$, and $\overline{1+x}|^{\frac{5}{n}}$, respectively), that the co-efficient of the second term of each of these powers of the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$, is always a fraction, of which the index of the power to which the value of $\overline{1+x}|^{\frac{1}{n}}$ is raised is the numerator, and n is the denominator. Thus, the co-efficient of the second term of the series that is equal to $\overline{1+x}|^{\frac{2}{n}}$, to wit, the series $1 + \frac{2}{n}x - \frac{1}{n} \times \frac{2n-4}{2n}x^2$ &c, is the fraction $\frac{2}{n}$; and the co-efficient of the second term of the series that is equal to $\overline{1+x}|^{\frac{3}{n}}$, to wit, the series $1 + \frac{3}{n}x - \frac{1}{n} \times \frac{3n-9}{2n}x^2$ &c, is the fraction $\frac{3}{n}$; and the co-efficient of the second

ſecond term of the ſeries that is equal to $\overline{1+x}\big|^{\frac{4}{n}}$, to wit, the ſeries $1 + \frac{4}{n}x - \frac{1}{n} \times \frac{4n-16}{2n}x^2$ &c, is the fraction $\frac{4}{n}$; and the co-efficient of the ſecond term of the ſeries that is equal to $\overline{1+x}\big|^{\frac{5}{n}}$, to wit, the ſeries $1 + \frac{5}{n}x - \frac{1}{n} \times \frac{5n-25}{2n}x^2$ &c, is the fraction $\frac{5}{n}$. And it is evident that the ſame thing will take place in all higher powers of $\overline{1+x}\big|^{\frac{1}{n}}$, or of the ſeries $1 + \frac{1}{n}x - \frac{1}{n} + \frac{n-1}{2n}x^2 +$ &c, and conſequently that the co-efficient of the ſecond term of the ſeries that is equal to $\overline{1+x}\big|^{\frac{m}{n}}$ will be $\frac{m}{n}$, and that the ſaid ſecond term itſelf will be $\frac{m}{n}x$. Therefore the two firſt terms of the ſeries that is equal to $\overline{1+x}\big|^{\frac{m}{n}}$ will be $1 + \frac{m}{n}x$. And theſe three obſervations will always be true, whether n be greater than m, or m be greater than n.

Theſe things being premiſed, the values of the following co-efficients, C, D, E, F, &c, of the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}\big|^{\frac{m}{n}}$ (m being any whole number whatſoever, and n any other whole number greater than m), may be inveſtigated in the manner following.

An Inveſtigation of the third, fourth, fifth, ſixth, and other following terms of the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}\big|^{\frac{m}{n}}$, or the m*th power of the* n*th root of the binomial quantity $1 + x$, when* m *is any whole number whatſoever, and* n *is any other whole number greater than* m.

58. Let y be put $=$ the ſeries $\frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, or $1 + y$ be equal the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}\big|^{\frac{m}{n}}$. Then will $\overline{1+y}\big|^{n}$ be equal to the nth power of $\overline{1+x}\big|^{\frac{m}{n}}$, or to $\overline{1+x}\big|^{m}$. But, by the binomial theorem in the caſe of integral powers (which has been demonſtrated above in the preceeding tract in pages 153, 154, &c — — 169,) $\overline{1+x}\big|^{m}$ is $=$ the ſeries $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2}$

 $\times \frac{m-2}{3}$

$\times \frac{m-2}{3} x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} x^5 +$ &c, and $\overline{1+y}|^n$ is $=$ the series $1 + \frac{n}{1} y + \frac{n}{1} \times \frac{n-1}{2} y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} y^5 +$ &c. Therefore the series $1 + \frac{m}{1} x + \frac{m}{1} \times \frac{m-1}{2} x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} x^5 +$ &c, will be equal to the series $1 + \frac{n}{1} y + \frac{n}{1} \times \frac{n-1}{2} y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} y^5 +$ &c; and consequently (subtracting 1 from both sides of the equation), the series $\frac{m}{1} x + \frac{m}{1} \times \frac{m-1}{2} x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} x^5 +$ &c, will be equal to the series $\frac{n}{1} y + \frac{n}{1} \times \frac{n-1}{2} y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} y^5 +$ &c.

59. To shorten the expressions of these co-efficients, let us put $Q = \frac{m}{1} \times \frac{m-1}{2}$, $R = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $S = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, and $T = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, and $q = \frac{n}{1} \times \frac{n-1}{2}$, $r = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, $s = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $t = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$. And we shall then have $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 +$ &c $= ny + qy^2 + ry^3 + sy^4 + ty^5 +$ &c.

60. We must now raise the several powers of the series $\frac{m}{n} x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, or Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to y), in order to have the values of y^2, y^3, y^4, y^5, &c, expressed in powers of x. This may be done in the manner following:

$$\begin{array}{rlllll}
y = & Bx, & Cx^2, & Dx^3, & Ex^4, & Fx^5, \ \&c \\
y = & Bx, & Cx^2, & Dx^3, & Ex^4, & Fx^5, \ \&c \\
\hline
 & B^2x^2, & BCx^3, & BDx^4, & BEx^5, & \&c \\
 & & BCx^3, & C^2x^4, & CDx^5, & \&c \\
 & & & BDx^4, & CDx^5, & \&c \\
 & & & & BEx^5, & \&c \\
\hline
y^2 = & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\
 & & & C^2x^4, & 2CDx^5, & \&c
\end{array}$$

$$
\begin{array}{rclll}
y & = & Bx, & Cx^2, & Dx^3,\ \&c \\
\hline
 & & B^3x^3, & 2B^2Cx^4, & 2B^2Dx^5,\ \&c \\
 & & & & BC^2x^5,\ \&c \\
 & & & B^2Cx^4, & 2BC^2x^5,\ \&c \\
 & & & & B^2Dx^5,\ \&c \\
\hline
y^3 & = & B^3x^3, & 3B^2Cx^4, & 3B^2Dx^5,\ \&c \\
 & & & & 3BC^2x^5,\ \&c \\
y & = & Bx, & Cx^2,\ \&c & \\
\hline
 & & B^4x^4, & 3B^3Cx^5,\ \&c & \\
 & & & B^3Cx^5,\ \&c & \\
\hline
y^4 & = & B^4x^4, & 4B^3Cx^5,\ \&c. & \\
y & = & Bx,\ \&c & & \\
\hline
y^5 & = & B^5x^5,\ \&c. & &
\end{array}
$$

61. By substituting these powers of the series Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c, instead of y, y^2, y^3, y^4, y^5, &c, in the equation $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c = ny + qy^2 + ry^3 + sy^4 + ty^5 + \&c$, we shall have the simple series $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c =$ the compound series

$$
\begin{array}{llllll}
nBx, & nCx^2, & nDx^3, & nEx^4, & nFx^5, & \&c \\
 & +qB^2x^2, & 2qBCx^3, & 2qBDx^4, & 2qBEx^5, & \&c \\
 & & & qC^2x^4, & 2qCDx^5, & \&c \\
 & & +rB^3x^3, & 3rB^2Cx^4, & 3rB^2Dx^5, & \&c \\
 & & & & 3rBC^2x^5, & \&c \\
 & & & +sB^4x^4, & 4sB^3Cx^5, & \&c \\
 & & & & +tB^5x^5, & \&c.
\end{array}
$$

Therefore, if we divide all the terms of this equation by x, we shall have the simple series $m + Qx + Rx^2 + Sx^3 + Tx^4 + \&c =$ the compound series

$$
\begin{array}{llllll}
nB, & nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\
 & +qB^2x, & 2qBCx^2, & 2qBDx^3, & 2qBEx^4, & \&c \\
 & & & qC^2x^3, & 2qCDx^4, & \&c \\
 & & +rB^3x^2, & 3rB^2Cx^3, & 3rB^2Dx^4, & \&c \\
 & & & & 3rBC^2x^4, & \&c \\
 & & & +sB^4x^3, & 4sB^3Cx^4, & \&c \\
 & & & & +tB^5x^4, & \&c.
\end{array}
$$

And this equation will be true, of however small a magnitude we suppose x to be taken: and consequently it will be true also when x is equal to o. But, when x is $=$ o, all the terms in this equation that involve x in them will be equal to o likewise, and consequently the equation will be as follows, to wit, $m = nB$. Therefore B will be $= \frac{m}{n}$, agreeably to what has been already shewn in art. 57, and consequently the two first terms of the series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$ will be $1 + \frac{m}{n}x$.

Q. E. I.

62. Since

62. Since m is equal to n B, it follows that, if we subtract m and n B from the opposite sides of the last equation, the remainders will be equal; that is, the simple series $Qx + Rx^2 + Sx^3 + Tx^4, + \&c$, will be equal to the compound series

$$\begin{array}{lllll}
nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\
+qB^2x, & 2qBCx^2, & 2qBDx^3, & 2qBEx^4, & \&c \\
 & & qC^2x^3, & 2qCDx^4, & \&c \\
 & +rB^3x^2, & 3rB^2Cx^3, & 3rB^2Dx^4, & \&c \\
 & & & 3rBC^2x^4, & \&c \\
 & & +sB^4x^3, & 4sB^3Cx^4, & \&c \\
 & & & +tB^5x^4, & \&c.
\end{array}$$

Therefore, if we divide all the terms by x, the quotients on both sides will be equal; that is, the simple series $Q + Rx + Sx^2 + Tx^3 + \&c$, will be equal to the compound series

$$\begin{array}{lllll}
nC, & nDx, & nEx^2, & nFx^3, & \&c \\
+qB^2, & 2qBCx, & 2qBDx^2, & 2qBEx^3, & \&c \\
 & & qC^2x^2, & 2qCDx^3, & \&c \\
 & +rB^3x, & 3rB^2Cx^2, & 3rB^2Dx^3, & \&c \\
 & & & 3rBC^2x^3, & \&c \\
 & & +sB^4x^2, & 4sB^3Cx^3, & \&c \\
 & & & +tB^5x^3, & \&c.
\end{array}$$

And this equation will always be true, to how small a quantity soever we suppose x to be diminished; and consequently it will be true also when x is $= 0$. But, when x is $= 0$, all the terms on both sides of the equation, that involve x, will also be equal to 0, and consequently the equation will be as follows, to wit, $Q = nC + qB^2$, that is, (because Q is $= \frac{m}{1} \times \frac{m-1}{2}$, and q is $= \frac{n}{1} \times \frac{n-1}{2}$) $\frac{m}{1} \times \frac{m-1}{2}$ will be $= nC + \frac{n}{1} \times \frac{n-1}{2} \times B^2$, or (because B is $= \frac{m}{n}$) $\frac{m}{1} \times \frac{m-1}{2}$ will be $= nC + \frac{n}{1} \times \frac{n-1}{2} \times \frac{m}{n} \times \frac{m}{n}$, or $\frac{m}{1} \times \frac{m-1}{2}$ will be $= nC + \frac{n-1}{2n} \times m^2$, or $mn \times \frac{m-1}{2n}$ will be $= nC + \frac{n-1}{2n} \times m^2$, or $\frac{m^2n-mn}{2n}$ will be $= nC + \frac{m^2n-m^2}{2n}$; whence (adding $\frac{mn}{2n}$ to both sides) we shall have $\frac{m^2n}{2n} = nC + \frac{m^2n-m^2+mn}{2n}$, and (adding $\frac{m^2}{2n}$ to both sides) $\frac{m^2n+m^2}{2n} = nC + \frac{m^2n+mn}{2n}$, and (subtracting $\frac{m^2n}{2n}$ from both sides), $\frac{m^2}{2n} = nC + \frac{mn}{2n}$; that is, $\frac{m^2}{2n}$ will be equal to $\frac{mn}{2n}$, together with nC, either added to it or subtracted from it, as may be necessary to produce such equality. But, because n is supposed to be greater than m, $\frac{mn}{2n}$ will be greater than $\frac{mm}{2n}$; and consequently nC must be subtracted from $\frac{mn}{2n}$ in order to make it equal to $\frac{mm}{2n}$, or $\frac{m^2}{2n}$. We shall therefore have $\frac{m^2}{2n} = -nC + \frac{mn}{2n}$, and consequently (adding nC to both sides) $\frac{m^2}{2n} + nC = \frac{mn}{2n}$, and (subtracting $\frac{m^2}{2n}$ from both sides) $nC = \frac{mn}{2n} - \frac{m^2}{2n} = \frac{mn-m^2}{2n} =$

$\frac{m\times\overline{n-m}}{2n} = m \times \frac{n-m}{2n}$, and consequently $C = \frac{m}{n} \times \frac{n-m}{2n}$. Therefore the third term, Cx^2, of the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, must have the sign $-$ prefixed to it, and will be $= \frac{m}{n} \times \frac{n-m}{2n} \times x^2$, and the three first terms of the said series will be $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2$. Q. E. I.

63. It has been shewn, in the last article, that Q is $= nC + qB^2$, or (because it has also been shewn that nC is to be subtracted from qB^2, or to have the sign $-$ prefixed to it) $-nC + qB^2$. Therefore, if we subtract Q and $-nC + qB^2$ from the opposite sides of the equation which was obtained in that article by the division of x, the remainders will be equal to each other; that is, the simple series $Rx + Sx^2 + Tx^3 +$ &c will be equal to the compound series

$$\begin{array}{llll}
nDx, & nEx^2, & nFx^3, & \&c \\
2qBCx, & 2qBDx^2, & 2qBEx^3, & \&c \\
 & qC^2x^2, & 2qCDx^3, & \&c \\
+rB^3x, & 3rB^2Cx^2, & 3rB^2Dx^3, & \&c \\
 & & 3rBC^2x^3, & \&c \\
 & +sB^4x^2, & 4sB^3Cx^3, & \&c \\
 & & +tB^5x^3, & \&c\text{: or, if}
\end{array}$$

we prefix the sign $-$ to the terms $2qBCx$, $3rB^2Cx^2$, and $4sB^3Cx^3$ (in which the simple power of C occurs with the powers of B), and the sign $+$ to the terms qC^2x^2, $3rBC^2x^3$ (in which the square of C occurs by itself and with B), the simple series $Rx + Sx^2 + Tx^3 +$ &c will be equal to the compound series

$$\begin{array}{llll}
nDx, & nEx^2, & nFx^3, & \&c \\
-2qBCx, & 2qBDx^2, & 2qBEx^3, & \&c \\
 & +qC^2x^2, & 2qCDx^3, & \&c \\
+rB^3x & -3rB^2Cx^2, & 3rB^2Dx^3, & \&c \\
 & & +3rBC^2x^3, & \&c \\
 & +sB^4x^2 & -4sB^3Cx^3, & \&c \\
 & & +tB^5x^3, & \&c.
\end{array}$$

Therefore, if we divide all the terms of this equation by x, we shall have the simple series $R + Sx + Tx^2 +$ &c $=$ the compound series

$$\begin{array}{llll}
nD, & nEx, & nFx^2, & \&c \\
-2qBC, & 2qBDx, & 2qBEx^2, & \&c \\
 & +qC^2x, & 2qCDx^2, & \&c \\
+rB^3 & -3rB^2Cx, & 3rB^2Dx^2, & \&c \\
 & & +3rBC^2x^2, & \&c \\
 & +sB^4x & -4sB^3Cx^2, & \&c \\
 & & +tB^5x^2, & \&c.
\end{array}$$

And this equation will be always true, to how small a quantity soever we suppose x to be diminished: and therefore it will be true also when x is $= 0$. But, when

when x is $= 0$, all the terms in the equation that involve x will be equal to 0 likewiſe, and conſequently the equation will be as follows, to wit, $R = nD - 2qBC + rB^3$; by examining and reſolving which equation we may both determine whether the ſign $+$ or the ſign $-$ is to be prefixed to the quantity nD, and conſequently to Dx^3, or the fourth term of the ſeries $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$ (from which fourth term the ſaid quantity nD has been derived by the operations of multiplication and diviſion in the courſe of the foregoing proceſſes), and likewiſe what is the magnitude of D, and conſequently that of Dx^3. Now this reſolution will be found to be a matter of ſome intricacy. It may, however, be performed in the manner following:

64. By ſubſtituting $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$ in this equation inſtead of R, and $\frac{n}{1} \times \frac{n-1}{2}$ inſtead of q, and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ inſtead of r, and $\frac{m}{n}$ inſtead of B, and $\frac{m}{n} \times \frac{n-m}{2n}$ inſtead of C, it will become as follows, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} = nD - n \times \overline{n-1} \times \frac{m}{n} \times \frac{m}{n} \times \frac{n-m}{2n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{m^3}{n^3}$, or $\frac{m^3-3m^2+2m}{6} = nD - m \times \overline{n-1} \times \frac{m}{n} \times \frac{n-m}{2n} + \frac{n^3-3n^2+2n}{6} \times \frac{m^3}{n^3}$, or $\frac{m^3n^2-3m^2n^2+2mn^2}{6nn} = nD - \frac{3m^2}{3n} \times \overline{n-1} \times \frac{n-m}{2n} + m^3 \times \overline{\frac{nn-3n+2}{6nn}}$ $= nD - 3m^2 \times \overline{\frac{nn-nm-n+m}{6nn}} + \frac{m^3n^2-3m^3n+2m^3}{6nn} = nD - \frac{3m^2n^2+3m^3n+3m^2n-3m^3}{6nn} + \frac{m^3n^2-3m^3n+2m^3}{6nn} = nD - \frac{3m^2n^2+3m^2n-m^3+m^3n^2}{6nn}$; and conſequently (adding $\frac{3m^2n^2}{6nn}$ to both ſides) we ſhall have $\frac{m^3n^2+2mn^2}{6nn} = nD + \frac{3m^2n-m^3+m^3n^2}{6nn}$, and (adding $\frac{m^3}{6nn}$ to both ſides) $\frac{m^3n^2+2mn^2+m^3}{6nn} = nD + \frac{3m^2n+m^3n^2}{6nn}$, and (ſubtracting $\frac{m^3n^2}{6nn}$ from both ſides) $\frac{2mn^2+m^3}{6nn} = nD + \frac{3m^2n}{6nn}$.

Now $\frac{2mn^2+m^3}{6nn}$ is greater than $\frac{3m^2n}{6nn}$, as may be thus demonſtrated. Since n is greater than m, mn^2 will be greater than m^2n, and conſequently $2m^2n + mn^2$ will be greater than $2m^2n + m^2n$, or than $3m^2n$. Further, ſince mn^2, m^2n, and m^3 are in continued proportion, the common ratio being that of n to m, it follows from Euclid's Elements, Book 5, Prop. 25, that the ſum of the two extreme terms will be greater than twice the middle term; that is, $mn^2 + m^3$ will be greater than $2m^2n$. Therefore $mn^2 + mn^2 + m^3$ will be greater than $mn^2 + 2m^2n$, or $2mn^2 + m^3$ will be greater than $mn^2 + 2m^2n$. But it has been ſhewn that $mn^2 + 2m^2n$ is greater than $m^2n + 2m^2n$, or $3m^2n$. Therefore $2mn^2 + m^3$ (which is greater than $mn^2 + 2m^2n$) will, *à fortiori*, be greater than $3m^2n$. Therefore $\frac{2mn^2+m^3}{6nn}$ will be greater than $\frac{3m^2n}{6nn}$. Q. E. D.

Since

Since therefore $\frac{2mn^2+m^3}{6nn}$ is greater than $\frac{3m^2n}{6nn}$, but equal to $n\text{D} + \frac{3m^2n}{6nn}$, it follows that $n\text{D}$ muſt be added to $\frac{3m^2n}{6nn}$ in order to produce the ſaid equality. Therefore the ſign + muſt be prefixed to $n\text{D}$, and conſequently to $\text{D}x^3$, or the fourth term of the ſeries $1 + \text{B}x$, $\text{C}x^2$, $\text{D}x^3$, $\text{E}x^4$, $\text{F}x^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, or the ſaid fourth term muſt be added to the firſt term 1. Therefore the four firſt terms of the ſaid ſeries will be $1 + \text{B}x - \text{C}x^2 + \text{D}x^3$, or $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + \text{D}x^3$; which was the firſt point that was to be determined.

And, ſecondly, the magnitude of D may be determined by means of the equation laſt-obtained, to wit, $\frac{2mn^2+m^3}{6nn} = n\text{D} + \frac{3m^2n}{6nn}$, or $\frac{2mn^2+m^3}{6nn} = +n\text{D} + \frac{3m^2n}{6nn}$. For, by ſubtracting $\frac{3m^2n}{6nn}$ from both ſides, we ſhall have $n\text{D} = \frac{2mn^2-3m^2n+m^3}{6nn} = \frac{\overline{n-m} \times \overline{2nm-m^2}}{6nn} = m \times \frac{\overline{n-m}}{2n} \times \frac{2n-m}{3n}$, and conſequently (dividing both ſides by n) $\text{D} = \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}$, or $\text{D} = \text{C} \times \frac{2n-m}{3n}$, or $\frac{2n-m}{3n} \times \text{C}$. Therefore the four firſt terms of the ſeries $1 + \text{B}x$, $\text{C}x^2$, $\text{D}x^3$, $\text{E}x^4$, $\text{F}x^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, are $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}x^3$, or $1 + \frac{m}{n}\text{A}x - \frac{n-m}{2n}\text{B}x^2 + \frac{2n-m}{3n}\text{C}x^3$.

Q. E. I.

65. It is poſſible, in the like manner, to determine the ſigns + or − that are to be prefixed to the following terms, $\text{E}x^4$, $\text{F}x^5$, $\text{G}x^6$, &c, of the ſeries $1 + \text{B}x$, $\text{C}x^2$, $\text{D}x^3$, $\text{E}x^4$, $\text{F}x^5$, $\text{G}x^6$, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), and to find the values of the co-efficients E, F, G, &c, by the reſolution of the like ſimple equations which may be derived from the fundamental equation obtained in art. 61. The equation for determining the ſign and value of the fifth term, $\text{E}x^4$, of the ſaid ſeries, is as follows, to wit, $\text{S} = n\text{E} + 2q\text{BD} + q\text{C}^2 - 3r\text{B}^2\text{C} + s\text{B}^4$; and the equation for determining the ſign and value of the ſixth term, $\text{F}x^5$, of the ſaid ſeries, is as follows, to wit, $\text{T} = n\text{F} - 2q\text{BE} - 2q\text{CD} + 3r\text{B}^2\text{D} + 3r\text{BC}^2 - 4s\text{B}^3\text{C} + t\text{B}^5$. But the computations that would be neceſſary to the reſolution of theſe equations would (as the reader will eaſily conceive from the difficulty we have found in reſolving the laſt equation for determining the ſign and magnitude of $\text{D}x^3$) be exceſſively complicated and troubleſome; and in the equations for determining the following terms $\text{G}x^6$, $\text{H}x^7$, $\text{I}x^8$, $\text{K}x^9$, $\text{L}x^{10}$, &c, of the ſaid ſeries, the intricacy of the calculations would be ſo great as to make the reſolution of thoſe equations abſolutely impracticable. And therefore I ſhall proceed no further in this method of in-

veſtgating

vestigating the terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which equal to $\overline{1+x}|^{\frac{m}{n}}$.

66. And further we may observe, concerning this method of investigating the terms of the said series, that, if it were not so excessively laborious in the practice as we have seen it to be after the two or three first terms of it, but we were able to compute twenty or thirty of its terms with only a moderate degree of trouble, it would still be impossible to ascertain, by means of the terms so computed, the law of their generation, or continuation, one from another, so as to be able to conclude with certainty that the said law must take place with respect to the terms that had not actually been computed, as well as with respect to those that had: which is the object we ought principally to aim at when we are investigating the terms of this series. I shall therefore on this account, as well as on account of the increasing intricacy of the calculations that occur in it, desist from all further prosecution of this method of investigating the terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is $= \overline{1+x}|^{\frac{m}{n}}$; and shall proceed to lay before the reader another method of investigating it, which will both be much easier to practise than the foregoing method, and will also enable us to discover the law by which the co-efficients B, C, D, E, F, G, H, I, K, L, &c, of the powers of x are generated, or formed, one from another; and to conclude that the said law must take place in the terms which have not been actually computed or investigated, as well as in those that have. This method is exactly similar to that by which we investigated the terms of the series that is equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, in art. 36, 37, 38, 39, 40, and 41.

Another Investigation of the third, fourth, fifth, sixth, and other following terms of the Series $1 + \frac{m}{n}, x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, or the m*th power of the* n*th root of the binomial quantity* $1 + x$, *when* m *is any whole number whatsoever, and* n *is any other whole number greater than* m.

---

67. It has been shewn above, in the three observations contained in art. 54, 55, 56, and 57, that, if x be of any magnitude less than 1, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of $\overline{1+x}|^{\frac{1}{n}}$ or of the nth root of $1 + x$, will be equal to the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, in which C, D, E, F, &c, represent certain numeral co-efficients of x^2, x^3, x^4, x^5, &c, that are always the same, whatever

whatever may be the magnitude of x. It follows, therefore, that if we ſuppoſe x to be increaſed from any one particular magnitude denoted by x, to any other magnitude (leſs than 1) that is denoted by y, the quantity $\overline{1+y}|^{\frac{m}{n}}$, or the mth power of $\overline{1+y}|^{\frac{1}{n}}$ or of the nth root of $1+y$, will be equal to the ſeries $1+\frac{m}{n}y$, Cy^2, Dy^3, Ey^4, Fy^5, &c. Now let d be equal to the difference by which y exceeds x; ſo that $x+d$ ſhall be equal to y: and let $x+d$ be ſubſtituted inſtead of y in the laſt equation $\overline{1+y}|^{\frac{m}{n}} = 1+\frac{m}{n}y$, Cy^2, Dy^3, Ey^4, Fy^5, &c. And we ſhall then have $\overline{1+x+d}|^{\frac{m}{n}} =$ the ſeries $1+\frac{m}{n}\times\overline{x+d}$, $C\times\overline{x+d}|^2$, $D\times\overline{x+d}|^3$, $E\times\overline{x+d}|^4$, $F\times\overline{x+d}|^5$, &c $=$ the ſeries $1+\frac{m}{n}\times\overline{x+d}$, $C\times\overline{x^2+2xd+d^2}$, $D\times\overline{x^3+3x^2d+3xd^2+d^3}$, $E\times\overline{x^4+4x^3d+6x^2d^2+4xd^3+d^4}$, $F\times\overline{x^5+5x^4d+10x^3d^2+10x^2d^3+5xd^4+d^5}$, $+$ &c $=$ the compound ſeries

$$\begin{array}{llllll}
1+\frac{m}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\
+\frac{m}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c.
\end{array}$$

68. Let f be $= 1+x$. And we ſhall have $f+d = 1+x+d$, and $\overline{f+d}|^{\frac{m}{n}} = \overline{1+x+d}|^{\frac{m}{n}}$. But $f+d$ is $= f\times\overline{1+\frac{d}{f}}$. Therefore $\overline{f+d}|^{\frac{m}{n}}$ will be $= f^{\frac{m}{n}}\times\overline{1+\frac{d}{f}}|^{\frac{m}{n}}$. But, becauſe $\overline{1+x}|^{\frac{m}{n}}$ is equal to the ſeries $1+\frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, it follows that $\overline{1+\frac{d}{f}}|^{\frac{m}{n}}$ will, in like manner, be equal to the ſeries $1+\frac{m}{n}\times\frac{d}{f}$, $C\times\overline{\frac{d}{f}}|^2$, $D\times\overline{\frac{d}{f}}|^3$, $E\times\overline{\frac{d}{f}}|^4$, $F\times\overline{\frac{d}{f}}|^5$, &c $=$ the ſeries $1+\frac{m}{n}\times\frac{d}{f}$, $C\times\frac{d^2}{f^2}$, $D\times\frac{d^3}{f^3}$, $E\times\frac{d^4}{f^4}$, $F\times\frac{d^5}{f^5}$, &c. Therefore $f^{\frac{m}{n}}\times\overline{1+\frac{d}{f}}|^{\frac{m}{n}}$ will be equal to $f^{\frac{m}{n}}\times$ the ſeries $1+\frac{m}{n}\times\frac{d}{f}$, $C\times\frac{d^2}{f^2}$, $D\times\frac{d^3}{f^3}$, $E\times\frac{d^4}{f^4}$, $F\times\frac{d^5}{f^5}$, &c $=$ the ſeries $f^{\frac{m}{n}}+\frac{m}{n}\times f^{\frac{m}{n}}\times\frac{d}{f}$, $C\times f^{\frac{m}{n}}\times\frac{d^2}{f^2}$, $D\times f^{\frac{m}{n}}\times\frac{d^3}{f^3}$, $E\times f^{\frac{m}{n}}\times\frac{d^4}{f^4}$, $F\times f^{\frac{m}{n}}\times\frac{d^5}{f^5}$, &c.

$\times f^{\frac{m}{n}} \times \frac{d^5}{f^5}$, &c. Therefore $\overline{f+d}|^{\frac{m}{n}}$ (which is $= f^{\frac{m}{n}} \times \overline{1+\frac{d}{f}}|^{\frac{m}{n}}$) will alſo be equal to the ſeries $f^{\frac{m}{n}} + \frac{m}{n} \times f^{\frac{m}{n}} \times \frac{d}{f}$, $C \times f^{\frac{m}{n}} \times \frac{d^2}{f^2}$, $D \times f^{\frac{m}{n}} \times \frac{d^3}{f^3}$, $E \times f^{\frac{m}{n}} \times \frac{d^4}{f^4}$, $F \times f^{\frac{m}{n}} \times \frac{d^5}{f^5}$, &c. Therefore, if we ſubſtitute $1 + x$ in this laſt equation inſtead of f, to which it is equal, we ſhall have $\overline{1+x+d}|^{\frac{m}{n}}$ = the ſeries $\overline{1+x}|^{\frac{m}{n}} + \frac{m}{n} \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c.

69. But it has been ſhewn in art. 67, that $\overline{1+x+d}|^{\frac{m}{n}}$ is equal to the compound ſeries

$$\begin{array}{llllll}
1 + \frac{m}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\
+ \frac{m}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c.
\end{array}$$

Therefore the ſeries $\overline{1+x}|^{\frac{m}{n}} + \frac{m}{n} \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c, will be equal to the ſaid compound ſeries

$$\begin{array}{llllll}
1 + \frac{m}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\
+ \frac{m}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c.
\end{array}$$

70. Now let $\overline{1+x}|^{\frac{m}{n}}$ be ſubtracted from the left-hand ſide of this laſt equation; and the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), be ſubtracted from the right-hand ſide of it. Then, it is evident, the remainders will be equal to each other; that is, the ſeries $\frac{m}{n} \times \overline{1+x}|^{\frac{m}{n}}$

$\times \frac{d}{1+x}$, $C \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^2}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^3}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^4}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^5}{\overline{1+x}|^5}$, &c, will be equal to the compound series

$$\begin{array}{llllll}
\frac{m}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\
 & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\
 & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\
 & & & Ed^4, & 5Fxd^4, & \&c \\
 & & & & Fd^5, & \&c.
\end{array}$$

And consequently, if we divide all the terms by d, we shall have the series $\frac{m}{n} \times \overline{1+x}|^{\frac{m}{n}} \times \frac{1}{1+x}$, $C \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d}{\overline{1+x}|^2}$, $D \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^2}{\overline{1+x}|^3}$, $E \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^3}{\overline{1+x}|^4}$, $F \times \overline{1+x}|^{\frac{m}{n}} \times \frac{d^4}{\overline{1+x}|^5}$, &c = the compound series

$$\begin{array}{llllll}
\frac{m}{n}, & 2Cx, & 3Dx^2, & 4Ex^3, & 5Fx^4, & \&c \\
 & Cd, & 3Dxd, & 6Ex^2d, & 10Fx^3d, & \&c \\
 & & Dd^2, & 4Exd^2, & 10Fx^2d^2, & \&c \\
 & & & Ed^3, & 5Fxd^3, & \&c \\
 & & & & Fd^4, & \&c.
\end{array}$$

71. This equation is always true, how small soever we may suppose d to be: and therefore it will be true also when d is $= 0$. But, when d is $= 0$, all the terms on both sides the equation that involve d will be equal to 0 likewise; and consequently the equation will then be as follows, to wit, $\frac{m}{n} \times \overline{1+x}|^{\frac{m}{n}} \times \frac{1}{1+x}$ = the series $\frac{m}{n}$, $2Cx$, $3Dx^2$, $4Ex^3$, $5Fx^4$, &c. Therefore, if we multiply all the terms by the fraction $\frac{n}{m}$, we shall have $\overline{1+x}|^{\frac{m}{n}} \times \frac{1}{1+x} =$ the series 1, $\frac{2n}{m}Cx$, $\frac{3n}{m}Dx^2$, $\frac{4n}{m}Ex^3$, $\frac{5n}{m}Fx^4$, &c; and, if we multiply both sides of this last equation by $1 + x$, we shall have $\overline{1+x}|^{\frac{m}{n}}$ = the compound series

$$\begin{array}{llllll}
1, & \frac{2n}{m}Cx, & \frac{3n}{m}Dx^2, & \frac{4n}{m}Ex^3, & \frac{5n}{m}Fx^4, & \&c. \\
 & +x, & \frac{2n}{m}Cx^2, & \frac{3n}{m}Dx^3, & \frac{4n}{m}Ex^4, & \&c.
\end{array}$$

72. But $\overline{1+x}|^{\frac{m}{n}}$ is also equal to the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c. Therefore the said series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, will be equal to the compound series

$$1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \frac{5n}{m}Fx^4, \&c.$$
$$+ x, \frac{2n}{m}Cx^2, \frac{3n}{m}Dx^3, \frac{4n}{m}Ex^4, \&c;$$

and consequently (subtracting 1 from both sides of the equation) we shall have the series $\frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c = the compound series

$$\frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \frac{5n}{m}Fx^4, \&c,$$
$$+ x, \frac{2n}{m}Cx^2, \frac{3n}{m}Dx^3, \frac{4n}{m}Ex^4, \&c;$$

and, lastly (dividing all the terms by x), we shall have the simple series $\frac{m}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c, equal to the compound series

$$\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c$$
$$+ 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c;$$

by the help of which equation we may discover both "which of the signs + and − are to be prefixed to the several co-efficients C, D, E, F, &c (and consequently to the corresponding terms Cx^2, Dx^3, Ex^4, Fx^5, &c, in the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$)," and also, "what are the magnitudes of the said co-efficients C, D, E, F, &c, respectively." This may be done by proceeding in the manner following:

73. In the first place, since the simple series $\frac{m}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c, is equal to the compound series

$$\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c$$
$$+ 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c,$$

and this equation is always true, of how small a magnitude soever we suppose x to be taken, it follows that it will also be true when x is = o. But, when x is = o, all the terms in the equation that involve x will be equal to o likewise; that is, all the terms on the left-hand side of the equation, except the first term $\frac{m}{n}$, and all the terms on the right-hand side of the equation, except the two first terms of the two lines of terms, to wit, $\frac{2n}{m}C + 1$, will be equal to o. Therefore $\frac{m}{n}$ will be = to $\frac{2n}{m}C + 1$, or to $1, \frac{2n}{m}C$, or to 1 together with $\frac{2n}{m}C$, either added to it or subtracted from it, as may be necessary to produce such equality. Now, because n is supposed to be greater than m, $\frac{m}{n}$ will be less than 1, and consequently $\frac{2n}{m}C$ must be subtracted from 1, in order to make it equal

to $\frac{m}{n}$. Therefore the sign $-$ must be prefixed to $\frac{2n}{m}$ C in the equation $\frac{m}{n} = \frac{2n}{m}C + 1$, or $\frac{m}{n} = 1, \frac{2n}{m}C$; and consequently the sign $-$ must also be prefixed to the third term Cx^2 of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), from which third term the said quantity $\frac{2n}{m}C$ has been derived by the operations of multiplication and division in the course of the foregoing processes. Therefore the three first terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x - Cx^2$. Q. E. I.

And the magnitude of the co-efficient C may likewise be determined by means of the equation $\frac{m}{n} = 1, \frac{2n}{m}C$, or $\frac{m}{n} = 1 - \frac{2n}{m}C$. For, by adding $\frac{2n}{m}C$ to both sides, we shall have $\frac{m}{n} + \frac{2n}{m}C = 1$; and, by subtracting $\frac{m}{n}$ from both sides, we shall have $\frac{2n}{m}C = 1 - \frac{m}{n} = \frac{n-m}{n}$; and (by dividing both sides by $\frac{2n}{m}$, or multiplying them into $\frac{m}{2n}$) $C = \frac{m}{2n} \times \frac{n-m}{n} = \frac{m}{n} \times \frac{n-m}{2n}$. Therefore the three first terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is $= \overline{1+x}|^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2$, or $1 + \frac{m}{n}Ax - \frac{n-m}{2n}Bx^2$. Q. E. I.

74. To determine the sign that is to be prefixed to the fourth term, Dx^3, of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, and to find the magnitude of its co-efficient D, we must proceed as follows:

Since the sign $-$ is to be prefixed to the co-efficient C, it must likewise be prefixed to all the terms which involve it in the grand, fundamental equation obtained in art. 72, to wit, the equation between the simple series $\frac{m}{n}, Cx, Dx^2, Ex^3, Fx^4$, &c, and the compound series

$$\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c$$
$$+ 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c;$$

which equation will therefore be as follows, to wit, the simple series $\frac{m}{n} - Cx, Dx^2, Ex^3, Fx^4$, &c $=$ the compound series

$$-\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c$$
$$+ 1 - \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c.$$

Let

Let $2\,C\,x$ be added to the two sides of this equation; and we shall then have the simple series $\frac{m}{n} + C\,x,\ D\,x^2,\ E\,x^3,\ F\,x^4$, &c = the compound series

$$\begin{array}{l} -\frac{2n}{m}C,\ \frac{3n}{m}D\,x,\ \frac{4n}{m}E\,x^2,\ \frac{5n}{m}F\,x^3,\ \&c \\ +\ 1,\ -\frac{2n}{m}C\,x,\ \frac{3n}{m}D\,x^2,\ \frac{4n}{m}E\,x^3,\ \&c \\ +\ 2\,C\,x. \end{array}$$

But it has been shewn in the last article, that $\frac{m}{n}$ is $= 1 - \frac{2n}{m}C$. Therefore if we subtract $\frac{m}{n}$ and $1 - \frac{2n}{m}C$ from the opposite sides of the last equation, the remainders will be equal to each other; that is, the simple series $C\,x,\ D\,x^2,\ E\,x^3,\ F\,x^4$, &c, will be equal to the compound series

$$\begin{array}{l} \frac{3n}{m}D\,x,\ \frac{4n}{m}E\,x^2,\ \frac{5n}{m}F\,x^3,\ \&c \\ -\frac{2n}{m}C\,x,\ \frac{3n}{m}D\,x^2,\ \frac{4n}{m}E\,x^3,\ \&c \\ +\ 2\,C\,x; \end{array}$$

and consequently (dividing all the terms by x), we shall have the simple series $C,\ D\,x,\ E\,x^2,\ F\,x^3$, &c = the compound series

$$\begin{array}{l} \frac{3n}{m}D,\ \frac{4n}{m}E\,x,\ \frac{5n}{m}F\,x^2,\ \&c \\ -\frac{2n}{m}C,\ \frac{3n}{m}D\,x,\ \frac{4n}{m}E\,x^2,\ \&c \\ +\ 2\,C. \end{array}$$

And this equation will be true, how small soever we may suppose x to be: and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will be $C = \frac{3n}{m}D - \frac{2n}{m}C + 2\,C$. Therefore $C + \frac{2n}{m}C$ will be $= \frac{3n}{m}D + 2\,C$, and $\frac{2n}{m}C$ will be $= \frac{3n}{m}D + C$, or $C,\ \frac{3n}{m}D$; that is, $\frac{2n}{m}C$ will be equal to C together with $\frac{3n}{m}D$ either added to it or subtracted from it, as may be necessary to produce the said equality. But, because n is greater than m, $\frac{2n}{m}C$ will be greater than $\frac{2m}{m}C$, or than $2\,C$, and, *à fortiori*, greater than C. Therefore, in order to make C and $\frac{3n}{m}D$ be equal to $\frac{2n}{m}C$, $\frac{3n}{m}D$ must be added to C, and consequently must have the sign + prefixed to it. Therefore the sign + must also be prefixed to the fourth term, $D\,x^3$, of the series $1 + \frac{m}{n}x,\ C\,x^2,\ D\,x^3,\ E\,x^4,\ F\,x^5$, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), from which fourth term the quantity $\frac{3n}{m}D$ has been derived by the operations of multiplication and division in the course of the foregoing processes. Therefore the four first terms of the said series will be $1 + \frac{m}{n}x - C\,x^2 + D\,x^3$. Q. E. I.

And

And the magnitude of the co-efficient D may likewise be determined by means of the equation $\frac{2n}{m}$ C = C + $\frac{3n}{m}$ D. For, by subtracting C from both sides, we shall have $\frac{3n}{m}D = \frac{2n}{m}C - C = \overline{\frac{2n}{m} - 1}| \times C = \frac{2n-m}{m} \times C$, and (by dividing both sides by $\frac{3n}{m}$, or multiplying them into $\frac{m}{3n}$) $D = \frac{2n-m}{3n} \times C$, or (because C is $= \frac{m}{n} \times \frac{n-m}{2n}$) $D = \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}$. Q. E. I.

Therefore the four first terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, are $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}x^3$, or $1 + \frac{m}{n}Ax - \frac{n-m}{2n}Bx^2 + \frac{2n-m}{3n}Cx^3$. Q. E. I.

75. To determine the sign that is to be prefixed to the fifth term, Ex^4, of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, and to find the magnitude of the co-efficient E, we must proceed as follows:

It has been shewn in the last article, that the simple series C, Dx, Ex^2, Fx^3, &c is = the compound series

$$\begin{array}{l} \frac{3n}{m}D, \frac{4n}{m}Ex, \frac{5n}{m}Fx^2, \&c \\ -\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \&c \\ +2C, \end{array}$$

or (prefixing the sign + to the terms which involve the co-efficient D), that the simple series $C + Dx, Ex^2, Fx^3$, &c, is = the compound series

$$\begin{array}{l} +\frac{3n}{m}D, \frac{4n}{m}Ex, \frac{5n}{m}Fx^2, \&c \\ -\frac{2n}{m}C + \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \&c \\ +2C. \end{array}$$

And it has been shewn also in the same article, that C is $= +\frac{3n}{m}D - \frac{2n}{m}C + 2C$. Therefore, if we subtract C from the left-hand side of the last equation, and $+\frac{3n}{m}D - \frac{2n}{m}C + 2C$ from the right-hand side of it, the remainders will be equal; that is, the simple series $+ Dx$, Ex^2, Fx^3, &c, will be = the compound series

$$\begin{array}{l} \frac{4n}{m}Ex, \frac{5n}{m}Fx^2, \&c \\ +\frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \&c; \end{array}$$

and consequently (dividing all the terms by x) the simple series + D, Ex, Fx^2, &c, will be = the compound series

$$\begin{array}{l} \frac{4n}{m}E, \frac{5n}{m}Fx, \&c \\ +\frac{3n}{m}D, \frac{4n}{m}Ex, \&c. \end{array}$$

And

And this equation is always true, of however ſmall a magnitude we ſuppoſe x to be taken: and therefore it will alſo be true when x is $= 0$. But, when x is $= 0$, all the terms in the equation that involve x will be equal to 0 likewiſe, and conſequently the equation will be $+ D = \frac{4n}{m} E + \frac{3n}{m} D$, or $+ D = + \frac{3n}{m} D, \frac{4n}{m} E$; that is, D will be equal to $\frac{3n}{m}$ D together with $\frac{4n}{m}$ E, either added to it or ſubtracted from it, as may be neceſſary to produce the ſaid equality. But, becauſe n is greater than m, $\frac{3n}{m}$ D will be greater than D, and conſequently $\frac{4n}{m}$ E muſt be ſubtracted from it, in order to make it equal to D. We muſt therefore prefix the ſign $-$ to the quantity $\frac{4n}{m}$ E in the equation $+ D = \frac{4n}{m} E + \frac{3n}{m} D$; which equation will therefore be $+ D = - \frac{4n}{m} E + \frac{3n}{m} D$, or $+ D = + \frac{3n}{m} D - \frac{4n}{m} E$. Therefore the ſign $-$ muſt alſo be prefixed to the fifth term, Ex^4, of the ſeries $1 + \frac{m}{n} x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1 + x}\rvert^{\frac{m}{n}}$; and conſequently the five firſt terms of the ſaid ſeries will be $1 + \frac{m}{n} x - Cx^2 + Dx^3 - Ex^4$. Q. E. I.

And the magnitude of the co-efficient E may likewiſe be determined by means of the equation $+ D = - \frac{4n}{m} E + \frac{3n}{m} D$, or $+ D = + \frac{3n}{m} D - \frac{4n}{m} E$. For, by adding $\frac{4n}{m}$ E to both ſides, we ſhall have $D + \frac{4n}{m} E = \frac{3n}{m} D$, and, by ſubtracting D from both ſides, $\frac{4n}{m} E = \frac{3n}{m} D - D = \overline{\frac{3n}{m} - 1}\rvert \times D = \frac{3n - m}{m} \times D$, and conſequently $E = \frac{m}{4n} \times \frac{3n - m}{m} \times D = \frac{3n - m}{4n} \times D = \frac{3n - m}{4n} \times \frac{2n - m}{3n} \times \frac{n - m}{2n} \times \frac{m}{n}$, or $\frac{m}{n} \times \frac{n - m}{2n} \times \frac{2n - m}{3n} \times \frac{3n - m}{4n}$. Q. E. I.

Therefore the five firſt terms of the ſeries $1 + \frac{m}{n} x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1 + x}\rvert^{\frac{m}{n}}$, will be $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n - m}{2n} x^2 + \frac{m}{n} \times \frac{n - m}{2n} \times \frac{2n - m}{3n} x^3 - \frac{m}{n} \times \frac{n - m}{2n} \times \frac{2n - m}{3n} \times \frac{3n - m}{4n} x^4$, or $1 + \frac{m}{n} Ax - \frac{n - m}{2n} Bx^2 + \frac{2n - m}{3n} Cx^3 - \frac{3n - m}{4n} Dx^4$.

76. To determine the ſign that is to be prefixed to Fx^5, the ſixth term of the ſeries $1 + \frac{m}{n} x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1 + x}\rvert^{\frac{m}{n}}$, and to find the magnitude of the co-efficient F, we muſt proceed as follows:

In the laſt article it was ſhewn that the ſimple ſeries $+ D, Ex, Fx^2, Gx^3$, &c, was equal to the compound ſeries

$\frac{4n}{m}$

$$\frac{4n}{m}E,\ \frac{5n}{m}Fx,\ \frac{6n}{m}Gx^2,\ \frac{7n}{m}Hx^3,\ \&c$$
$$+\ \frac{3n}{m}D,\ \frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2,\ \frac{6n}{m}Gx^3,\ \&c,$$

or (prefixing the sign — to the terms that involve the co-efficient E) that the simple series $+ D - Ex, Fx^2, Gx^3$, &c, was equal to the compound series

$$-\frac{4n}{m}E,\quad \frac{5n}{m}Fx,\ \frac{6n}{m}Gx^2,\ \frac{7n}{m}Hx^3,\ \&c$$
$$+\ \frac{3n}{m}D - \frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2,\ \frac{6n}{m}Gx^3,\ \&c.$$

Add $2Ex$ to both sides of this equation. And we shall then have the simple series $D + Ex, Fx^2, Gx^3$, &c = the compound series

$$-\frac{4n}{m}E,\quad \frac{5n}{m}Fx,\ \frac{6n}{m}Gx^2,\ \frac{7n}{m}Hx^3,\ \&c$$
$$+\ \frac{3n}{m}D - \frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2,\ \frac{6n}{m}Gx^3,\ \&c$$
$$+\ 2Ex.$$

But it has been shewn in the last article that D is $= -\frac{4n}{m}E + \frac{3n}{m}D$, or $\frac{3n}{m}D - \frac{4n}{m}E$. Therefore, if we subtract D from the left-hand side of the last equation, and $\frac{3n}{m}D - \frac{4n}{m}E$ from its right-hand side, the remainders will be equal; that is, the simple series $+ Ex, Fx^2, Gx^3$, &c, will be = the compound series

$$\frac{5n}{m}Fx,\ \frac{6n}{m}Gx^2,\ \frac{7n}{m}Hx^3,\ \&c$$
$$-\frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2,\ \frac{6n}{m}Gx^3,\ \&c$$

$+\ 2Ex$; and consequently (dividing all the terms by x) the simple series E, Fx, Gx^2, &c will be = the compound series

$$\frac{5n}{m}F,\ \frac{6n}{m}Gx,\ \frac{7n}{m}Hx^2,\ \&c$$
$$-\frac{4n}{m}E,\ \frac{5n}{m}Fx,\ \frac{6n}{m}Gx^2,\ \&c$$
$$+\ 2E.$$

And this equation is always true, of however small a magnitude we suppose x to be taken; and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms in the equation that involve x will be equal to 0 likewise, and consequently the equation will be $E = \frac{5n}{m}F - \frac{4n}{m}E + 2E$. Therefore (adding $\frac{4n}{m}E$ to both sides) we shall have $E + \frac{4n}{m}E = \frac{5n}{m}F + 2E$, and (subtracting E from both sides) we shall have $\frac{4n}{m}E = \frac{5n}{m}F + E$, or $\frac{4n}{m}E = E, \frac{5n}{m}F$; that is, $\frac{4n}{m}E$ will be equal to E together with $\frac{5n}{m}F$ either added to it or subtracted from it, as may be necessary to produce the said equality. But, because n is greater than m, $\frac{4n}{m}E$ will be much greater than E; and therefore

fore $\frac{5n}{m}$ F muſt be added to E in order to make it equal to $\frac{4n}{m}$ E. We muſt therefore prefix the ſign + to the quantity $\frac{5n}{m}$ F in the equation $\frac{4n}{m} \text{E} = \text{E}, \frac{5n}{m}$ F; and conſequently the ſaid equation will be $\frac{4n}{m} \text{E} = \text{E} + \frac{5n}{m} \text{F}$. Therefore the ſign + muſt alſo be prefixed to the ſixth term, Fx^5, of the ſeries that is equal to $\overline{1 + x}|^{\frac{m}{n}}$, to wit, the ſeries $1 + \frac{m}{n} x, \text{C}x^2, \text{D}x^3, \text{E}x^4, \text{F}x^5$, &c; from which ſixth term the quantity $\frac{5n}{m}$ F has been derived above by the operations of multiplication and diviſion in the courſe of the foregoing proceſſes. Therefore the firſt ſix terms of the ſaid ſeries will be $1 + \frac{m}{n} x - \text{C}x^2 + \text{D}x^3 - \text{E}x^4 + \text{F}x^5$. Q. E. I.

And the magnitude of the co-efficient F may likewiſe be determined by means of the equation $\frac{4n}{m} \text{E} = \text{E} + \frac{5n}{m} \text{F}$. For, by ſubtracting E from both ſides, we ſhall have $\frac{5n}{m} \text{F} = \frac{4n}{m} \text{E} - \text{E} = \overline{\frac{4n}{m} - 1}| \times \text{E} = \frac{4n - m}{m} \times \text{E}$; and conſequently F will be $= \frac{m}{5n} \times \frac{4n - m}{m} \times \text{E} = \frac{4n - m}{5n} \times \text{E} = \frac{4n - m}{5n} \times \frac{3n - m}{4n} \times \frac{2n - m}{3n} \times \frac{n - m}{2n} \times \frac{m}{n}$, or $\frac{m}{n} \times \frac{n - m}{2n} \times \frac{2n - m}{3n} \times \frac{3n - m}{4n} \times \frac{4n - m}{5n}$. Q. E. I.

Therefore the firſt ſix terms of the ſeries $1 + \frac{m}{n} x, \text{C}x^2, \text{D}x^3, \text{E}x^4, \text{F}x^5$, &c, which is equal to $\overline{1 + x}|^{\frac{m}{n}}$, are $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n - m}{2n} x^2 + \frac{m}{n} \times \frac{n - m}{2n} \times \frac{2n - m}{3n} x^3 - \frac{m}{n} \times \frac{n - m}{2n} \times \frac{2n - m}{3n} \times \frac{3n - m}{4n} x^4 + \frac{m}{n} \times \frac{n - m}{2n} \times \frac{2n - m}{3n} \times \frac{3n - m}{4n} \times \frac{4n - m}{5n} x^5$, or $1 + \frac{m}{n} \text{A}x - \left|\overline{\frac{n - m}{2n}}\right. \text{B}x^2 + \frac{2n - m}{3n} \text{C}x^3 - \left|\overline{\frac{3n - m}{4n}}\right. \text{D}x^4 + \frac{4n - m}{5n} \text{E}x^5$.

77. Having thus gone through the inveſtigations of the values of the four co-efficients C, D, E, F, at conſiderable length, I ſhall treat more conciſely of the inveſtigation of the following co-efficients G, H, I, K, L, M, N, O, P, Q, R, S, T, &c, and ſhall only obſerve concerning them, that the ſigns + and −, which are to be prefixed to them, may be determined, and their magnitudes diſcovered, by means of the following ſhort and eaſy ſimple equations, which may be eaſily derived from the grand, fundamental, equation obtained above in art. 72; to wit, the equations

$$\text{F} = \frac{6n}{m} \text{G}, \frac{5n}{m} \text{F},$$
$$\text{G} = \frac{7n}{m} \text{H}, \frac{6n}{m} \text{G},$$
$$\text{H} = \frac{8n}{m} \text{I}, \frac{7n}{m} \text{H},$$

$$I = \frac{9n}{m} K, \frac{8n}{m} I,$$
$$K = \frac{10n}{m} L, \frac{9n}{m} K,$$
$$L = \frac{11n}{m} M, \frac{10n}{m} L,$$
$$M = \frac{12n}{m} N, \frac{11n}{m} M,$$
$$N = \frac{13n}{m} O, \frac{12n}{m} N,$$
$$O = \frac{14n}{m} P, \frac{13n}{m} O,$$
$$P = \frac{15n}{m} Q, \frac{14n}{m} P,$$
$$Q = \frac{16n}{m} R, \frac{15n}{m} Q,$$
$$R = \frac{17n}{m} S, \frac{16n}{m} R,$$
$$S = \frac{18n}{m} T, \frac{17n}{m} S, \&c,$$

or, (becauſe F has the ſign + prefixed to it, and $\frac{5n}{m}$ F is greater than F, and conſequently $\frac{6n}{m}$ G muſt be ſubtracted from $\frac{5n}{m}$ F in order to make it equal to F),

$$F = -\frac{6n}{m} G + \frac{5n}{m} F,$$

and (for the like reaſons),

$$-G = +\frac{7n}{m} H - \frac{6n}{m} G,$$
$$\text{and } +H = -\frac{8n}{m} I + \frac{7n}{m} H,$$
$$-I = +\frac{9n}{m} K - \frac{8n}{m} I,$$
$$+K = -\frac{10n}{m} L + \frac{9n}{m} K,$$
$$-L = +\frac{11n}{m} M - \frac{10n}{m} L,$$
$$+M = -\frac{12n}{m} N + \frac{11n}{m} M,$$
$$-N = +\frac{13n}{m} O - \frac{12n}{m} N,$$
$$+O = -\frac{14n}{m} P + \frac{13n}{m} O,$$
$$-P = +\frac{15n}{m} Q - \frac{14n}{m} P,$$
$$+Q = -\frac{16n}{m} R + \frac{15n}{m} Q,$$
$$-R = +\frac{17n}{m} S - \frac{16n}{m} R,$$
$$+S = -\frac{18n}{m} T + \frac{17n}{m} S,$$

&c;

&c; in all which equations the signs to be prefixed to the several co-efficients G, H, I, K, L, M, N, O, P, Q, R, S, T, &c, are alternately − and +. And the same thing, it is evident, must take place in all the following co-efficients of the powers of x in the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}\big|^{\frac{m}{n}}$, to whatever number of terms the said series shall be continued. We may therefore conclude that the said series will, when n is greater than m, be $1 + \frac{m}{n}x - Cx^2 + Dx^3 - Ex^4 + Fx^5 - Gx^6 + Hx^7 - Ix^8 + Kx^9 - Lx^{10} + Mx^{11} - Nx^{12} + Ox^{13} - Px^{14} + Qx^{15} - Rx^{16} + Sx^{17} - Tx^{18} +$ &c, in which all the terms after the two first are marked with the signs − and + alternately, or are to be alternately subtracted from, and added to, the said two first terms. Q. E. I.

And, secondly, the magnitudes of the co-efficients G, H, I, K, L, M, N, O, P, Q, R, S, T, &c, may be discovered by resolving the foregoing short simple equations, to wit,

$$+F = -\frac{6n}{m}G + \frac{5n}{m}F,$$
$$-G = +\frac{7n}{m}H - \frac{6n}{m}G,$$
$$+H = -\frac{8n}{m}I + \frac{7n}{m}H,$$
$$-I = +\frac{9n}{m}K - \frac{8n}{m}I,$$
$$+K = -\frac{10n}{m}L + \frac{9n}{m}K,$$
$$-L = +\frac{11n}{m}M - \frac{10n}{m}L,$$
$$+M = -\frac{12n}{m}N + \frac{11n}{m}M,$$
$$-N = +\frac{13n}{m}O - \frac{12n}{m}N,$$
$$+O = -\frac{14n}{m}P + \frac{13n}{m}O,$$
$$-P = +\frac{15n}{m}Q - \frac{14n}{m}P,$$
$$+Q = -\frac{16n}{m}R + \frac{15n}{m}Q,$$
$$-R = +\frac{17n}{m}S - \frac{16n}{m}R,$$
$$+S = -\frac{18n}{m}T + \frac{17n}{m}S, \text{ \&c.}$$

For, since $+F$ is $= -\frac{6n}{m}G + \frac{5n}{m}F$, we shall have $F + \frac{6n}{m}G = \frac{5n}{m}F$, and $\frac{6n}{m}G = \frac{5n}{m}F - F = \overline{\frac{5n}{m} - 1}\big| \times F = \frac{5n-m}{m} \times F$, and consequently G $(= \frac{m}{6n} \times \frac{6n}{m}G = \frac{m}{6n} \times \frac{5n-m}{m} \times F) = \frac{5n-m}{6n} \times F$. Q. E. I.

And,

And, ſince $-$ G is $= +\frac{7n}{m}H - \frac{6n}{m}G$, we ſhall have $\frac{6n}{m}G - G = +\frac{7n}{m}H$, or $\overline{\frac{6n}{m} - 1}| \times G = \frac{7n}{m}H$, or $\frac{6n-m}{m} \times G = \frac{7n}{m} \times H$, and conſequently H $(= \frac{m}{7n} \times \frac{7n}{m}H = \frac{m}{7n} \times \frac{6n-m}{m} \times G) = \frac{6n-m}{7n} \times G$. Q. E. I.

And, ſince $+$ H is $= -\frac{8n}{m}I + \frac{7n}{m}H$, we ſhall have $H + \frac{8n}{m}I = \frac{7n}{m}H$, and $\frac{8n}{m}I = \frac{7n}{m}H - H = \overline{\frac{7n}{m} - 1}| \times H = \frac{7n-m}{m} \times H$, and conſequently I $(= \frac{m}{8n} \times \frac{8n}{m}I = \frac{m}{8n} \times \frac{7n-m}{m} \times H) = \frac{7n-m}{8n} \times H$. Q. E. I.

And, ſince $-$ I is $= +\frac{9n}{m}K - \frac{8n}{m}I$, we ſhall have $\frac{8n}{m}I - I = \frac{9n}{m}K$, or $\overline{\frac{8n}{m} - 1}| \times I = \frac{9n}{m}K$, or $\frac{8n-m}{m} \times I = \frac{9n}{m}K$, and conſequently K $(= \frac{m}{9n} \times \frac{9n}{m}K = \frac{m}{9n} \times \frac{8n-m}{m} \times I) = \frac{8n-m}{9n} \times I$. Q. E. I.

And, ſince $+$ K is $= -\frac{10n}{m}L + \frac{9n}{m}K$, we ſhall have $K + \frac{10n}{m}L = \frac{9n}{m}K$, and $\frac{10n}{m}L = \frac{9n}{m}K - K = \overline{\frac{9n}{m} - 1}| \times K = \frac{9n-m}{m} \times K$, and conſequently L $(= \frac{m}{10n} \times \frac{10n}{m}L = \frac{m}{10n} \times \frac{9n-m}{m} \times K) = \frac{9n-m}{10n} \times K$. Q. E. I.

And, ſince $-$ L is $= +\frac{11n}{m}M - \frac{10n}{m}L$, we ſhall have $\frac{10n}{m}L - L = +\frac{11n}{m}M$, or $\frac{11n}{m}M = \frac{10n}{m}L - L = \overline{\frac{10n}{m} - 1}| \times L = \frac{10n-m}{m} \times L$, and conſequently M $(= \frac{m}{11n} \times \frac{11n}{m}M = \frac{m}{11n} \times \frac{10n-m}{m} \times L) = \frac{10n-m}{11n} \times L$.

Q. E. I.

And, ſince $+$ M is $= -\frac{12n}{m}N + \frac{11n}{m}M$, we ſhall have $M + \frac{12n}{m}N = \frac{11n}{m}M$, and $\frac{12n}{m}N = \frac{11n}{m}M - M = \overline{\frac{11n}{m} - 1}| \times M = \frac{11n-m}{m} \times M$, and conſequently N $(= \frac{m}{12n} \times \frac{12n}{m}N = \frac{m}{12n} = \frac{11n-m}{m} \times M) = \frac{11n-m}{12n} \times M$.

Q. E. I.

And, ſince $-$ N is $= +\frac{13n}{m}O - \frac{12n}{m}N$, we ſhall have $\frac{12n}{m}N - N = \frac{13n}{m}O$, or $\frac{13n}{m}O = \frac{12n}{m}N - N = \overline{\frac{12n}{m} - 1}| \times N = \frac{12n-m}{m} \times N$, and conſequently O $(= \frac{m}{13n} \times \frac{13n}{m}O = \frac{m}{13n} \times \frac{12n-m}{m} \times N) = \frac{12n-m}{13n} \times N$.

Q. E. I.

And, ſince $+$ O is $= -\frac{14n}{m}P + \frac{13n}{m}O$, we ſhall have $O + \frac{14n}{m}P = \frac{13n}{m}O$, and $\frac{14n}{m}P = \frac{13n}{m}O - O = \overline{\frac{13n}{m} - 1}| \times O = \frac{13n-m}{m} \times O$, and conſequently

consequently P $(= \frac{m}{14n} \times \frac{14n}{m}$ P $= \frac{m}{14n} \times \frac{13n - m}{m} \times$ O $= \frac{13n - m}{14n} \times$ O.

Q. E. I.

And, since $-$ P is $= + \frac{15n}{m}$ Q $- \frac{14n}{m}$ P, we shall have $\frac{15n}{m}$ Q $= \frac{14n}{m}$ P $-$ P $= \overline{\frac{14n}{m} - 1}| \times$ P $= \frac{14n - m}{m} \times$ P, and consequently Q $(= \frac{m}{15n} \times \frac{15n}{m}$ Q $= \frac{m}{15n} \times \frac{14n - m}{m} \times$ P) $= \frac{14n - m}{15n} \times$ P.

Q. E. I.

And, since $+$ Q is $= - \frac{16n}{m}$ R $+ \frac{15n}{m}$ Q, we shall have Q $+ \frac{16n}{m}$ R $= \frac{15n}{m}$ Q, and $\frac{16n}{m}$ R $= \frac{15n}{m}$ Q $-$ Q $= \overline{\frac{15n}{m} - 1}| \times$ Q $= \frac{15n - m}{m} \times$ Q, and consequently R $(= \frac{m}{16n} \times \frac{16n}{m}$ R $= \frac{m}{16n} \times \frac{15n - m}{m} \times$ Q) $= \frac{15n - m}{16n} \times$ Q.

Q. E. I.

And, since $-$ R is $= + \frac{17n}{m}$ S $- \frac{16n}{m}$ R, we shall have $\frac{17n}{m}$ S $= \frac{16n}{m}$ R $-$ R $= \overline{\frac{16n}{m} - 1}| \times$ R $= \frac{16n - m}{m} \times$ R, and consequently S $(= \frac{m}{17n} \times \frac{17n}{m}$ S $= \frac{m}{17n} \times \frac{16n - m}{m}$ R) $= \frac{16n - m}{17n} \times$ R.

Q. E. I.

And, since $+$ S is $= - \frac{18n}{m}$ T $+ \frac{17n}{m}$ S, we shall have S $+ \frac{18n}{m}$ T $= \frac{17n}{m}$ S, and $\frac{18n}{m}$ T $= \frac{17n}{m}$ S $-$ S $= \overline{\frac{17n}{m} - 1}| \times$ S $= \frac{17n - m}{m} \times$ S, and consequently T $(= \frac{m}{18n} \times \frac{18n}{m}$ T $= \frac{m}{18n} \times \frac{17n - m}{m} \times$ S) $= \frac{17n - m}{18n} \times$ S.

Q. E. I.

Therefore the quantity $\overline{1 + x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$, will, when n is greater than m, be equal to the series $1 + \frac{m}{n} Ax - \frac{n - m}{2n} Bx^2 + \frac{2n - m}{3n} Cx^3 - \frac{3n - m}{4n} Dx^4 + \frac{4n - m}{5n} Ex^5 - \frac{5n - m}{6n} Fx^6 + \frac{6n - m}{7n} Gx^7 - \frac{7n - m}{8n} Hx^8 + \frac{8n - m}{9n} Ix^9 - \frac{9n - m}{10n} Kx^{10} + \frac{10n - m}{11n} Lx^{11} - \frac{11n - m}{12n} Mx^{12} + \frac{12n - m}{13n} Nx^{13} - \frac{13n - m}{14n} Ox^{14} + \frac{14n - m}{15n} Px^{15} - \frac{15n - m}{16n} Qx^{16} + \frac{16n - m}{17n} Rx^{17} - \frac{17n - m}{18n} Sx^{18} +$ &c *ad infinitum*, in which all the terms, after the two first, are marked with the signs $-$ and $+$ alternately, or are alternately to be subtracted from, and added to, the two first terms, and the co-efficients of the third, fourth, fifth, sixth, and other following terms are generated from $\frac{m}{n}$, the co-efficient of the second term, and from each other, by the continual multiplication of the fractions $\frac{n - m}{2n}$, $\frac{2n - m}{3n}$, $\frac{3n - m}{4n}$, $\frac{4n - m}{5n}$, $\frac{5n - m}{6n}$, $\frac{6n - m}{7n}$, $\frac{7n - m}{8n}$, $\frac{8n - m}{9n}$, $\frac{9n - m}{10n}$, &c, in which every new fraction

tion is derived from that which immediately preceeds it, by adding n to both the numerator and its denominator.

This series is exactly the same with that which is derived above in art. 3 from the binomial theorem, in the case of integral powers.

78. When the denominator n of the fraction $\frac{m}{n}$ (which is the index of the binomial quantity $1 + x$), is an exact multiple of the numerator m, the foregoing series $1 + \frac{m}{n} Ax - \left|\frac{n-m}{2n}\right. Bx^2 + \frac{2n-m}{3n} Cx^3 - \left|\frac{3n-m}{4n}\right. Dx^4 + \frac{4n-m}{5n} Ex^5 -$ &c, will co-incide with, or may be reduced to, the series found in the former part of this discourse for the value of $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, or the nth root of the quantity $1 + x$.

For, if n is an exact multiple of m, let it contain m exactly p times, or be equal to p times m, or pm. Then will $\overline{1+x}|^{\frac{m}{n}}$ be $= \overline{1+x}|^{\frac{m}{pm}} = \overline{1+x}|^{\frac{1}{p}}$; and the series $1 + \frac{m}{n} Ax - \left|\frac{n-m}{2n}\right. Bx^2 + \frac{2n-m}{3n} Cx^3 - \left|\frac{3n-m}{4n}\right. Dx^4 + \frac{4n-m}{5n} Ex^5 -$ &c, will be $= 1 + \frac{m}{pm} Ax - \left|\frac{pm-m}{2pm}\right. Bx^2 + \frac{2pm-m}{3pm} Cx^3 - \left|\frac{3pm-m}{4pm}\right. Dx^4 + \frac{4pm-m}{5pm} Ex^5 -$ &c $= 1 + \frac{1}{p} Ax - \left|\frac{p-1}{2p}\right. Bx^2 + \frac{2p-1}{3p} Cx^3 - \left|\frac{3p-1}{4p}\right. Dx^4 + \frac{4p-1}{5p} Ex^5 -$ &c. Therefore $\overline{1+x}|^{\frac{1}{p}}$, or $\sqrt[p]{1+x}$, or the pth root of the binomial quantity $1 + x$, will be equal to the series $1 + \frac{1}{p} Ax - \left|\frac{p-1}{2p}\right. Bx^2 + \frac{2p-1}{3p} Cx^3 - \left|\frac{3p-1}{4p}\right. Dx^4 + \frac{4p-1}{5p} Ex^5 -$ &c; which is the same series that was found above, in art. 47, for the value of any root of $1 + x$, excepting that the letter p is here used, instead of the letter n, to denote the index of the root that is to be extracted. Q. E. D.

79. And, if we suppose n to be exactly equal to m, and consequently $\frac{m}{n}$ to be equal to $\frac{m}{m}$, or 1, and $\overline{1+x}|^{\frac{m}{n}}$ to be equal to $\overline{1+x}|^{\frac{m}{m}}$, or $\overline{1+x}|^{1}$, or $1 + x$, the foregoing series $1 + \frac{m}{n} Ax - \left|\frac{n-m}{2n}\right. Bx^2 + \frac{2n-m}{3n} Cx^3 - \left|\frac{3n-m}{4n}\right. Dx^4 + \frac{4n-m}{5n} Ex^5 -$ &c will be $= 1 + \frac{m}{m} Ax - \left|\frac{m-m}{2m}\right. Bx^2 + \frac{2m-m}{3m} Cx^3 - \left|\frac{3m-m}{4m}\right. Dx^4 + \frac{4m-m}{5m} Ex^5 -$ &c $= 1 + 1 \times Ax - \left|\frac{1-1}{2}\right. Bx^2 + \frac{2-1}{3} Cx^3 - \left|\frac{3-1}{4}\right. Dx^4 + \frac{4-1}{5} Ex^5 -$ &c $= 1 + 1 \times Ax - \frac{0}{2} Bx^2 + \frac{1}{3} Cx^3 - \frac{2}{4} Dx^4 + \frac{3}{5} Ex^5 -$ &c $= 1 + 1 \times 1 \times x - 0 \times x^2 + \frac{1}{3} \times 0 \times x^3 - \frac{2}{4} \times 0 \times x^4 + \frac{3}{5} \times 0 \times x^5 -$ &c $= 1 + x - 0 + 0 - 0 + 0 -$ &c $= 1 + x$; as it ought to be.

80. We

80. We come now to the third caſe of fractional indexes, in which the numerator m of the index $\frac{m}{n}$ of the power of the binomial quantity $1 + x$, is greater than the denominator.

Of the Binomial Theorem in the caſe of $\overline{1 + x}|^{\frac{m}{n}}$, *or of the* n*th root of the* m*th power of the binomial quantity* $1 + x$, *or the* m*th power of its* n*th root, when* m *is any whole number whatſoever, and* n *any other whole number leſs than* m.

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81. The methods of inveſtigating the ſeries that is equal to $\overline{1 + x}|^{\frac{m}{n}}$ when the numerator m is greater than the denominator n, are the ſame with thoſe which have been employed in the inveſtigation of the ſeries that is equal to the ſame quantity $\overline{1 + x}|^{\frac{m}{n}}$ when the numerator m is leſs than the denominator n. But the ſeries that will be obtained by theſe methods for the value of $\overline{1 + x}|^{\frac{m}{n}}$ will not be always exactly the ſame, of whatever magnitude greater than n we ſuppoſe x to be taken; as was the caſe with the ſeries that was found for the value of $\overline{1 + x}|^{\frac{m}{n}}$ when m is leſs than n: but it will be different for every new multiple of n that is contained in m. For, if m is greater than n, but leſs than $2n$, the third term of the ſeries will be marked with the ſign $+$, or added to the firſt term, and all the following terms will be marked with the ſigns $-$ and $+$ alternately: and, if m is greater than $2n$, but leſs than $3n$, the third and fourth terms of the ſeries will be marked with the ſign $+$, or added to the firſt term, and all the following terms will be marked with the ſigns $-$ and $+$ alternately: and, if m is greater than $3n$, but leſs than $4n$, the third and fourth and fifth terms of the ſeries will be marked with the ſign $+$, or added to the firſt term, and all the following terms will be marked with the ſigns $-$ and $+$ alternately: and, if m is greater than $4n$, but leſs than $5n$, the third, fourth, fifth and ſixth terms of the ſeries will be marked with the ſign $+$, or added to the firſt term, and all the following terms will be marked with the ſigns $-$ and $+$ alternately; and in general, if m is greater than p times n, or pn, but leſs than $p + 1$ times n, or $pn + n$, the third, fourth, fifth, ſixth, and other following terms of the ſeries, to the number of p terms, will be marked with the ſign $+$, or added to the firſt term, and all the following terms will be marked with the ſigns $-$ and $+$ alternately; as was ſet forth in the beginning of this diſcourſe in art. 5 and 6. Now, if we were to apply the two foregoing inveſtigations, contained in art. 54, 55, 56, 57, 58, &c — 65, and in art. 67, 68, 69, 70, 71, 72, &c — 77, to the diſcovery of the ſeries which is equal to $\overline{1 + x}|^{\frac{m}{n}}$, when

m is greater than n; and, in the resolution of each of the simple equations by which the co-efficients of the several powers of x would be to be determined, were to attend to all the different relative magnitudes of m and n, and to suppose m, first, to be greater than n, but less than $2n$, and, secondly, to be greater than $2n$, but less than $3n$, and, thirdly, to be greater than $3n$, but less than $4n$, and, fourthly, to be greater than $4n$, but less than $5n$, and so on with respect to all the co-efficients we were to investigate; we should find the examination of such a variety of cases intolerably tedious and laborious. And therefore, in applying the foregoing methods of investigation to the discovery of the series which is equal to $\overline{1+x}|^{\frac{m}{n}}$ in this second case, in which m is supposed to be greater than n, I shall make only the first of these suppositions, to wit, that m, though it is greater than n, is less than $2n$; which will reduce the investigation of this series to the same degree of difficulty, and no more, as was found in the investigation of the series that is equal to $\overline{1+x}|^{\frac{m}{n}}$ in the former case, in which m was supposed to be less n. Now upon this supposition, the series that is equal to $\overline{1+x}|^{\frac{m}{n}}$ may be investigated in the following manner.

82. The reasonings used above in art. 54, 55, 56, 57, in the three observations preparatory to the foregoing investigations, are evidently true when m, or the numerator of the index $\frac{m}{n}$ of the power of $1+x$ in the quantity $\overline{1+x}|^{\frac{m}{n}}$, is of any magnitude greater than the denominator n, as well as when n is greater than m. And therefore it follows that in both cases the quantity $\overline{1+x}|^{\frac{m}{n}}$ will be equal to the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, of which 1 is the first term and $\frac{m}{n}x$ is the second term, and is to be added to the first term, and consequently marked with the sign +, and the following terms are Cx^2, Dx^3, Ex^4, Fx^5, &c, or the second and other following powers of x in their natural order, without interruption, to wit, x^2, x^3, x^4, x^5, &c, multiplied into certain fixed numeral co-efficients denoted by the capital letters C, D, E, F, &c, and are to be connected with the two first terms $1 + \frac{m}{n}x$ either by addition or subtraction, and consequently are to have either the sign + or the sign − prefixed to them. It therefore only remains that we determine which of the said terms Cx^2, Dx^3, Ex^4, Fx^5, &c, are to be added to the two first terms $1 + \frac{m}{n}x$, and consequently are to have the sign + prefixed to them, and which of them are to be subtracted from the said two first terms, and consequently are to have the sign − prefixed to them, and what are the several values, or magnitudes, of the co-efficients C, D, E, F, &c, respectively.

83. Now let y be put = the series $\frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (as in art. 58), or $1+y$ be = the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is

is equal to $\overline{1+x}|^{\frac{m}{n}}$. Then will $\overline{1+y}|^{n}$ be $=$ the nth power of $\overline{1+x}|^{\frac{m}{n}}$, that is, to $\overline{1+x}|^{m}$. And conſequently the ſeries $1 + \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times y^5 +$ &c, will be $=$ the ſeries $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 +$ &c, and conſequently (ſubtracting 1 from both ſides of the equation), the ſeries $\frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 +$ &c, will be $=$ the ſeries $\frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 +$ &c, or (if, for the ſake of brevity, we make $\frac{m}{1} \times \frac{m-1}{2} = Q$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} = R$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} = S$, and $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} = T$, and $\frac{n}{1} \times \frac{n-1}{2} = q$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = r$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} = s$, and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} = t$), the ſeries $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 +$ &c, will be $=$ the ſeries $ny + qy^2 + ry^3 + sy^4 + ty^5 +$ &c.

84. Now let the ſeveral powers of the ſeries $\frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, or (putting $B = \frac{m}{n}$) of the ſeries Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to y), be raiſed by multiplication, to the end that we may have the values of y^2, y^3, y^4, y^5, &c, expreſſed in powers of x; and let the ſaid values of y^2, y^3, y^4, y^5, &c, obtained by the ſaid multiplications, be ſubſtituted inſtead of y^2, y^3, y^4, y^5, &c, reſpectively in the laſt equation. And we ſhall then have the ſimple ſeries $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 +$ &c $=$ the compound ſeries

$$\begin{array}{llllll}
nBx, & nCx^2, & nDx^3, & nEx^4, & nFx^5, & \&c \\
 & +qB^2x^2, & 2qBCx^3, & 2qBDx^4, & 2qBEx^5, & \&c \\
 & & & qC^2x^4, & 2qCDx^5, & \&c \\
 & & +rB^3x^3, & 3rB^2Cx^4, & 3rB^2Dx^5, & \&c \\
 & & & & 3rBC^2x^5, & \&c \\
 & & & +sB^4x^4, & 4sB^3Cx^5, & \&c \\
 & & & & +tB^5x^5; &
\end{array}$$

 and

and consequently (dividing all the terms by x) the simple series $m + Qx + Rx^2 + Sx^3 + Tx^4 + \&c =$ the compound series

$$\begin{array}{llllll}
nB, & nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\
 & +qB^2x, & 2qBCx^2, & 2qBDx^3, & 2qBEx^4, & \&c \\
 & & & qC^2x^3, & 2qCDx^4, & \&c \\
 & & +rB^3x^2, & 3rB^2Cx^3, & 3rB^2Dx^4, & \&c \\
 & & & & 3rBC^2x^4, & \&c \\
 & & & +sB^4x^3, & 4sB^3Cx^4, & \&c \\
 & & & & +tB^5x^4, & \&c;
\end{array}$$

as is shewn more at large above, in art. 58, 59, 60, and 61.

85. From this general and fundamental equation we may, by repeating the reasonings used above in art. 61, 62, 63, 64, derive the following particular equations for the determination of the magnitudes of the several co-efficients B, C, D, E, F, &c, and of the signs + and −, that are to be prefixed to them; to wit,

1st, $m = nB$;
2dly, $Q = nC + qB^2$;
3dly, $R = nD,\ 2qBC + rB^3$;
4thly, $S = nE,\ 2qBD,\ qC^2,\ 3rB^2C + sB^4$,
and 5thly, $T = nF,\ 2qBE,\ 2qCD,\ 3rB^2D,\ 3rBC^2,\ 4sB^3C + tB^5$.

86. From the first of these equations, to wit, $m = nB$, it follows that the co-efficient B is $= \frac{m}{n}$, as it has already been shewn to be in art. 56 and 57.

87. From the second of these equations, to wit, $Q = nC + qB^2$, or $Q = qB^2,\ nC$, we may deduce both the sign that is to be prefixed to nC and the magnitude of C, by proceeding in the manner following:

Since Q is $= m \times \frac{m-1}{2}$, and q is $n \times \frac{n-1}{2}$, and B is $= \frac{m}{n}$, we shall have $m \times \frac{m-1}{2} = n \times \frac{n-1}{2} \times \frac{m}{n} B,\ nC$, or $\frac{mm-m}{2} = \frac{mn-m}{2} B,\ nC$, or $\frac{mm-m}{2} \times \frac{B}{B} = \frac{mn-m}{2} B,\ nC$, or $\frac{mm-m}{2} \times \frac{n}{m} \times B = \frac{mn-m}{2} B,\ nC$, or $\frac{m^2n-mn}{2m} B = \frac{m^2n-m^2}{2m} \times B,\ nC$, and (adding $\frac{mn}{2m} B$ to both sides) $\frac{m^2n}{2m} B = \frac{m^2n+mn-m^2}{2m} B,\ nC$, and (adding $\frac{m^2}{2m} B$ to both sides) $\frac{m^2n+m^2}{2m} B = \frac{m^2n+mn}{2m} B,\ nC$, and (subtracting $\frac{m^2n}{2m} B$ from both sides) $\frac{m^2}{2m} B = \frac{mn}{2m} B,\ nC$, or $\frac{m}{2} B = \frac{n}{2} B,\ nC$; and therefore (because m is now supposed to be greater than n, and consequently $\frac{m}{2} B$ is greater than $\frac{n}{2} B$) $\frac{m}{2} B$ will be $= \frac{n}{2} B + nC$; and consequently nC will be $= \frac{m}{2} B - \frac{n}{2} B = \frac{m-n}{2} B$, and C will be $= \frac{m-n}{2n} B = \frac{m-n}{2n} \times \frac{m}{n}$, or $\frac{m}{n} \times \frac{m-n}{2n}$. Therefore Cx^2, or the third term of the series $1 + Bx,\ Cx^2,\ Dx^3,\ Ex^4,\ Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, must be added to the two first

firſt terms $1 + Bx$, or $1 + \frac{m}{n}x$, and the three firſt terms of the ſaid ſeries will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2$. Q. E. I.

88. From the third of the foregoing equations, to wit, $R = nD, 2qBC + rB^3$, or (becauſe we now know that the ſign $+$ is to be prefixed to the term $2qBC$), $R = nD + 2qBC + rB^3$, we may deduce both the ſign that is to be prefixed to nD, and the magnitude of D, by proceeding in the manner following:

By ſubſtituting $m \times \frac{m-1}{2} \times \frac{m-2}{3}$ in this equation inſtead of R, and $n \times \frac{n-1}{2}$ inſtead of q, and $n \times \frac{n-1}{2} \times \frac{n-2}{3}$ inſtead of r, and $\frac{m}{n}$ inſtead of B, and $\frac{m}{n}\,\frac{m-n}{2n}$ inſtead of C, it will become as follows, to wit, $m \times \frac{m-1}{2} \times \frac{m-2}{3} = nD + n \times \overline{n-1} \times \frac{m}{n} \times \frac{m}{n} \times \frac{m-n}{2n} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{m^3}{n^3}$, or $\frac{m^3-3m^2+2m}{6} = nD + m \times \overline{n-1} \times \frac{m}{n} \times \frac{m-n}{2n} + \frac{n^3-3n^2+2n}{6} \times \frac{m^3}{n^3}$, or $\frac{m^3n^2-3m^2n^2+2mn^2}{6nn} = nD + \frac{3m^2}{3n} \times \overline{n-1} \times \frac{m-n}{2n} + m^3 \times \overline{\frac{nn-3n+2}{6nn}} = nD + 3m^2 \times \overline{\frac{mn-nn-m+n}{6nn}} + \frac{m^3n^2-3m^3n+2m^3}{6nn} = nD + \frac{3m^3n-3m^2n^2-3m^3+3m^2n}{6nn} + \frac{m^3n^2-3m^3n+2m^3}{6nn} = nD\ \frac{-3m^2n^2-m^3+3m^2n+m^3n^2}{6nn}$; and conſequently (adding $\frac{3m^2n^2}{6nn}$ to both ſides) we ſhall have $\frac{m^3n^2+2mn^2}{6nn} = nD + \frac{3m^2n-m^3+m^3n^2}{6nn}$, and (adding $\frac{m^3}{6nn}$ to both ſides) $\frac{m^3n^2+2mn^2+m^3}{6nn} = nD + \frac{3m^2n+m^3n^2}{6nn}$, and (ſubtracting $\frac{m^3n^2}{6nn}$ from both ſides) $\frac{2mn^2+m^3}{6nn} = nD + \frac{3m^2n}{6nn}$, or $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn}, nD$; that is, $\frac{2mn^2+m^3}{6nn}$ will be equal to $\frac{3m^2n}{6nn}$ together with nD either added to it, or ſubtracted from it, as may be neceſſary to produce ſuch equality.

89. Now, becauſe m (though greater than n) is ſuppoſed to be leſs than $2n$, it follows that $\overline{m-n}$ will be leſs than $\overline{2n-n}$, or than n; and conſequently that $m \times \overline{m-n} \times \overline{m-n}$ will be leſs than $m \times \overline{m-n} \times n$, or that $m \times \overline{m^2-2mn+n^2}$ will be leſs than $mn \times \overline{m-n}$, or that $m^3-2m^2n+mn^2$ will be leſs than m^2n-mn^2. Therefore (adding mn^2 to both ſides) $m^3-2m^2n+2mn^2$ will be leſs than m^2n, and (adding $2m^2n$ to both ſides) m^3+2mn^2 will be leſs than $3m^2n$, and conſequently $\frac{2mn^2+m^3}{6nn}$ will be leſs than $\frac{3m^2n}{6nn}$. Therefore, in order to make $\frac{3m^2n}{6nn}, nD$ be equal to $\frac{2mn^2+m^3}{6nn}$, we muſt ſubtract nD from $\frac{3m^2n}{6nn}$, and conſequently muſt prefix the ſign $-$ to nD; ſo that the equation $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn}, nD$ will be $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn} - nD$. Therefore the ſign $-$ muſt alſo be prefixed

prefixed to Dx^3, or the fourth term of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$); from which fourth term the quantity nD was derived, by means of the operations of multiplication and division, in the course of the foregoing investigation. Therefore the four first terms of the said series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, will be $1 + Bx + Cx^2 - Dx^3$, or $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - Dx^3$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - Dx^3$. Q. E. I.

90. And to determine the magnitude of D, we shall have the aforesaid equation $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn} - nD$; whence (by adding nD to both sides) we shall have $\frac{2mn^2+m^3}{6nn} + nD = \frac{3m^2n}{6nn}$, and (by subtracting $\frac{2mn^2+m^3}{6nn}$ from both sides) $nD = \frac{3m^2n - 2mn^2 - m^3}{6nn} = m \times \frac{3mn - 2n^2 - m^2}{6nn} = m \times \frac{\overline{m-n} \times \overline{2n-m}}{6nn} = m \times \frac{m-n}{2n} \times \frac{2n-m}{3n}$, and consequently (by dividing both sides by n) $D = \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}$. Therefore the four first terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5$, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}x^3$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \left|\frac{2n-m}{3n}Cx^3\right.$.

Q. E. I.

91. These four terms of the series which is equal to $\overline{1+x}|^{\frac{m}{n}}$ in this case of the relative magnitudes of m and n, in which m is greater than n, but less than $2n$, are the same with the four first terms of the series given above in the beginning of this discourse, in art. 6, for the value of $\overline{1+x}|^{\frac{m}{n}}$ in the same case. Therefore the said series, in art. 6 is true, at least in its four first terms.

92. I shall not attempt to find the values of the co-efficients E and F by resolving the fourth and fifth equations set down in art. 85, to wit, the equation $S = nE, 2qBD, qC^2, 3rB^2C + sB^4$, or (because it has been shewn that the co-efficient C is to have the sign + prefixed to it, and the co-efficient D is to have the sign − prefixed to it) $S = nE - 2qBD + qC^2 + 3rB^2C + sB^4$, and the equation $T = nF, 2qBE, 2qCD, 3rB^2D, 3rBC^2, 4sB^3C + tB^5$, or $T = nF, 2qBE - 2qCD - 3rB^2D + 3rBC^2 + 4sB^3C + tB^5$: I say, I shall not attempt to find the values of E and F by resolving these two equations, for the reasons given above in art. 65 and 66; but shall proceed to apply the other method of investigating the values of the third and fourth, and other following terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4$,

Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, to the prefent cafe, in which m is fuppofed to be greater than n, but lefs than $2n$. This may be done in the manner following:

93. It has been fhewn in art. 82 that the two firft terms of the feries that is equal to $\overline{1+x}|^{\frac{m}{n}}$ will be 1 and $\frac{m}{n}x$ when m is greater than n, as well as when n is greater than m; and that the fecond term $\frac{m}{n}x$ is to be added to the firft term 1, and confequently to have the fign + prefixed to it, in this cafe as well as in the former; and likewife that the following terms of the faid feries will be Cx^2, Dx^3, Ex^4, Fx^5, &c (or the fquare and cube and other following powers of x in their natural order, without interruption, multiplied into certain fixt numeral co-efficients, which may be denoted by the capital letters C, D, E, F, &c), and connected with the two firft terms $1 + \frac{m}{n}x$ by either addition or fubtraction: fo that the feries that is equal to $\overline{1+x}|^{\frac{m}{n}}$ is in this cafe, as well as in the former, $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c; in which the feveral terms Cx^2, Dx^3, Ex^4, Fx^5, &c, have a comma prefixed to them, inftead of either of the figns + and −, becaufe we do not as yet know to which of them we are to prefix the fign +, and to which we are to prefix the fign −.

94. Now, fince $\overline{1+x}|^{\frac{m}{n}}$ is in this cafe, as well as in the former, = the feries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, it follows from art. 67, 68, 69, 70, 71, 72, that the fimple feries $\frac{m}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c, will be equal to the compound feries

$$\frac{2n}{m}C,\ \frac{3n}{m}Dx,\ \frac{4n}{m}Ex^2,\ \frac{5n}{m}Fx^3,\ \&c$$
$$+\ 1,\ \frac{2n}{m}Cx,\ \frac{3n}{m}Dx^2,\ \frac{4n}{m}Ex^3,\ \&c;\ \text{by}$$

the help of which equation we may difcover both "which of the figns + and − are to be prefixed to the feveral co-efficients C, D, E, F, &c (and confequently to the correfponding terms Cx^2, Dx^3, Ex^4, Fx^5, &c, in the feries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$)," and alfo "what are the magnitudes of the faid co-efficients C, D, E, F, &c, refpectively." This may be done by proceeding in the manner following:

95. In the firft place, fince the fimple feries $\frac{m}{n}$ Cx, Dx^2, Ex^3, Fx^4, &c, is equal to the compound feries

$$\frac{2n}{m}C,\ \frac{3n}{m}Dx,\ \frac{4n}{m}Ex^2,\ \frac{5n}{m}Fx^3,\ \&c$$
$$+\ 1,\ \frac{2n}{m}Cx,\ \frac{3n}{m}Dx^2,\ \frac{4n}{m}Ex^3,\ \&c;\ \text{and}$$

this

this equation is always true, of how ſmall a magnitude ſoever we ſuppoſe x to be taken, it follows that it will alſo be true when x is $= 0$. But, when x is $= 0$, all the terms in the equation that involve x will be equal to 0 likewiſe; that is, all the terms on the left-hand ſide of the equation, except the firſt term $\frac{m}{n}$, and all the terms on the right-hand ſide of the equation, except the two firſt terms of the two lines of terms, to wit, $\frac{2n}{m}C + 1$, will be $= 0$. Therefore $\frac{m}{n}$ will be $=$ to $\frac{2n}{m}C + 1$, or to $1, \frac{2n}{m}C$, or to 1, together with $\frac{2n}{m}C$, either added to it, or ſubtracted from it, as may be neceſſary to produce ſuch equality. Now, becauſe m is ſuppoſed in the preſent caſe to be greater than n, $\frac{m}{n}$ will be greater than 1, and conſequently $\frac{2n}{m}C$ muſt be added to 1, in order to make it equal to $\frac{m}{n}$. Therefore the ſign $+$ muſt be prefixed to $\frac{2n}{m}C$ in the equation $\frac{m}{n} = \frac{2n}{m}C + 1$, or $\frac{m}{n} = 1, \frac{2n}{m}C$; and conſequently the ſign $+$ muſt alſo be prefixed to the third term Cx^2 of the ſeries $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c (which is equal to $\overline{1+x}\rvert^{\frac{m}{n}}$); from which third term the ſaid quantity $\frac{2n}{m}C$ has been derived, by the operations of multiplication and diviſion, in the proceſſes of the inveſtigation contained in art. 67, 68, —— 72. Therefore the three firſt terms of the ſaid ſeries $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}\rvert^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x + Cx^2$. Q. E. I.

And the magnitude of the co-efficient C may likewiſe be determined by means of the equation $\frac{m}{n} = 1, \frac{2n}{m}C$, or $\frac{m}{n} = 1 + \frac{2n}{m}C$. For (by ſubtracting 1 from both ſides) we ſhall have $\frac{2n}{m}C = \frac{m}{n} - 1 = \frac{m-n}{n}$; and (by dividing both ſides by $\frac{2n}{m}$, or multiplying them into $\frac{m}{2n}$) we ſhall have $C = (\frac{m}{2n} \times \frac{m-n}{n} =) \frac{m}{n} \times \frac{m-n}{2n}$. Therefore the three firſt terms of the ſeries $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}\rvert^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2$. Q. E. I.

96. To determine the ſign that is to be prefixed to the fourth term, Dx^3, of the ſeries $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5$, &c, which is equal to $\overline{1+x}\rvert^{\frac{m}{n}}$, and to find the magnitude of its co-efficient D, we muſt proceed as follows:

Since

Since the sign $+$ is to be prefixed to the co-efficient C, it must likewise be prefixed to all the terms which involve it in the grand fundamental equation set down in art. 94, to wit, the equation between the simple series $\frac{m}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c, and the compound series

$$\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c$$
$$+ 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c; \text{ which}$$

equation will therefore be as follows, to wit, the simple series $\frac{m}{n} + Cx$, Dx^2, Ex^3, Fx^4, &c $=$ the compound series

$$+ \frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c$$
$$+ 1 + \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c.$$

But it has been shewn in the last article that $\frac{m}{n}$ is $= 1 + \frac{2n}{m}C$. Therefore, if we subtract $\frac{m}{n}$ and $1 + \frac{2n}{m}C$ from the opposite sides of the last equation, the remainders will be equal to each other; that is, the simple series Cx, Dx^2, Ex^3, Fx^4, &c, will be equal to the compound series

$$\frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c$$
$$+ \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c;$$

and consequently (dividing all the terms by x) we shall have the simple series C, Dx, Ex^2, Fx^3, &c $=$ the compound series

$$\frac{3n}{m}D, \frac{4n}{m}Ex, \frac{5n}{m}Fx^2, \&c$$
$$+ \frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \&c.$$

And this equation will be true, of how small a magnitude soever we suppose x to be taken: and therefore it will also be true when x is $= 0$. But, when x is equal to 0, all the terms that involve x will be equal to 0 likewise, and consequently the equation will be $C = \frac{3n}{m}D + \frac{2n}{m}C$, or $C = + \frac{2n}{m}C, \frac{3n}{m}D$; that is, C will be equal to $\frac{2n}{m}C$, together with $\frac{3n}{m}D$ either added to it or subtracted from it, as may be necessary to produce the said equality. But, because m is supposed to be less than $2n$, C will be less than $\frac{2n}{m}C$; and consequently $\frac{3n}{m}D$ must be subtracted from $\frac{2n}{m}C$, in order to make it equal to C. Therefore the sign $-$ must be prefixed to the quantity $\frac{3n}{m}D$; and consequently the same sign must also be prefixed to the fourth term Dx^3, of the series $1 + \frac{m}{n}x$, Cx^2, Dx^3,

Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$); from which fourth term the said quantity $\frac{3n}{m}$ D has been derived, by the operations of multiplication and division, in the course of the investigation by which the fundamental equation, set down in art. 94, was obtained. Therefore the four first terms of the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x + Cx^2 - Dx^3$, or $1 + \frac{m}{n}x + \times \frac{m-n}{2n}x^2 - Dx^3$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - Dx^3$. Q. E. I.

And, to determine the magnitude of D, we shall have the equation $C = \frac{2n}{m}C - \frac{3n}{m}D$; whence (adding $\frac{3n}{m}D$ to both sides) we shall have $C + \frac{3n}{m}D = \frac{2n}{m}C$, and (subtracting C from both sides) $\frac{3n}{m}D = \frac{2n}{m}C - C = \overline{\frac{2n}{m} - 1}| \times C = \frac{2n-m}{m} \times C$, and (multiplying both sides by $\frac{m}{3n}$) $D = \frac{2n-m}{3n} \times C = \frac{2n-m}{3n} \times \frac{m}{n} \times \frac{m-n}{2n}$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}$. Therefore the four first terms of the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}x^3$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \frac{2n-m}{3n}Cx^3$. Q. E. I.

97. To determine the sign that is to be prefixed to the fifth term, Ex^4, of the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, and to find the magnitude of the co-efficient E, we must proceed as follows:

It has been shewn in the last article that the simple series C, Dx, Ex^2, Fx^3, &c, is equal to the compound series

$$\frac{3n}{m}D,\ \frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2,\ \&c$$
$$+\frac{2n}{m}C,\ \frac{3n}{m}Dx,\ \frac{4n}{m}Ex^2,\ \&c;$$

and that the sign $-$ is to be prefixed to the terms that involve the co-efficient D. It therefore follows that the simple series $C - Dx$, Ex^2, Fx^3, &c, will be equal to the compound series

$$-\frac{3n}{m}D,\ \frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2,\ \&c$$
$$+\frac{2n}{m}C - \frac{3n}{m}Dx,\ \frac{4n}{m}Ex^2,\ \&c.$$

Therefore, if we add $2Dx$ to both sides, we shall have the simple series $C + Dx$, Ex^2, Fx^3, &c = the compound series

$$-\frac{3n}{m}D,$$

$$
\begin{array}{l}
-\frac{3n}{m}D,\ \frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2, \&c \\
+\frac{2n}{m}C - \frac{3n}{m}Dx,\ \frac{4n}{m}Ex^2, \&c \\
\qquad +2Dx.
\end{array}
$$

Further, it has alſo been ſhewn, in the laſt article, that C is $= \frac{2n}{m}C - \frac{3n}{m}D$. Therefore, if we ſubtract C and $\frac{2n}{m}C - \frac{3n}{m}D$ from the oppoſite ſides of the laſt equation, the remainders will be equal; that is, the ſimple ſeries Dx, Ex^2, Fx^3, &c, will be equal to the compound ſeries

$$
\begin{array}{l}
\frac{4n}{m}Ex,\ \frac{5n}{m}Fx^2, \&c \\
-\frac{3n}{m}Dx,\ \frac{4n}{m}Ex^2, \&c \\
+2Dx.
\end{array}
$$

Therefore (dividing all the terms by x) we ſhall have the ſimple ſeries $+D$, Ex, Fx^2, &c = the compound ſeries

$$
\begin{array}{l}
\frac{4n}{m}E,\ \frac{5n}{m}Fx, \&c \\
-\frac{3n}{m}D,\ \frac{4n}{m}Ex, \&c \\
+2D.
\end{array}
$$

And this equation will be true, of how ſmall a magnitude ſoever we ſuppoſe x to be taken: and therefore it will alſo be true when x is $= 0$. But when x is $= 0$, all the terms that involve x will be equal to 0 likewiſe, and conſequently the equation will then be $+D = \frac{4n}{m}E - \frac{3n}{m}D + 2D$. Therefore (adding $\frac{3n}{m}D$ to both ſides) we ſhall have $D + \frac{3n}{m}D = \frac{4n}{m}E + 2D$, and (ſubtracting D from both ſides) $\frac{3n}{m}D = \frac{4n}{m}E + D$, or $\frac{3n}{m}D = D, \frac{4n}{m}E$, or $\frac{3n}{m}D = D$, together with $\frac{4n}{m}E$, either added to it, or ſubtracted from it, as may be neceſſary to produce ſuch equality. But, becauſe m is ſuppoſed to be leſs than $2n$, $\frac{3n}{m}$ will be greater than 1, and $\frac{3n}{m}D$ will be greater than D; and conſequently $\frac{4n}{m}E$ muſt be added to D, in order to make it equal to $\frac{3n}{m}D$. Therefore the ſign $+$ muſt be prefixed to the quantity $\frac{4n}{m}E$ in the equation $\frac{3n}{m}D = D, \frac{4n}{m}E$, and therefore it muſt alſo be prefixed to the fifth term, Ex^4, of the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}\big|^{\frac{m}{n}}$), from which fifth term the ſaid quantity $\frac{4n}{m}E$ has been derived above in art. 67, 68, ... 72, by the operations of multiplication and diviſion. Therefore the five firſt terms of the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, which is equal to $\overline{1+x}\big|^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x$

$+ \frac{m}{n} \times \frac{m-n}{2n} x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} x^3 + E x^4$, or $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 - \overline{\frac{2n-m}{3n}} C x^3 + E x^4$. Q. E. I.

And to determine the magnitude of the co-efficient E, we ſhall have the equation $\frac{3n}{m} D = D + \frac{4n}{m} E$; whence (ſubtracting D from both ſides) $\frac{4n}{m} E$ will be $= \frac{3n}{m} D - D = \overline{\frac{3n}{m} - 1} \times D = \frac{3n-m}{m} \times D$, and conſequently (multiplying both ſides into $\frac{m}{4n}$) E will be $= \frac{3n-m}{4n} \times D = \frac{3n-m}{4n} \times \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}$. Therefore the five firſt terms of the ſeries $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, will be $1 + \frac{m}{n} x + \frac{m}{n} \times \frac{m-n}{2n} x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4$, or $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 - \overline{\frac{2n-m}{3n}} C x^3 + \frac{3n-m}{4n} D x^4$. Q. E. I.

98. To determine the ſign that is to be prefixed to the ſixth term, $F x^5$, of the ſeries $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5$, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, and to find the magnitude of the co-efficient F, we muſt proceed as follows:

It has been ſhewn, in the laſt article, that the ſimple ſeries D, $E x$, $F x^2$, &c, is equal to the compound ſeries

$$\begin{array}{ll} \frac{4n}{m} E, & \frac{5n}{m} F x, \text{ \&c} \\ -\frac{3n}{m} D, & \frac{4n}{m} E x, \\ +2D, & \end{array}$$

&c, and alſo that the ſign + is to be prefixed to the terms $\frac{4n}{m} E$ and $\frac{4n}{m} E x$, and $E x$. Therefore the ſaid equation will, when theſe three terms have their proper ſigns prefixed to them, be as follows; to wit, the ſimple ſeries $D + E x, F x^2$, &c = the compound ſeries

$$\begin{array}{lll} + \frac{4n}{m} E, & \frac{5n}{m} F x, & \frac{6n}{m} G x^2, \text{ \&c} \\ - \frac{3n}{m} D + & \frac{4n}{m} E x, & \frac{5n}{m} F x^2, \text{ \&c} \\ + 2D. & & \end{array}$$

Further, it has been ſhewn, in the laſt article, that D is equal to $+ \frac{4n}{m} E - \frac{3n}{m} D + 2D$.

Therefore, if we ſubtract D and $+ \frac{4n}{m} E - \frac{3n}{m} D + 2D$ from the oppoſite ſides of the foregoing equation, the remainders will be equal, that is, the ſimple ſeries $+ E x, F x^2$, &c, will be equal to the compound ſeries

2

$\frac{5n}{m} F x$,

$$\frac{5n}{m} F x, \frac{6n}{m} G x^2, \&c$$

$+ \frac{4n}{m} E x, \frac{5n}{m} F x^2$, &c; and consequently (dividing all the terms on both sides by x) the simple series $+ E, F x$, &c, will be equal to the compound series

$$\frac{5n}{m} F, \frac{6n}{m} G x, \&c$$
$$+ \frac{4n}{m} E, \frac{5n}{m} F x, \&c.$$

And this equation will be true, of how small a magnitude soever we may suppose x to be taken; and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will then be $+ E = \frac{5n}{m} F + \frac{4n}{m} E$, or $+ E = + \frac{4n}{m} E, \frac{5n}{m} F$; that is, E will be equal to $\frac{4n}{m} E$, together with $\frac{5n}{m} F$, either added to it or subtracted from it, as may be necessary to produce such equality. But, because $2n$ is supposed to be greater than m, $4n$ will, *à fortiori*, be greater than m, and consequently $\frac{4n}{m} E$ will be greater than E. Therefore, in order to make $\frac{4n}{m} E, \frac{5n}{m} F$ be equal to E, we must subtract $\frac{5n}{m} F$ from $\frac{4n}{m} E$, or prefix the sign $-$ to the term $\frac{5n}{m} F$. Therefore the same sign $-$ must also be prefixed to the sixth term, $F x^5$, of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5$, &c (which is equal to $\overline{1 + x}|^{\frac{m}{n}}$), from which sixth term the said quantity $\frac{5n}{m} F$ has been derived by the operations of multiplication and division in the course of the foregoing investigation. Therefore the first six terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5$, &c (which is equal to $\overline{1 + x}|^{\frac{m}{n}}$), will in this case be $1 + \frac{m}{n} x + \frac{m}{n} \times \frac{m-n}{2n} x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 - F x^5$, or $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 - \frac{2n-m}{3n} C x^3 + \frac{3n-m}{4n} D x^4 - F x^5$.

Q. E. I.

And to determine the magnitude of the co-efficient F, we shall have the equation $E = \frac{4n}{m} E - \frac{5n}{m} F$; whence (adding $\frac{5n}{m} F$ to both sides) we shall have $E + \frac{5n}{m} F = \frac{4n}{m} E$, and (subtracting E from both sides) $\frac{5n}{m} F$ ($= \frac{4n}{m} E - E = \overline{\frac{4n}{m} - 1}| \times E$) $= \frac{4n-m}{m} \times E$, and (multiplying both sides into $\frac{m}{5n}$) $F = \frac{4n-m}{5n} \times E = \frac{4n-m}{5n} \times \frac{3n-m}{4n} \times \frac{2n-m}{3n} \times \frac{m-n}{2n} \times \frac{m}{n}$. Therefore the first six terms

terms of the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), will (upon the ſuppoſition here made, that m is greater than n, but leſs than $2n$) be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}x^4 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \frac{2n-m}{3n}Cx^3 + \frac{3n-m}{4n}Dx^4 - \frac{4n-m}{5n}Ex^5$.

Q. E. I.

99. In like manner we may derive from the general equation in art. 94, to wit, the equation between the ſimple ſeries $\frac{m}{n}$, Cx, Dx^2, Ex^3, Fx^4, Gx^5, &c, and the compound ſeries

$$\frac{2n}{m}C,\ \frac{3n}{m}Dx,\ \frac{4n}{m}Ex^2,\ \frac{5n}{m}Fx^3,\ \frac{6n}{m}Gx^4,\ \frac{7n}{m}Hx^5,\ \&c$$
$$+\ 1,\ \frac{2n}{m}Cx,\ \frac{3n}{m}Dx^2,\ \frac{4n}{m}Ex^3,\ \frac{5n}{m}Fx^4,\ \frac{6n}{m}Gx^5,\ \&c,$$

the following particular equations for determining both the ſigns that are to be prefixed to the following terms, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c, of the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), and the values of the co-efficients G, H, I, K, L, M, &c; to wit, the equations

$$F = \frac{6n}{m}G,\ \frac{5n}{m}F,$$
$$G = \frac{7n}{m}H,\ \frac{6n}{m}G,$$
$$H = \frac{8n}{m}I,\ \frac{7n}{m}H,$$
$$I = \frac{9n}{m}K,\ \frac{8n}{m}I,$$
$$K = \frac{10n}{m}L,\ \frac{9n}{m}K,$$
$$L = \frac{11n}{m}M,\ \frac{10n}{m}L,$$

&c,

$$\text{or } F = \frac{5n}{m}F,\ \frac{6n}{m}G,$$
$$G = \frac{6n}{m}G,\ \frac{7n}{m}H,$$
$$H = \frac{7n}{m}H,\ \frac{8n}{m}I,$$
$$I = \frac{8n}{m}I,\ \frac{9n}{m}K,$$

K =

$$K = \frac{9n}{m} K, \frac{10n}{m} L,$$
$$L = \frac{10n}{m} L, \frac{11n}{m} M,$$

&c, as far as we pleaſe to continue them.

100. And, in all theſe equations, the ſingle term which forms the left-hand ſide of the equation, will always have the ſame ſign + or − prefixed to it as is prefixed to the term on the right-hand ſide of the equation that involves in it the ſame letter. Thus, the two terms F and $\frac{5n}{m}$ F, in the firſt equation, will have the ſame ſign prefixed to them; and the two terms G and $\frac{6n}{m}$ G, in the ſecond equation, will have the ſame ſign prefixed to them; and the two terms H and $\frac{7n}{m}$ H, in the third equation, will have the ſame ſign prefixed to them; and the following letters, I, K, and L, in the fourth, fifth, and ſixth equations, will, in like manner, have the ſame ſigns prefixed to them as are prefixed to the terms $\frac{8n}{m}$ I, $\frac{9n}{m}$ K, $\frac{10n}{m}$ L, reſpectively. This follows evidently from the obſervation that has been often repeated in the foregoing articles, to wit, that the terms that involve the ſame letters in theſe latter, or particular equations, are all derived from the ſame terms of the ſame original ſeries, to wit, the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), by the operations of multiplication and diviſion in the courſe of the foregoing inveſtigation; by which operations the ſigns + and −, that are to be prefixed to them, cannot be affected.

101. Further, the terms denoted by the ſingle letters F, G, H, I, K, L, &c, which form the left-hand ſides of theſe equations, are uniformly leſs than the terms $\frac{5n}{m}$ F, $\frac{6n}{m}$ G, $\frac{7n}{m}$ H, $\frac{8n}{m}$ I, $\frac{9n}{m}$ K, $\frac{10n}{m}$ L, &c, on the right-hand ſides of the ſame equations which involve the ſame letters reſpectively; becauſe m is ſuppoſed to be leſs than $2n$, and, *à fortiori*, leſs than $5n$, $6n$, $7n$, $8n$, $9n$, $10n$, &c. It follows, therefore, that, in order to make the right-hand ſides of theſe ſeveral equations be equal to the left-hand ſides of them reſpectively, it is neceſſary that the ſign + or −, that is to be prefixed to the ſecond term on the right-hand ſide of each of theſe equations, ſhould be contrary to the ſign which is prefixed to the firſt term. And hence we may determine the ſeveral ſigns that are to be prefixed to the ſeveral terms of the ſaid particular equations, in the manner following.

102. It having been ſhewn, in art. 98, that the ſign − is to be prefixed to the co-efficient F, and to the term $\frac{5n}{m}$ F, which involves it, it follows, from the laſt article, that the ſign + muſt be prefixed to the remaining term $\frac{6n}{m}$ G, in the firſt of

of the foregoing particular equations, to wit, the equation $F = \frac{5n}{m} F, \frac{6n}{m} G$; and consequently the said equation, when its terms have their proper signs prefixed to them, will be $- F = - \frac{5n}{m} F + \frac{6n}{m} G$.

And, secondly, since $\frac{6n}{m} G$ has the sign + prefixed to it, it follows, from art. 100, that the term G in the second equation, $G = \frac{6n}{m} G, \frac{7n}{m} H$, must also have the sign + prefixed to it; and consequently, by art. 101, that the remaining term, $\frac{7n}{m} H$, must have the sign − prefixed to it; and consequently the said second equation, when its terms have their proper signs prefixed to them, will be $+ G = + \frac{6n}{m} G - \frac{7n}{m} H$.

And, thirdly, since $\frac{7n}{m} H$ has the sign − prefixed to it, it follows, from art. 100, that the term H in the third equation, $H = \frac{7n}{m} H, \frac{8n}{m} I$, must also have the sign − prefixed to it, and consequently (by art. 101) that the remaining term, $\frac{8n}{m} I$, must have the sign + prefixed to it; and consequently the said third equation, when its terms have their proper signs prefixed to them, will be $- H = - \frac{7n}{m} H + \frac{8n}{m} I$.

And in like manner it may be shewn that the fourth equation, $I = \frac{8n}{m} I, \frac{9n}{m} K$, when the proper signs are prefixed to its terms, will be $+ I = + \frac{8n}{m} I - \frac{9n}{m} K$; and that the fifth equation, $K = \frac{9n}{m} K, \frac{10n}{m} L$, when the proper signs are prefixed to its terms, will be $- K = - \frac{9n}{m} K + \frac{10n}{m} L$; and that the sixth equation, $L = \frac{10n}{m} L, \frac{11n}{m} M$, when the proper signs are prefixed to its terms, will be $+ L = + \frac{10n}{m} L - \frac{11n}{m} M$.

Therefore the six foregoing equations, when their proper signs + and − are prefixed to their terms, will be as follows; to wit,

$$- F = - \frac{5n}{m} F + \frac{6n}{m} G,$$
$$+ G = + \frac{6n}{m} G - \frac{7n}{m} H,$$
$$- H = - \frac{7n}{m} H + \frac{8n}{m} I,$$
$$+ I = + \frac{8n}{m} I - \frac{9n}{m} K,$$
$$- K = - \frac{9n}{m} K + \frac{10n}{m} L,$$
$$+ L = \frac{10n}{m} L - \frac{11n}{m} M.$$

In

In these equations the several co-efficients F, G, H, I, K, and L, have the signs $-$ and $+$ prefixed to them alternately; and it is easy to see that the same thing must take place in all the following co-efficients, M, N, O, P, Q, R, S, T, &c, to whatever number of terms the series be continued. We may therefore conclude that in the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, Nx^{12}, Ox^{13}, Px^{14}, Qx^{15}, Rx^{16}, Sx^{17}, Tx^{18}, &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), when m is greater than n, but less than $2n$, the third term, Cx^2, is to be added to the two first $1 + \frac{m}{n}x$, and consequently marked with the sign $+$, and all the following terms Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, Nx^{12}, Ox^{13}, Px^{14}, Qx^{15}, Rx^{16}, Sx^{17}, Tx^{18}, &c, to whatever number of terms the series be continued, are to be marked with the signs $-$ and $+$ alternately; and consequently that, with respect to the signs of its terms, the said series will be as follows, to wit, $1 + \frac{m}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + Rx^{16} - Sx^{17} + Tx^{18} - $ &c. Q. E. I.

103. It remains that we determine the magnitudes of the several co-efficients F, G, H, I, K, L, &c. Now these may be easily found by resolving the short simple equations obtained in the foregoing article; to wit, the equations

$$-F = -\frac{5n}{m}F + \frac{6n}{m}G,$$

$$+G = +\frac{6n}{m}G - \frac{7n}{m}H,$$

$$-H = -\frac{7n}{m}H + \frac{8n}{m}I,$$

$$+I = +\frac{8n}{m}I - \frac{9n}{m}K,$$

$$-K = -\frac{9n}{m}K + \frac{10n}{m}L,$$

$$+L = +\frac{10n}{m}L - \frac{11n}{m}M,$$

&c; which may be done in the manner following:

To resolve the first equation, $-F = -\frac{5n}{m}F + \frac{6n}{m}G$, let $\frac{5n}{m}F$ be added to both sides; and we shall have $\frac{6n}{m}G = \frac{5n}{m}F - F = \overline{\frac{5n}{m} - 1}| \times F = \frac{5n-m}{m} \times F$, and consequently (multiplying both sides into $\frac{m}{6n}$) $G = \frac{5n-m}{6n} \times F$.

Q. E. I.

To resolve the second equation $+G = \frac{6n}{m}G - \frac{7n}{m}H$, let $\frac{7n}{m}H$ be added to both sides; and we shall have $+G + \frac{7n}{m}H = \frac{6n}{m}G$, and consequently (subtracting G from both sides) $\frac{7n}{m}H = \frac{6n}{m}G - G = \overline{\frac{6n}{m} - 1}| \times G = \frac{6n-m}{m} \times G$, and (multiplying both sides by $\frac{m}{7n}$) $H = \frac{6n-m}{7n} \times G$. Q. E. I.

To

To resolve the third equation $-H = -\frac{7n}{m}H + \frac{8n}{m}I$, let $\frac{7n}{m}H$ be added to both sides; and we shall have $\frac{8n}{m}I = \frac{7n}{m}H - H = \overline{\frac{7n}{m} - 1} \times H = \frac{7n - m}{m} \times H$, and consequently (multiplying both sides by $\frac{m}{8n}$) $I = \frac{7n - m}{8n} \times H$.

Q. E. I.

To resolve the fourth equation $+I = \frac{8n}{m}I - \frac{9n}{m}K$, let $\frac{9n}{m}K$ be added to both sides; and we shall have $I + \frac{9n}{m}K = \frac{8n}{m}I$, and consequently (subtracting I from both sides) $\frac{9n}{m}K = \frac{8n}{m}I - I = \overline{\frac{8n}{m} - 1} \times I = \frac{8n - m}{m} \times I$, and (multiplying both sides by $\frac{m}{9n}$) $K = \frac{8n - m}{9n} \times I$. Q. E. I.

To resolve the fifth equation $-K = -\frac{9n}{m}K + \frac{10n}{m}L$, let $\frac{9n}{m}K$ be added to both sides; and we shall have $\frac{10n}{m}L = \frac{9n}{m}K - K = \overline{\frac{9n}{m} - 1} \times K = \frac{9n - m}{m} \times K$, and consequently (multiplying both sides by $\frac{m}{10n}$) $L = \frac{9n - m}{10n} \times K$.

Q. E. I.

And, to resolve the sixth equation, $+L = +\frac{10n}{m}L - \frac{11n}{m}M$, let $\frac{11n}{m}M$ be added to both sides; and we shall have $L + \frac{11n}{m}M = \frac{10n}{m}L$, and consequently (subtracting L from both sides) $\frac{11n}{m}M = \frac{10n}{m}L - L = \overline{\frac{10n}{m} - 1} \times L = \frac{10n - m}{m} \times L$, and (multiplying both sides by $\frac{m}{11n}$) $M = \frac{10n - m}{11n} \times L$.

Q. E. I.

It appears therefore that

$$\text{G will be} = \frac{5n - m}{6n} \times F,$$

$$\text{And that H will be} = \frac{6n - m}{7n} \times G,$$

$$\text{and } I = \frac{7n - m}{8n} \times H,$$

$$\text{and } K = \frac{8n - m}{9n} \times I,$$

$$\text{and } L = \frac{9n - m}{10n} \times K,$$

$$\text{and } M = \frac{10n - m}{11n} \times L.$$

And it is easy to see that, if we were to investigate the values of the following co-efficients, N, O, P, Q, R, S, T, &c, by the resolution of the following short, simple, equations which relate to them respectively, we should find

N to

$$\begin{aligned}
\text{N to be} &= \frac{11n-m}{12n} \times \text{M},\\
\text{and O} &= \frac{12n-m}{13n} \times \text{N},\\
\text{and P} &= \frac{13n-m}{14n} \times \text{O},\\
\text{and Q} &= \frac{14n-m}{15n} \times \text{P},\\
\text{and R} &= \frac{15n-m}{16n} \times \text{Q},\\
\text{and S} &= \frac{16n-m}{17n} \times \text{R},\\
\text{and T} &= \frac{17n-m}{18n} \times \text{S},
\end{aligned}$$

and so on *ad infinitum*, every new generating fraction being derived from that which immediately preceeds it by adding n to both its numerator and its denominator.

104. We may therefore now conclude that when m is greater than n, but less than $2n$, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$, will be equal to the series $1 + \frac{m}{n} \text{A} x + \frac{m-n}{2n} \text{B} x^2 - \Big|\frac{2n-m}{3n} \text{C} x^3 + \frac{3n-m}{4n} \text{D} x^4 - \Big|\frac{4n-m}{5n} \text{E} x^5 + \frac{5n-m}{6n} \text{F} x^6 - \Big|\frac{6n-m}{7n} \text{G} x^7 + \frac{7n-m}{8n} \text{H} x^8 - \Big|\frac{8n-m}{9n} \text{I} x^9 + \frac{9n-m}{10n} \text{K} x^{10} - \Big|\frac{10n-m}{11n} \text{L} x^{11} + \frac{11n-m}{12n} \text{M} x^{12} - \Big|\frac{12n-m}{13n} \text{N} x^{13} + \frac{13n-m}{14n} \text{O} x^{14} - \Big|\frac{14n-m}{15n} \text{P} x^{15} + \frac{15n-m}{16n} \text{Q} x^{16} - \Big|\frac{16n-m}{17n} \text{R} x^{17} - \frac{17n-m}{18n} \text{S} x^{18} -$ &c, *ad infinitum*; in which series the third term $\frac{m-n}{2n} \text{B} x^2$ is added to the two first terms $1 + \frac{m}{n} x$, and the fourth, fifth, sixth, seventh, and other following terms are alternately subtracted from the three first terms $1 + \frac{m}{n} \text{A} x + \frac{m-n}{2n} \text{B} x^2$, and added to them, and consequently marked with the signs $-$ and $+$ alternately; and the several generating fractions $\frac{2n-m}{3n}$, $\frac{3n-m}{4n}$, $\frac{4n-m}{5n}$, $\frac{5n-m}{6n}$, $\frac{6n-m}{7n}$, &c, are formed one from the other by the continual addition of n to both their numerators and their denominators.

Q. E. I.

105. We have now shewn the binomial theorem to be true in the case of $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$, when m, the numerator of the index $\frac{m}{n}$, is greater than n, its denominator, but less than $2n$. It remains that we shew it to be true also when m is greater than $2n$,

 but

but lefs than $3n$, and when it is greater than $3n$ but lefs than $4n$, and when it is greater than $4n$ but lefs than $5n$, and, in general, when it is of any greater magnitude whatfoever.

106. Now for this purpofe we need only obferve that in all thefe different magnitudes of m with refpect to n, as well as when m is either lefs than n, or greater than n, but lefs than $2n$, the quantity $\overline{1+x}|^{\frac{m}{n}}$ will be equal to the fame feries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, of which the firft term is 1, and the fecond term is $\frac{m}{n}x$ and is added to the firft term, and the third and fourth and fifth and fixth, and other following terms, Cx^2, Dx^3, Ex^4, Fx^5, &c, confift of the fquare, and cube, and fourth power, and fifth power and other following powers of x in their natural order without interruption, multiplied into certain fixt numbers, or numeral co-efficients, which may be denoted by the capital letters C, D, E, F, &c, and are to be connected with the two firft terms $1 + \frac{m}{n}x$ either by addition or fubtraction, and confequently marked with either the fign + or the fign −. For from hence it follows that, if we apply the reafonings ufed above in art. 67, 68, 69, 70, 71, 72, to the inveftigation of the terms Cx^2, Dx^3, Ex^4, Fx^5, &c, in the feries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, and of the figns + and −, which are to be prefixed to them, in any of thefe new relative magnitudes of m and n, we fhall always come to the fame final equation which was obtained for this purpofe in art. 72, to wit, the equation between the fimple feries $\frac{m}{n}$, Cx, Dx^2, Ex^3, Fx^4, &c, and the compound feries

$$\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \text{&c}$$
$$+ 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \text{&c.}$$

And from this general and fundamental equation we may derive particular equations for the determination of the figns that are to be prefixed to the feveral terms Cx^2, Dx^3, Ex^4, Fx^5, &c, and of the magnitudes of the feveral co-efficients C, D, E, F, &c, which will confift of the very fame terms as the particular equations derived from the faid general equation in the former relative magnitudes of m and n; to wit, the particular equations following:

1ft, $\frac{m}{n} = \frac{2n}{m}C, + 1,$

2dly, $C = \frac{3n}{m}D, \frac{2n}{m}C,$

3dly, $D = \frac{4n}{m}E, \frac{3n}{m}D,$

4thly, $E = \frac{5n}{m}F, \frac{4n}{m}E,$

&c,

Or (as they may be expreffed with more convenience)

1ft,

1st, $\frac{m}{n} = 1, \frac{2n}{m} C,$

2dly, $C = \frac{2n}{m} C, \frac{3n}{m} D,$

3dly, $D = \frac{3n}{m} D, \frac{4n}{m} E,$

4thly, $E = \frac{4n}{m} E, \frac{5n}{m} F,$

&c. And the only difference between these particular equations in the cases of these new relative magnitudes of m and n (in which m is supposed to be greater than $2n$) and the former particular equations, consisting of the same terms, in the cases of the former relative magnitudes of m and n (in which m was supposed to be either less than n, or greater than n, but less than $2n$) will be in the signs + and − that are to be prefixed to their terms. This difference we must now proceed to investigate.

107. In the aforesaid particular equations, by means of which the signs to be prefixed to the terms Cx^2, Dx^3, Ex^4, Fx^5, &c, and the values or magnitudes of the co-efficients C, D, E, F, &c, are to be determined, to wit, the equations

$$\frac{m}{n} = 1, \frac{2n}{m} C,$$
$$C = \frac{2n}{m} C, \frac{3n}{m} D,$$
$$D = \frac{3n}{m} D, \frac{4n}{m} E,$$
$$E = \frac{4n}{m} E, \frac{5n}{m} F,$$
$$F = \frac{5n}{m} F, \frac{6n}{m} G,$$

&c, the common denominator m of the several fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, &c, will, on the present supposition, be greater than the numerator, $2n$, of the first of the said fractions; and it may be greater than $3n$, $4n$, $5n$, $6n$, $7n$, and many more of the numerators of the following fractions, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c. But, however great we may suppose m to be in comparison of n, we may always, by continuing the aforesaid particular equations to a sufficient number, come at last to an equation in which the numerator of the fraction involved in the first term on the right-hand side of the equation will be greater than the denominator m. For let pn be the greatest multiple of n that is less than m: then will $\overline{p+1} \times n$, or $pn + n$, be greater than m; and consequently, when the fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c, are continued to the term $\frac{\overline{p+1} \times n}{m}$, or $\frac{pn+n}{m}$ (to which, it is evident, they may be continued), the numerator of the said fraction $\frac{pn+n}{m}$ will be greater than its denominator. Now, so long as the numerators of these fractions, $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c, are less than their denominator m, the fractions themselves will be less than 1: but when we come to the fraction

 $\frac{pn+n}{m}$,

$\frac{pn+n}{m}$, of which the numerator is greater than the denominator, the ſaid fraction will be greater than 1; and ſo will, *à fortiori*, all the following fractions; to wit, $\frac{\overline{p+2}\times n}{m}$, $\frac{\overline{p+3}\times n}{m}$, $\frac{\overline{p+4}\times n}{m}$, $\frac{\overline{p+5}\times n}{m}$, &c, or $\frac{pn+2n}{m}$, $\frac{pn+3n}{m}$, $\frac{pn+4n}{m}$, $\frac{pn+5n}{m}$, &c. Therefore, ſo long as the numerators of the ſaid fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c, continue to be leſs than their common denominator m, the quantities $\frac{2n}{m}$ C, $\frac{3n}{m}$ D, $\frac{4n}{m}$ E, $\frac{5n}{m}$ F, $\frac{6n}{m}$ G, $\frac{7n}{m}$ H, &c (which are the firſt terms on the right-hand ſides of the 2d, 3d, 4th, 5th, 6th, 7th, &c, particular equations above-mentioned) will be leſs than the ſingle quantities C, D, E, F, G, H, &c, which form the left-hand ſides of the ſame equations: but when we are come to the fractions $\frac{pn+n}{m}$, $\frac{pn+2n}{m}$, $\frac{pn+3n}{m}$, $\frac{pn+4n}{m}$, $\frac{pn+5n}{m}$, &c (which are greater than 1), the quantities of which theſe fractions will be the co-efficients, and which will be the firſt terms on the right-hand ſides of the ſeveral particular equations in which they will appear, will be greater than the correſponding ſingle quantities, which will form the left-hand ſides of the ſame particular equations; or, if we ſuppoſe the ſaid particular equations to be as follows, to wit,

$$C' = \frac{pn+n}{m}C', \frac{pn+2n}{m}D',$$
$$D' = \frac{pn+2n}{m}D', \frac{pn+3n}{m}E',$$
$$E' = \frac{pn+3n}{m}E', \frac{pn+4n}{m}F',$$
$$\text{and } F' = \frac{pn+4n}{m}F', \frac{pn+5n}{m}G',$$

&c (in which the capital letters C, D, E, F, G, &c, have an accent ′ placed over them, in order to diſtinguiſh them from the former co-efficients C, D, E, F, G, &c), the quantities $\frac{pn+n}{m}C'$, $\frac{pn+2n}{m}D'$, $\frac{pn+3n}{m}E'$, and $\frac{pn+4n}{m}F'$, &c, which are the firſt terms on the right-hand ſides of the ſaid particular equations, will be greater than the correſponding ſingle terms C′, D′, E′, F′, &c, which form the left hand ſides of the ſame equations.

108. Further, it follows, from what has been repeatedly obſerved in the courſe of the foregoing inveſtigations, that in all theſe particular equations the terms that involve the ſame co-efficients, or capital letters, C, D, E, F, G, &c, or C′, D′, E′, F′, G′, &c, muſt have the ſame ſign + or − prefixed to them; becauſe they are derived from the ſame original terms Cx^2, Dx^3, Ex^4, Fx^5, &c, in the aſſumed ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, by the operations of multiplication and diviſion, by which the ſigns + and −, that are to be prefixed to them, cannot be affected. Therefore the firſt terms on the right-hand ſides of all theſe particular equations muſt always have the ſame ſigns + and − prefixed to them as are to be prefixed to the ſingle quantities which form the left-hand ſides of the ſame equations.

109. And hence it follows that, so long as the first terms on the right-hand sides of these particular equations, to wit, the terms $\frac{2n}{m}$ C, $\frac{3n}{m}$ D, $\frac{4n}{m}$ E, $\frac{5n}{m}$ F, $\frac{6n}{m}$ G, &c, are less than the single terms that form the left-hand sides of the same equations, to wit, the terms C, D, E, F, G, &c, respectively, the second terms on the right-hand sides of the said particular equations, to wit, the terms $\frac{3n}{m}$ D, $\frac{4n}{m}$ E, $\frac{5n}{m}$ F, $\frac{6n}{m}$ G, $\frac{7n}{m}$ H, &c, must be marked with the same signs as the first terms on the same right-hand sides of the same equations, to wit, $\frac{2n}{m}$ C, $\frac{3n}{m}$ D, $\frac{4n}{m}$ E, $\frac{5n}{m}$ F, $\frac{6n}{m}$ G, &c, or must be added to the said first terms, so as thereby to increase their magnitudes. For otherwise the two quantities on the right-hand sides of the said equations cannot be equal to the single quantities on the left-hand sides of them. But, when the first terms on the right-hand sides of these particular equations become greater than the corresponding single terms that form the left-hand sides of the same equations, which happens in the equations

$$C' = \frac{pn+n}{m}\ C', \frac{pn+2n}{m}\ D',$$
$$D' = \frac{pn+2n}{m}\ D', \frac{pn+3n}{m}\ E',$$
$$E' = \frac{pn+3n}{m}\ E', \frac{pn+4n}{m}\ F',$$
$$\text{and } F' = \frac{pn+4n}{m}\ F', \frac{pn+5n}{m}\ G',$$

&c, the second terms on the right-hand sides of these equations, to wit, the terms $\frac{pn+2n}{m}$ D′, $\frac{pn+3n}{m}$ E′, $\frac{pn+4n}{m}$ F′, $\frac{pn+5n}{m}$ G′, &c, must be marked with the contrary signs to those which are to be prefixed to the first terms on the right-hand sides of those equations, to wit, the terms $\frac{pn+n}{m}$ C′, $\frac{pn+2n}{m}$ D′, $\frac{pn+3n}{m}$ E′, $\frac{pn+4n}{m}$ F′, &c, respectively, so as to lessen the too great magnitudes of the said first terms, and reduce them to an equality with the single terms C′, D′, E′, F′, &c, which form the left-hand sides of the said equations.

110. From these observations it follows that, if m is greater than $2n$, but less than $3n$, or, if p is $= 2$, and consequently $\frac{pn+n}{m}$ is $= \frac{2n+n}{m}$, or $\frac{3n}{m}$, the particular equations set down above in the beginning of art. 107, to wit, the equations

$$\frac{m}{n} = 1, \frac{2n}{m}\ C,$$
$$C = \frac{2n}{m}\ C, \frac{3n}{m}\ D,$$
$$D = \frac{3n}{m}\ D, \frac{4n}{m}\ E,$$

E =

$$E = \frac{4n}{m} E, \frac{5n}{m} F,$$

$$F = \frac{5n}{m} F, \frac{6n}{m} G,$$

&c, will, when the proper figns + and − are prefixed to their terms, be as follows, to wit,

$$\frac{m}{n} = 1 + \frac{2n}{m} C,$$

$$+ C = + \frac{2n}{m} C + \frac{3n}{m} D,$$

$$+ D = + \frac{3n}{m} D - \frac{4n}{m} E,$$

$$- E = - \frac{4n}{m} E + \frac{5n}{m} F,$$

$$+ F = + \frac{5n}{m} F - \frac{6n}{m} G,$$

&c; and, if m is greater than $3n$, but lefs than $4n$, or p is $= 3$, and confequently $\frac{pn + n}{m}$ is $= \frac{3n + n}{m} = \frac{4n}{m}$, the faid equations will, when the proper figns + and − are prefixed to their terms, be as follows, to wit,

$$\frac{m}{n} = 1 + \frac{2n}{m} C,$$

$$+ C = + \frac{2n}{m} C + \frac{3n}{m} D,$$

$$+ D = + \frac{3n}{m} D + \frac{4n}{m} E,$$

$$+ E = + \frac{4n}{m} E - \frac{5n}{m} F,$$

$$- F = - \frac{5n}{m} F + \frac{6n}{m} G,$$

And the next equation to thefe will be

$$+ G = + \frac{6n}{m} G - \frac{7n}{m} H,$$

&c.

And, in general, all the co-efficients C, D, E, &c, will be marked with the fign + till we come to the equation in which the fraction by which the co-efficient is multiplied in the firft term on the right-hand fide of the equation is $\frac{pn + n}{m}$, of which the numerator $pn + n$ is greater than the denominator m; after which the next co-efficient will be marked with the fign −, and the next to that will be marked with the fign +, and the next again with the fign −, and fo on to as many co-efficients as thefe equations fhall be extended to, the figns + and − being to be prefixed to all the faid following co-efficients alternately. It follows therefore that in the general feries $1 + \frac{m}{n} x$, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, &c, *ad infinitum*, which is $= \overline{1 + x}|^{\frac{m}{n}}$, all the terms Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, &c, will have the fign + prefixed to them, or will be to be added to the two firft terms $1 + \frac{m}{n} x$, till we come to the $\overline{pn + n}|$th term,

term, $pn + n$ being the first multiple of n that is greater than m; and the said $\overline{pn + n}|$th term will be to be subtracted from the said two first terms, and marked with the sign $-$; and all the subsequent terms after the $\overline{pn + n}|$th term, will be marked with the sign $+$ and the sign $-$ alternately, or will be alternately to be added to, and subtracted from, the said two first terms. Q. E. I.

Note. This conclusion is agreeable to what was asserted above in the beginning of this discourse, in art. 5 and 6.

111. It remains that we investigate the values, or magnitudes, of the co-efficients C, D, E, F, G, &c, when m is greater than $2n$.

Now these magnitudes are to be determined by resolving the particular equations above-mentioned in art. 107, to wit, the equations

$$\frac{m}{n} = 1, \frac{2n}{m} C,$$
$$C = \frac{2n}{m} C, \frac{3n}{m} D,$$
$$D = \frac{3n}{m} D, \frac{4n}{m} E,$$
$$E = \frac{4n}{m} E, \frac{5n}{m} F,$$
$$F = \frac{5n}{m} F, \frac{6n}{m} G,$$

&c, and

$$C' = \frac{pn + n}{m} C', \frac{pn + 2n}{m} D',$$
$$D' = \frac{pn + 2n}{m} D', \frac{pn + 3n}{m} E',$$
$$E' = \frac{pn + 3n}{m} E', \frac{pn + 4n}{m} F',$$
$$F' = \frac{pn + 4n}{m} F', \frac{pn + 5n}{m} G',$$

&c, which, if the several terms of them have the proper signs $+$ and $-$ prefixed to them, will (according to what has been demonstrated in the four last articles, 107, 108, 109, 110) be as follows, to wit,

$$\frac{m}{n} = 1 + \frac{2n}{m} C,$$
$$+ C = + \frac{2n}{m} C + \frac{3n}{m} D,$$
$$+ D = + \frac{3n}{m} D + \frac{4n}{m} E,$$
$$+ E = + \frac{4n}{m} E + \frac{5n}{m} F,$$
$$+ F = + \frac{5n}{m} F + \frac{6n}{m} G,$$

&c, and

$$- C' = - \frac{pn + n}{m} C' + \frac{pn + 2n}{m} D',$$
$$+ D' = + \frac{pn + 2n}{m} D' - \frac{pn + 3n}{m} E',$$

$- E' =$

$$-\ E' = -\overline{\frac{pn+3n}{m}}\ E' + \frac{pn+4n}{m}\ F',$$

$$+\ F' = \frac{pn+4n}{m}\ F' - \overline{\frac{pn+5n}{m}}\ G',$$

&c; and theſe equations may be reſolved in the manner following:

112. To reſolve the firſt equation, $\frac{m}{n} = 1 + \frac{2n}{m}$ C, ſubtract 1 from both ſides; and we ſhall have $\frac{2n}{m}$ C $= \frac{m}{n} - 1 = \frac{m-n}{n}$, and conſequently (multiplying both ſides by $\frac{m}{2n}$) C $= \frac{m}{2n} \times \frac{m-n}{n} = \frac{m}{n} \times \frac{m-n}{2n}$. Q. E. I.

To reſolve the ſecond equation, $+$ C $= + \frac{2n}{m}$ C $+ \frac{3n}{m}$ D, ſubtract $\frac{2n}{m}$ C from C; and we ſhall have $\frac{3n}{m}$ D $=$ C $- \frac{2n}{m}$ C $= \overline{1 - \frac{2n}{m}} \times$ C $= \frac{m-2n}{m}$ $\times$ C, and conſequently (multiplying both ſides by $\frac{m}{3n}$) D $= \frac{m-2n}{3n} \times$ C.

Q. E. I.

To reſolve the third equation, $+$ D $= + \frac{3n}{m}$ D $+ \frac{4n}{m}$ E, ſubtract $\frac{3n}{m}$ D from D; and we ſhall have $\frac{4n}{m}$ E $=$ D $- \frac{3n}{m}$ D $= \overline{1 - \frac{3n}{m}} \times$ D $= \frac{m-3n}{m}$ $\times$ D, and conſequently (multiplying both ſides by $\frac{m}{4n}$) E $= \frac{m-3n}{4n} \times$ D.

Q. E. I.

To reſolve the fourth equation, $+$ E $= + \frac{4n}{m}$ E $+ \frac{5n}{m}$ F, ſubtract $\frac{4n}{m}$ E from both ſides; and we ſhall have $\frac{5n}{m}$ F $=$ E $- \frac{4n}{m}$ E $= \overline{1 - \frac{4n}{m}} \times$ E $=$ $\frac{m-4n}{m} \times$ E, and conſequently (multiplying both ſides by $\frac{m}{5n}$) F $= \frac{m-4n}{5n}$ $\times$ E. Q. E. I.

To reſolve the fifth equation, $+$ F $= + \frac{5n}{m}$ F $+ \frac{6n}{m}$ G, ſubtract $\frac{5n}{m}$ F from both ſides; and we ſhall have $\frac{6n}{m}$ G $=$ F $- \frac{5n}{m}$ F $= \overline{1 - \frac{5n}{m}} \times$ F $= \frac{m-5n}{m}$ $\times$ F, and conſequently (multiplying both ſides into $\frac{m}{6n}$) G $= \frac{m-5n}{6n} \times$ F.

Q. E. I.

Thus it appears that, ſo long as the numerators of the ſeveral fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, &c, continue to be leſs than their common denominator m, the ſeveral terms Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, &c, of the ſeries $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4,

Ex^4, Fx^5, Gx^6, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, will be all marked with the sign +, or added to the two first terms $1 + \frac{m}{n}x$, and that the co-efficient C will be $= \frac{m-n}{2n} \times \frac{m}{n}$, or $\frac{m-n}{2n} \times B$, and D will be $= \frac{m-2n}{3n} \times C$, and E will be $= \frac{m-3n}{4n} \times D$, and F will be $= \frac{m-4n}{5n} \times E$, and G will be $= \frac{m-5n}{6n} \times F$; which co-efficients are derived from $\frac{m}{n}$, or B (the co-efficient of the second term $\frac{m}{n}x$, or Bx), by the continual multiplication of the generating fractions $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, $\frac{m-4n}{5n}$, $\frac{m-5n}{6n}$, &c, of which the numerators decrease by the continual subtraction of n, and the denominators increase by the continual addition of the same quantity.

113. And, when the numerators of the fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c, are greater than the common denominator m, and the equations to be resolved are consequently

$$-C' = -\overline{\frac{pn+n}{m}}|C' + \frac{pn+2n}{m}D',$$
$$+D' = \frac{pn+2n}{m}D' - \overline{\frac{pn+3n}{m}}|E',$$
$$-E' = -\overline{\frac{pn+3n}{m}}|E' + \frac{pn+4n}{m}F',$$
$$+F' = +\frac{pn+4n}{m}F' - \overline{\frac{pn+5n}{m}}|G',$$
&c,

these equations may be resolved in the manner following:

To resolve the first equation, $-C' = -\overline{\frac{pn+n}{m}}|C' + \frac{pn+2n}{m}D'$, let $\frac{pn+n}{m}C'$ be added to both sides; and we shall have $\frac{pn+2n}{m}D' = \frac{pn+n}{m}C' - C' = \overline{\frac{pn+n}{m} - 1}| \times C' = \frac{pn+n-m}{m} \times C'$, and consequently (multiplying both sides by $\frac{m}{pn+2n}$) $D' = \frac{pn+n-m}{pn+2n} \times C'$. Q. E. I.

To resolve the second equation, $+D' = \frac{pn+2n}{m}D' - \overline{\frac{pn+3n}{m}}|E'$, let $\frac{pn+3n}{m}E'$ be added to both sides; and we shall have $D' + \frac{pn+3n}{m}E' = \frac{pn+2n}{m}D'$; and (subtracting D' from both sides) $\frac{pn+3n}{m}E' = \frac{pn+2n}{m}D' - D' = \overline{\frac{pn+2n}{m} - 1} \times D' = \frac{pn+2n-m}{m} \times D'$, and (multiplying both sides into the fraction $\frac{m}{pn+3n}$) $E' = \frac{pn+2n-m}{pn+3n} \times D'$. Q. E. I.

To resolve the third equation, $-E' = -\overline{\frac{pn+3n}{m}} \times E' + \frac{pn+4n}{m}F'$, let $\frac{pn+3n}{m}E'$ be added to both sides; and we shall have $\frac{pn+4n}{m}F' = \frac{pn+3n}{m}E' - E' = \overline{\frac{pn+3n}{m} - 1} \times E' = \frac{pn+3n-m}{m} \times E'$, and consequently (multiplying both sides into the fraction $\frac{m}{pn+4n}$) $F' = \frac{pn+3n-m}{pn+4n} \times E'$. Q. E. I.

And, to resolve the fourth equation, $+F' = +\frac{pn+4n}{m}F' - \overline{\frac{pn+5n}{m}}G'$, let $\frac{pn+5n}{m} \times G'$ be added to both sides; and we shall have $F' + \frac{pn+5n}{m}G' = \frac{pn+4n}{m}F'$, and (subtracting F' from both sides) $\frac{pn+5n}{m}G' = \frac{pn+4n}{m}F' - F'$, $= \overline{\frac{pn+4n}{m} - 1} \times F' = \frac{pn+4n-m}{m} \times F'$, and consequently (multiplying both sides by the fraction $\frac{m}{pn+5n}$) $G' = \frac{pn+4n-m}{pn+5n} \times F'$. Q. E. I.

It therefore appears that, when we are come to the equation $-C' = -\overline{\frac{pn+n}{m}}C' + \frac{pn+2n}{m}D'$, in which the first term on the right-hand side of the equation, to wit, $\frac{pn+n}{m}C'$, is greater than the single term on the left-hand side of the same equation, to wit, C', the several terms, D', E', F', G', &c, will be generated, or derived, from C' by the continual multiplication of the fractions $\frac{pn+n-m}{pn+2n}$, $\frac{pn+2n-m}{pn+3n}$, $\frac{pn+3n-m}{pn+4n}$, $\frac{pn+4n-m}{pn+5n}$, &c, of which both the numerators and denominators increase by the continual addition of n; whereas, before we came to the term $\frac{pn+n-m}{pn+2n} \times C'$, the numerators of the generating fractions $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, $\frac{m-4n}{5n}$, $\frac{m-5n}{6n}$, &c, decreased by the continual subtraction of n, while their denominators increased by the continual addition of it.

114. The numerators of the first set of generating fractions, $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, $\frac{m-4n}{5n}$, $\frac{m-5n}{6n}$, &c, till we come to a multiple of n that is greater than m, are the excesses of m above the several successive multiples of n, taken in their natural order, to wit, n, $2n$, $3n$, $4n$, $5n$, &c; and the numerators of the second set of generating fractions, $\frac{pn+n-m}{pn+2n}$, $\frac{pn+2n-m}{pn+3n}$, $\frac{pn+3n-m}{pn+4n}$, $\frac{pn+4n-m}{pn+5n}$, &c, after we are come to $\overline{p+1} \times n$, or $pn+n$, or the first multiple of n that is greater than m, are the excesses of the several following multiples of n, to wit, $\overline{p+1} \times n$, $\overline{p+2} \times n$, $\overline{p+3} \times n$, $\overline{p+4} \times n$, &c, or $pn+n$, $pn+2n$, $pn+3n$, $pn+4n$, &c, taken in their natural order, above m: so that in both sets of generating fractions the numerators of the said fractions are the differences of m from the several successive multiples of n, taken in their natural order, to

 wit,

wit, $2n$, $3n$, $4n$, $5n$, $6n$, &c, and pn, $\overline{p+1}|\times n$, $\overline{p+2}|\times n$, $\overline{p+3}|\times n$, $\overline{p+4}|\times n$, $\overline{p+5}|\times n$, &c, or $2n$, $3n$, $4n$, $5n$, $6n$, &c, and pn, $pn+n$, $pn+2n$, $pn+3n$, $pn+4n$, $pn+5n$, &c, *ad infinitum*.

115. It therefore has now been demonſtrated in art. 105, 106, &c .... 114, that, when m is greater than $2n$, and pn is the greateſt multiple of n that is leſs than m, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1+x$, will be equal to the ſeries $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 + \frac{m-5n}{6n}Fx^6 + \&c - \frac{pn+n-m}{pn+2n}C'x^{p+2} + \frac{pn+2n-m}{pn+3n}D'x^{p+3} - \frac{pn+3n-m}{pn+4n}E'x^{p+4} + \frac{pn+4n-m}{pn+5n}F'x^{p+5} - \frac{pn+5n-m}{pn+6n}G'x^{p+6} + \&c$, *ad infinitum*; in which ſeries all the terms after the firſt term 1 are to be added to the ſaid firſt term till we come to the term $\frac{pn+n-m}{pn+2n}C'x^{p+2}$, which is to be ſubtracted from the ſaid firſt term; and all the terms after the ſaid term $\frac{pn+n-m}{pn+2n}C'x^{p+2}$ are to be added to and ſubtracted from the ſaid firſt term 1 alternately. All which is agreeable to what was aſſerted above in the beginning of this diſcourſe, art. 5 and 6, concerning the ſeries that was equal to $\overline{1+x}|^{\frac{m}{n}}$ upon theſe ſuppoſitions of the relative magnitudes of m and n.

116. We have therefore now compleated the demonſtration of Sir Iſaac Newton's famous binomial theorem, in all the caſes of fractional powers whatſoever; to wit, 1ſt, in the caſe of the roots of a binomial quantity $1+x$, in art. 21, 22, 23, 24, 25, &c ...... 51, where we inveſtigated the terms of an infinite ſeries that would exhibit the value of $\sqrt[n]{1+x}$, or $\overline{1+x}|^{\frac{1}{n}}$, when n, the index of the root, was equal to any whole number whatſoever; and, 2dly, in the firſt caſe of the powers of the roots of a binomial quantity, in art. 54, 55, 56, 57, 58, 59, ... 79, where we inveſtigated the terms of an infinite ſeries that would exhibit the value of $\overline{1+x}|^{\frac{m}{n}}$, or of the mth power of the nth root of the binomial quantity $1+x$, when n, the index of the root, was equal to any whole number whatſoever, and m, the index of the power, was equal to any other whole number leſs than n; and 3dly, in the ſecond caſe of the powers of the roots of a binomial quantity, in art. 80, 81, 82, 83, ... 104, where we inveſtigated the terms of an infinite ſeries that would exhibit the value of $\overline{1+x}|^{\frac{m}{n}}$, or of the mth power of the nth root of the binomial quantity $1+x$, when n, the index of the root, was any whole number whatſoever, and m, the index of the power, was any other whole number greater than n, but leſs than $2n$; and, 4thly, in the laſt caſe of the powers of the roots of a binomial quantity in art. 105, 106,

 107,

107, 108, &c . . . . . 115, where we investigated the terms of an infinite series that would exhibit the value of $\overline{1+x}\big|^{\frac{m}{n}}$, or of the *m*th power of the *n*th root of the binomial quantity $1 + x$, when n, the index of the root, was any whole number whatsoever, and m, the index of the power, was any other whole number whatsoever greater than $2n$. Here therefore we might with propriety put an end to this discourse; but as the last case above-mentioned of this theorem, in which m is greater than $2n$, is attended with rather more difficulty than the former cases of it, on account of the different numbers of terms of the series $1 + \frac{m}{n}x$, Cx^2, Dx^3, Ex^4, Fx^5, &c, after the two first terms $1 + \frac{m}{n}x$, which are to be marked with the sign $+$, or added to the said two first terms, according to the different magnitudes of m with respect to n, I shall now proceed to lay before the reader another demonstration of this last case of the said theorem (in which m is greater than $2n$), and likewise of the next preceeding case of it (in which m is greater than n, but less than $2n$); which is grounded on the supposition that the said theorem is true in the first case of the quantity $\overline{1+x}\big|^{\frac{m}{n}}$, or when m, the numerator of the index $\frac{m}{n}$, is less than its denominator n.

Another demonstration of the Binomial Theorem in the case of the quantity $\overline{1+x}\big|^{\frac{m}{n}}$, *or the* m*th power of the* n*th root of the binomial quantity* $1 + x$, *when* m, *the index of the power, is greater than* n, *the index of the root; deduced from the series that is equal to* $\overline{1+x}\big|^{\frac{m}{n}}$, *when* m *is less than* n.

117. It has been shewn above, in a pretty full, and, I hope, satisfactory manner (in art. 54, 55, 56, 57, 58, 59, &c . . . . . 79), that when m, the numerator of the index $\frac{m}{n}$, is less than n, its denominator, the quantity $\overline{1+x}\big|^{\frac{m}{n}}$, or the *m*th power of the *n*th root of the binomial quantity $1 + x$, will be equal to the series $1 + \frac{m}{n}Ax - \overline{\frac{n-m}{2n}}Bx^2 + \frac{2n-m}{3n}Cx^3 - \overline{\frac{3n-m}{4n}}Dx^4 + \frac{4n-m}{5n}Ex^5 -$ &c, *ad infinitum* in which all the terms after the two first terms $1 + \frac{m}{n}Ax$ are marked alternately with the signs $-$ and $+$, or are to be alternately subtracted from, and added to, the said two first terms. Now from this proposition we may deduce a proof that, when m is greater than n, but less than $2n$, the quantity

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tity $\overline{1+x}|^{\frac{m}{n}}$ will be equal to the ſeries above found for it in art. 80, 81, 82, 83, &c, ... 104; and that, when m is of any magnitude greater than $2n$, the quantity $\overline{1+x}|^{\frac{m}{n}}$ will be equal to the ſeries found for it above in art. 105, 106, 107, 108, &c . . . . . 115. This may be done in the manner following:

118. In the firſt place let us ſuppoſe the index $\frac{m}{n}$ of the power to which the binomial quantity $1+x$ is to be raiſed, to be $\frac{m}{n}+1$, or $\frac{m+n}{n}$, inſtead of $\frac{m}{n}$; and let p be put $= m+n$. Then will $\overline{1+x}|^{\frac{m}{n}+1}$, or $\overline{1+x}|^{\frac{m+n}{n}}$, be $= \overline{1+x}|^{\frac{p}{n}}$; in which quantity p, the numerator of the index $\frac{p}{n}$, will be greater than its denominator n, but leſs than $2n$. For $m+n$ (to which p is equal) is greater than n, but (becauſe m alone is ſuppoſed to be leſs than n) leſs than $2n$.

We muſt therefore ſhew that $\overline{1+x}|^{\frac{p}{n}}$ will, upon theſe ſuppoſitions, be equal to the ſeries $1 + \frac{p}{n}Ax + \frac{p-n}{2n}Bx^2 - \frac{2n-p}{3n}Cx^3 + \frac{3n-p}{4n}Dx^4 - \frac{4n-p}{5n}Ex^5$ + &c, *ad infinitum*, in which the third term $\frac{p-n}{2n}Bx^2$ is marked with the ſign +, or is added to the two firſt terms $1+\frac{m}{n}Ax$; and all the following terms are marked with the ſigns − and + alternately, or are to be alternately ſubtracted from, and added to, the ſaid two firſt terms, agreeably to art. 104.

119. Now $\overline{1+x}|^{\frac{p}{n}}$ is $= \overline{1+x}|^{\frac{m+n}{n}} = \overline{1+x}|^{\frac{m}{n}+1} = \overline{1+x}|^{\frac{m}{n}} \times \overline{1+x}|^{1} = \overline{1+x}|^{\frac{m}{n}} \times \overline{1+x}$.

But, by the ſuppoſition $\overline{1+x}|^{\frac{m}{n}}$ is = the ſeries $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5$ − &c, as is ſhewn in art. 54, 55, 56, 57, 58, 59, &c . . . . . . 79; or, if we denote the ſeveral co-efficients of x, x^2, x^3, x^4, x^5, &c, in this ſeries, by the capital letters B, C, D, E, F, &c, reſpectively, $\overline{1+x}|^{\frac{m}{n}}$ will be = the ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5$ − &c.

Therefore $\overline{1+x}|^{\frac{p}{n}}$ (which is $= \overline{1+x}|^{\frac{m}{n}} \times \overline{1+x}$) will be = the ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5$ − &c $\times \overline{1+x}$ = the compound ſeries

1 +

$$1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$$
$$+ \quad x + Bx^2 - Cx^3 + Dx^4 - Ex^5 + \&c.$$

But, because B is greater than C, and C is greater than D, and D than E, and E than F, and every following co-efficient of a power of x in the second line is greater than the co-efficient of the same power of x in the first line, it is evident that this compound series will be equal to the series $1 + \overline{1 + B} \times x + \overline{B - C} \times x^2 - \overline{C - D} \times x^3 + \overline{D - E} \times x^4 - \overline{E - F} \times x^5 +$ &c; in which series the third term $\overline{B - C} \times x^2$ is added to the two first terms $1 + \overline{1 + B} \times x$, and the several following terms $\overline{C - D} \times x^3$, $\overline{D - E} \times x^4$, $\overline{E - F} \times x^5$, &c, are marked with the signs $-$ and $+$ alternately, or are alternately subtracted from, and added to, the said two first terms.

Therefore $\overline{1 + x}|^{\frac{p}{n}}$ will be equal to the said series $1 + \overline{1 + B} \times x + \overline{B - C} \times x^2 - \overline{C - D} \times x^3 + \overline{D - E} \times x^4 - \overline{E - F} \times x^5 +$ &c, in which the third term $\overline{B - C} \times x^2$ is added to the two first terms $1 + \overline{1 + B} \times x$, and the following terms are alternately subtracted from, and added to, the said two first terms; which is one of the properties that ought to belong to the series that is equal to $\overline{1 + x}|^{\frac{p}{n}}$, according to art. 102.

120. It remains that we shew that the co-efficients $1 + B$, $B - C$, $C - D$, $D - E$, and $E - F$, &c, of the powers of x in the said series $1 + \overline{1 + B} \times x + \overline{B - C} \times x^2 - \overline{C - D} \times x^3 + \overline{D - E} \times x^4 - \overline{E - F} \times x^5$ &c (which is equal to $\overline{1 + x}|^{\frac{p}{n}}$) are equal to $\frac{p}{n}$, $\frac{p}{n} \times \frac{p-n}{2n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}$ and $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}$, &c, respectively; agreeably to what is shewn in art. 104. This may be done in the manner following:

In the first place it is evident that $1 + B$ (being $= 1 + \frac{m}{n}$) will be $= \frac{n+m}{n} = \frac{p}{n}$. Q. E. D.

Secondly, since C is $= B \times \frac{n-m}{2n}$, it follows that $B - C$ will be $= B - B \times \frac{n-m}{2n} = B \times \frac{2n}{2n} - B \times \frac{n-m}{2n} = B \times \overline{\frac{2n}{2n} - \frac{n-m}{2n}} = B \times \frac{2n-n+m}{2n} = B \times \frac{n+m}{2n} = B \times \frac{p}{2n}$.

Thirdly, since D is $= C \times \frac{2n-m}{3n}$, we shall have $C - D = C - C \times \frac{2n-m}{3n} = C \times \frac{3n}{3n} - C \times \frac{2n-m}{3n} = C \times \overline{\frac{3n}{3n} - \frac{2n-m}{3n}} = C \times \frac{3n-2n+m}{3n} = C \times \frac{n+m}{3n} = C \times \frac{p}{3n}$.

Fourthly,

Fourthly, since E is $= D \times \frac{3n-m}{4n}$, we shall have $D - E = D - D \times \frac{3n-m}{4n} = D \times \frac{4n}{4n} - D \times \frac{3n-m}{4n} = D \times \overline{\frac{4n}{4n} - \overline{\frac{3n-m}{4n}}} = D \times \frac{4n-3n+m}{4n} = D \times \frac{n+m}{4n} = D \times \frac{p}{4n}$.

And, fifthly, since F is $= E \times \frac{4n-m}{5n}$, we shall have $E - F = E - E \times \frac{4n-m}{5n} = E \times \frac{5n}{5n} - E \times \frac{4n-m}{5n} = E \times \overline{\frac{5n}{5n} - \overline{\frac{4n-m}{5n}}} = E \times \frac{5n-4n+m}{5n} = E \times \frac{n+m}{5n} = E \times \frac{p}{5n}$.

And it is easy to see that, if we were to proceed in the same manner to examine the following differences, $F - G$, $G - H$, $H - I$, $I - K$, $K - L$, &c, continued to any number of terms, we should find them to be respectively equal to $F \times \frac{p}{6n}$, $G \times \frac{p}{7n}$, $H \times \frac{p}{8n}$, $I \times \frac{p}{9n}$, $K \times \frac{p}{10n}$, &c, in which quantities the denominators of the several fractions $\frac{p}{6n}$, $\frac{p}{7n}$, $\frac{p}{8n}$, $\frac{p}{9n}$, $\frac{p}{10n}$, &c, of which p is the numerator, increase continually by the addition of n.

It follows therefore that the series $1 + \overline{1 + B}| \times x + \overline{B - C}| \times x^2 - \overline{C - D} \times x^3 + \overline{D - E}| \times x^4 - \overline{E - F} \times x^5 +$ &c, which is $= \overline{1 + x}|^{\frac{p}{n}}$, will be $= 1 + \frac{p}{n}x + B \times \frac{p}{2n}x^2 - C \times \frac{p}{3n}x^3 + D \times \frac{p}{4n}x^4 - E \times \frac{p}{5n}x^5 +$ &c; and consequently this last series will be $= \overline{1 + x}|^{\frac{p}{n}}$. We must therefore shew that the several co-efficients $B \times \frac{p}{2n}$, $C \times \frac{p}{3n}$, $D \times \frac{p}{4n}$, and $E \times \frac{p}{5n}$, &c, of the powers of x in this last series are equal, respectively, to $\frac{p}{n} \times \frac{p-n}{2n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}$, and $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}$, &c. This may be proved in the manner following:

121. Since p is $= n + m$, we shall have $2n - p$ $(= 2n - n - m) = n - m$, and consequently (adding n continually to both sides)

$$3n - p = 2n - m,$$
and $4n - p = 3n - m$,
and $5n - p = 4n - m$,
and $6n - p = 5n - m$,
and $7n - p = 6n - m$,
and $8n - p = 7n - m$,
and so on *ad infinitum*.

We shall therefore have

$$B \left(= \frac{m}{n}\right) = \frac{p-n}{n},$$

and

$$\text{and } C\ (= \frac{m}{n} \times \frac{n-m}{2n}) = \frac{p-n}{n} \times \frac{2n-p}{2n},$$

$$\text{and } D\ (= \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}) = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n},$$

$$\text{and } E\ (= \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}) = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{4n-p}{4n},$$

$$\text{and } F\ (= \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}) = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{4n-p}{4n} \times \frac{5n-p}{5n},$$

and so on *ad infinitum*, every new capital letter being equal to the next preceeding capital letter multiplied into a new generating fraction (as $\frac{6n-p}{6n}$, $\frac{7n-p}{7n}$, $\frac{8n-p}{8n}$, $\frac{9n-p}{9n}$, $\frac{10n-p}{10n}$, &c), which is derived from the generating fraction next before it, by adding n to both its numerator and its denominator.

It follows therefore that the co-efficients $B \times \frac{p}{2n}$, $C \times \frac{p}{3n}$, $D \times \frac{p}{4n}$, $E \times \frac{p}{5n}$, &c, of x^2, x^3, x^4, x^5, and the following powers of x, in the third, fourth, fifth, sixth, and other following terms of the series $1 + \frac{p}{n}x + B \times \frac{p}{2n}x^2 - C \times \frac{p}{3n}x^3 + D \times \frac{p}{4n}x^4 - E \times \frac{p}{5n}x^5 + \&c$ (which has been shewn to be $= \overline{1+x}|^{\frac{p}{n}}$) will be equal to the following quantities; to wit,

$$B \times \frac{p}{2n} \text{ will be } = \frac{p-n}{n} \times \frac{p}{2n} = \frac{p}{n} \times \frac{p-n}{2n};$$

$$\text{and } C \times \frac{p}{3n} \text{ will be } = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{p}{3n} = \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n};$$

$$\text{and } D \times \frac{p}{4n} \text{ will be } = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{p}{4n} = \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n};$$

$$\text{and } E \times \frac{p}{5n} \text{ will be } = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{4n-p}{4n} \times \frac{p}{5n} = \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}.$$

And it is easy to see that, if we were to compute the values of the following co-efficients $F \times \frac{p}{6n}$, $G \times \frac{p}{7n}$, $H \times \frac{p}{8n}$, $I \times \frac{p}{9n}$, $K \times \frac{p}{10n}$, &c, in the same manner by substituting in them, instead of the letters F, G, H, I, K, &c, the respective values of those letters, they would be such as would follow from the value of the last co-efficient $E \times \frac{p}{5n}$, by multiplying it continually into the fractions $\frac{5n-p}{6n}$, $\frac{6n-p}{7n}$, $\frac{7n-p}{8n}$, $\frac{8n-p}{9n}$, $\frac{9n-p}{10n}$, &c, which are formed from each other by the continual addition of n to both their numerators and their denominators.

Therefore the series $1 + \frac{p}{n}x + B \times \frac{p}{2n}x^2 - C \times \frac{p}{3n}x^3 + D \times \frac{p}{4n}x^4 - E \times \frac{p}{5n}x^5 + \&c$, which has been shewn to be equal to $\overline{1+x}|^{\frac{p}{n}}$, will be

$= 1 +$

$= 1 + \frac{p}{n}x + \frac{p}{n} \times \frac{p-n}{2n}x^2 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}x^3 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}x^4 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}x^5 +$ &c, and consequently the quantity $\overline{1+x}|^{\frac{p}{n}}$ is = the series $1 + \frac{p}{n}x + \frac{p}{n} \times \frac{p-n}{2n}x^2 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}x^3 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}x^4 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}x^5 +$ &c, *ad infinitum*, agreeably to what was shewn above in art. 104. Q. E. D.

122. Having thus found that, when p is greater than n, but less than $2n$, the quantity $\overline{1+x}|^{\frac{p}{n}}$ is equal to the series $1 + \frac{p}{n}x + \frac{p}{n} \times \frac{p-n}{2n}x^2 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}x^3 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}x^4 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}x^5 +$ &c, *ad infinitum*, in which the third term $\frac{p}{n} \times \frac{p-n}{2n}x^2$ is marked with the sign +, or added to the two first terms, and all the following terms are marked with the signs − and + alternately, or are alternately subtracted from, and added to, the said two first terms, we must proceed to consider the remaining cases of the quantity $\overline{1+x}|^{\frac{m}{n}}$, in which m, the index of the power to which the nth root of the binomial quantity $1 + x$ is to be raised, is greater than $2n$, or twice the index of the said root. And, in order to do this with the more distinctness, I shall use the letter q, instead of the letter m, to denote the index of the said power, and shall continue to suppose m to be less than n (as it has been supposed to be in the course of the five last articles, 117, 118, 119, 120, and 121), and p to be $= m + n$, as before, and consequently $\frac{p}{n}$ to be $= \frac{m+n}{n}$, or $\frac{m}{n} + 1$, and shall suppose the new index q to be equal to $p + n$, or $m + n + n$, or $m + 2n$, and consequently $\frac{q}{n}$ to be $= \frac{p+n}{n}$, or $\frac{p}{n} + 1$; and, upon these suppositions, shall proceed to shew that $\overline{1+x}|^{\frac{q}{n}}$ will be = the series $1 + \frac{q}{n}x + \frac{q}{n} \times \frac{q-n}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}x^5 -$ &c, in which the fourth term, as well as the third, is marked with the sign +, or is added to the first term, agreeably to what is shewn above in art. 115, and asserted in the beginning of this discourse in art. 5 and 6.

123. Now, upon these suppositions, the quantity $\overline{1+x}|^{\frac{q}{n}}$, or the qth power of the nth root of the binomial quantity $1 + x$, will be $= \overline{1+x}|^{\frac{p+n}{n}} =$

$\overline{1+x}|^{\frac{p}{n}+1}$

$\overline{1+x}\big|^{\frac{p}{n}+1} = \overline{1+x}\big|^{\frac{p}{n}} \times \overline{1+x}\big|^{1} = \overline{1+x}\big|^{\frac{p}{n}} \times \overline{1+x} =$ the ſeries $1 + \frac{p}{n}x + \frac{p}{n} \times \frac{p-n}{2n}x^2 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}x^3 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} x^4 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}x^5 + \&c \times \overline{1+x} =$ (if the co-efficients of x, x^2, x^3, x^4, x^5, &c, in the terms of this ſeries, be denoted by B′, C′, D′, E′, F′, &c, or the capital letters B, C, D, E, F, &c, with an accent placed near them at the top) the ſeries $1 + B'x + C'x^2 - D'x^3 + E'x^4 - F'x^5 - \&c \times \overline{1+x} =$ the compound ſeries

$$\begin{array}{l} 1 + B'x + C'x^2 - D'x^3 + E'x^4 - F'x^5 + \&c \\ \quad + x + B'x^2 + C'x^3 - D'x^4 + E'x^5 - \&c. \end{array}$$

But, becauſe C′ is greater than D′, and D′ is greater than E′, and E′ is greater than F′, and every following co-efficient of a power of x in the lower line of this compound ſeries is greater than the co-efficient of the ſame power of x in the upper line of the ſame, it is evident that the ſaid compound ſeries will be equal to the ſeries $1 + \overline{1+B'}| \times x + \overline{B'+C'}| \times x^2 + \overline{C'-D'}| \times x^3 - \overline{D'-E'}| \times x^4 + \overline{E'-F'}| \times x^5 - \&c.$ Therefore the quantity $\overline{1+x}\big|^{\frac{q}{n}}$ will be equal to the ſaid ſeries $1 + \overline{1+B'}| \times x + \overline{B'+C'}| \times x^2 + \overline{C'-D'}| \times x^3 - \overline{D'-E'}| \times x^4 + \overline{E'-F'}| \times x^5 - \&c$; in which ſeries the fourth term $\overline{C'-D'}| \times x^3$, as well as the third term $\overline{B'+C'}| \times x^2$, is marked with the ſign +, or is to be added to the two firſt terms $1 + \overline{1+B'}| \times x$; and all the following terms are marked with the ſigns − and + alternately, or are to be alternately ſubtracted from, and added to, the ſaid firſt terms; which is one of the properties of the ſeries $1 + \frac{q}{n}x + \frac{q}{n} \times \frac{q-n}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}x^5 - \&c$, to which we are now to ſhew the ſaid quantity $\overline{1+x}\big|^{\frac{q}{n}}$ to be equal.

124. It remains that we ſhew that the co-efficients $1 + B'$, $B' + C'$, $C' - D'$, $D' - E'$, $E' - F'$, &c, of the ſeveral powers of x in the ſeries $1 + \overline{1+B'}| \times x + \overline{B'+C'}| \times x^2 + \overline{C'-D'}| \times x^3 - \overline{D'-E'}| \times x^4 + \overline{E'-F'}| \times x^5 - \&c$ (which has been ſhewn to be equal to $\overline{1+x}\big|^{\frac{q}{n}}$), are reſpectively equal to the co-efficients $\frac{q}{n}$, $\frac{q}{n} \times \frac{q-n}{2n}$, $\frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}$, $\frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}$, and $\frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}$, &c, of the ſame powers of x in the ſeries $1 + \frac{q}{n}x + \frac{q}{n} \times \frac{q-n}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}x^5 - \&c$, to which we are

are now to prove the quantity $\overline{1+x}|^{\frac{q}{n}}$ to be equal. For then it will follow that this laſt ſeries being thus ſhewn to be equal to the foregoing ſeries $1 + \overline{1+B'}| \times x + \overline{B'+C'}| \times x^2 + \overline{C'-D'}| \times x^3 - \overline{D'-E'}| \times x^4 + \overline{E'-F'}| \times x^5 -$ &c, which has been ſhewn to be equal to $\overline{1+x}|^{\frac{q}{n}}$, will alſo be equal to $\overline{1+x}|^{\frac{q}{n}}$. We muſt therefore endeavour to ſhew that

$$1 + B' \text{ will be } = \frac{q}{n},$$

$$\text{and } B' + C' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n},$$

$$\text{and } C' - D' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n},$$

$$\text{and } D' - E' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n},$$

$$\text{and } E' - F' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n},$$

and that the following co-efficients $F'-G'$, $G'-H'$, $H'-I'$, $I'-K'$, $K'-L'$, &c, of the powers of x in the former ſeries will, in like manner, be equal, reſpectively, to the following co-efficients of the ſame powers of x in the latter ſeries. This may be done in the manner following.

125. Since $p + n$ is $= q$, we ſhall have

$$p = q - n,$$

$$\text{and } p - n \ (= q - n - n) = q - 2n,$$

$$\text{and } 2n - p \ (= 2n - \overline{q-n} = 2n - q + n) = 3n - q,$$

and (adding n continually to both ſides)

$$\begin{aligned} 3n - p &= 4n - q, \\ 4n - p &= 5n - q, \\ 5n - p &= 6n - q, \\ 6n - p &= 7n - q, \\ 7n - p &= 8n - q, \\ 8n - p &= 9n - q; \end{aligned}$$

and ſo on *ad infinitum*.

Therefore

$$B' \text{ will be } (= \tfrac{p}{n}) = \frac{q-n}{n},$$

$$\text{and } C' \text{ will be } (= \frac{p}{n} \times \frac{p-n}{2n} = \frac{q-n}{n} \times \frac{q-2n}{2n}) = B' \times \frac{q-2n}{2n},$$

$$\text{and } D' \text{ will be } (= \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} = C' \times \frac{2n-p}{3n}) = C' \times \frac{3n-q}{3n},$$

$$\text{and } E' \ (= \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} = D' \times \frac{3n-p}{4n}) = D' \times \frac{4n-q}{4n},$$

$$\text{and } F' \ (= \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n} = E' \times \frac{4n-p}{5n}) = E' \times \frac{5n-q}{5n},$$

And, in like manner,

$$G' \text{ will be } (= F' \times \frac{5n-p}{6n}) = F' \times \frac{6n-q}{6n},$$

$$\text{and } H' \text{ will be } (= G' \times \frac{6n-p}{7n}) = G' \times \frac{7n-q}{7n},$$

and

and I′ will be $(= H' \times \frac{7n-p}{8n}) = H' \times \frac{8n-q}{8n}$,

and K′ will be $(= I' \times \frac{8n-p}{9n}) = I' \times \frac{9n-q}{9n}$;

and so on *ad infinitum*.

Therefore the co-efficient of x, to wit,

$1 + B'$, will be $(= 1 + \frac{q-n}{n} = \frac{n}{n} + \frac{q-n}{n} = \frac{n+q-n}{n}) = \frac{q}{n}$;

and $B' + C'$ will be $(= B' + B' \times \frac{q-2n}{2n} = B' \times \frac{2n}{2n} + B' \times \frac{q-2n}{2n} = B' \times \frac{2n+q-2n}{2n}) = B' \times \frac{q}{2n}$;

and $C' - D'$ will be $(= C' - C' \times \frac{3n-q}{3n} = C' \times \frac{3n}{3n} - C' \times \frac{3n-q}{3n} = C' \times \overline{\frac{3n}{3n} - \overline{\frac{3n-q}{3n}}|}| = C' \times \frac{3n - \overline{3n-q}|}{3n} = C' \times \frac{3n-3n+q}{3n}) = C' \times \frac{q}{3n}$;

and $D' - E'$ will be $(= D' - D' \times \frac{4n-q}{4n} = D' \times \frac{4n}{4n} - D' \times \frac{4n-q}{4n} = D' \times \overline{\frac{4n}{4n} - \overline{\frac{4n-q}{4n}}|}| = D' \times \frac{4n - \overline{4n-q}|}{4n} = D' \times \frac{4n-4n+q}{4n}) = D' \times \frac{q}{4n}$;

and $E' - F'$ will be $(= E' - E' \times \frac{5n-q}{5n} = E' \times \frac{5n}{5n} - E' \times \frac{5n-q}{5n} = E' \times \overline{\frac{5n}{5n} - \overline{\frac{5n-q}{5n}}|}| = E' \times \frac{5n - \overline{5n-q}|}{5n} = E' \times \frac{5n-5n+q}{5n}) = E' \times \frac{q}{5n}$;

and, in like manner,

$F' - G'$ will be $= F' \times \frac{q}{6n}$,

and $G' - H'$ will be $= G' \times \frac{q}{7n}$,

and $H' - I'$ will be $= H' \times \frac{q}{8n}$,

and $I' - K'$ will be $= I' \times \frac{q}{9n}$;

and so on *ad infinitum*.

Therefore the series $1 + \overline{1 + B'}| \times x + \overline{B' + C'}| \times x^2 + \overline{C' - D'}| \times x^3 - \overline{D' - E'}| \times x^4 + \overline{E' - F'}| \times x^5 -$ &c (which has been shewn to be equal to $\overline{1 + x}|^{\frac{q}{n}}$) will be $= 1 + \frac{q}{n}x + B' \times \frac{q}{2n}x^2 + C' \times \frac{q}{3n}x^3 - D' \times \frac{q}{4n}x^4 + E' \times \frac{q}{5n} \times x^5 -$ &c $= 1 + \frac{q}{n}x + \frac{q-n}{n} \times \frac{q}{2n}x^2 + \frac{q-n}{n} \times \frac{q-2n}{2n} \times \frac{q}{3n}x^3 - \overline{\frac{q-n}{n} \times \frac{q-2n}{2n} \times \frac{3n-q}{3n} \times \frac{q}{4n}x^4}| + \frac{q-n}{n} \times \frac{q-2n}{2n} \times \frac{3n-q}{3n} \times \frac{4n-q}{4n} \times \frac{q}{5n}x^5 -$ &c $= 1 + \frac{q}{n}x + \frac{q}{n} \times \frac{q-n}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}x^5 -$ &c. Therefore the quantity $\overline{1 + x}|^{\frac{q}{n}}$, or the qth, or $\overline{p + n}|$th, or $\overline{m + 2n}|$th, power

power of the nth root of the binomial quantity $1 + x$, will be equal to the series $1 + \frac{q}{n}x + \frac{q}{n} \times \frac{q-n}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}x^5 - \&c$, or (if we denote the co-efficients of x, x^2, x^3, x^4, x^5, &c, in this series, by B'', C'', D'', E'', F'', &c, or the capital letters B, C, D, E, F, &c, with two accents placed near them at the top) to the series $1 + \frac{q}{n}x + \frac{q-n}{2n}B''x^2 + \frac{q-2n}{3n}C''x^3 - \overline{\frac{3n-q}{4n}}\,D''x^4 + \frac{4n-q}{5n}E''x^5 - \&c$, *ad infinitum*; in which the fourth term $\frac{q-2n}{3n}C''x^3$, as well as the third term $\frac{q-n}{2n}B''x^2$, is marked with the sign $+$, or is to be added to the first term 1, and all the following terms $\frac{3n-q}{4n}D''x^4$, $\frac{4n-q}{5n}E''x^5$, &c, are marked with the signs $-$ and $+$ alternately; or are to be alternately subtracted from, and added to, the said first term, agreeably to what is shewn above in art. 115, and asserted in the beginning of this discourse in art. 5 and 6.

Q. E. D.

126. And, in like manner, if r be $= q + n$, or $p + n + n$, or $m + 3n$, we shall have $\overline{1+x}|^{\frac{r}{n}} = \overline{1+x}|^{\frac{q+n}{n}} = \overline{1+x}|^{\frac{q}{n}+1} = \overline{1+x}|^{\frac{q}{n}} \times \overline{1+x}|^1 = \overline{1+x}|^{\frac{q}{n}} \times \overline{1+x} =$ the series $1 + B''x + C''x^2 + D''x^3 - E''x^4 + F''x^5 - \&c$, $\times\ \overline{1+x} =$ the compound series

$$\begin{array}{l} 1 + B''x + C''x^2 + D''x^3 - E''x^4 + F''x^5 - \&c \\ + \quad x + B''x^2 + C''x^3 + D''x^4 - E''x^5 + \&c \end{array}$$

$= 1 + \overline{1+B''}| \times x + \overline{B''+C''}| \times x^2 + \overline{C''+D''}| \times x^3 + \overline{D''-E''}| \times x^4 - \overline{E''-F''}| \times x^5 + \&c = 1 + \overline{1+\frac{q}{n}}| \times x + \overline{B'' + B'' \times \frac{q-n}{2n}}| \times x^2 + \overline{C'' + C'' \times \frac{q-2n}{3n}}| \times x^3 + \overline{D'' - D'' \times \frac{3n-q}{4n}}| \times x^4 - \overline{E'' - E'' \times \frac{4n-q}{5n}}| \times x^5 + \&c = 1 + \frac{n+q}{n}x + B'' \times \frac{2n+q-n}{2n}x^2 + C'' \times \frac{3n+q-2n}{3n}x^3 + D'' \times \frac{4n-\overline{3n-q}}{4n}x^4 - E'' \times \frac{5n-\overline{4n-q}}{5n}x^5 + \&c = 1 + \frac{r}{n}x + B'' \times \frac{n+q}{2n}x^2 + C'' \times \frac{n+q}{3n}x^3 + D'' \times \frac{n+q}{4n}x^4 - E'' \times \frac{n+q}{5n}x^5 + \&c = 1 + \frac{r}{n}x + B'' \times \frac{r}{2n}x^2 + C'' \times \frac{r}{3n}x^3 + D'' \times \frac{r}{4n}x^4 - E'' \times \frac{r}{5n}x^5 + \&c = 1 + \frac{r}{n}x + \frac{q}{n} \times \frac{r}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{r}{3n}x^3 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{r}{4n}x^4 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{r}{5n}x^5 + \&c = 1 + \frac{r}{n}x + \frac{r-n}{n} \times \frac{r}{2n}x^2 + \frac{r-n}{n} \times \frac{r-2n}{2n} \times \frac{r}{3n}x^3 + \frac{r-n}{n} \times \frac{r-2n}{2n} \times \frac{r-3n}{3n} \times \frac{r}{4n}x^4 - \overline{\frac{r-n}{n} \times \frac{r-2n}{2n} \times \frac{r-3n}{3n}}$

$\frac{r-3n}{3n} \times \frac{4n-r}{4n} \times \frac{r}{5n} x^5 + \&c = 1 + \frac{r}{n}x + \frac{r}{n} \times \frac{r-n}{2n} x^2 + \frac{r}{n} \times \frac{r-n}{2n} \times \frac{r-2n}{3n} x^3 + \frac{r}{n} \times \frac{r-n}{2n} \times \frac{r-2n}{3n} \times \frac{r-3n}{4n} x^4 - \frac{r}{n} \times \frac{r-n}{2n} \times \frac{r-2n}{3n} \times \frac{r-3n}{4n} \times \frac{4n-r}{5n} \times x^5 + \&c$, or (if we put B''', C''', D'', E''', F''', &c, or the capital letters B, C, D, E, F, &c, with three accents placed near them at the top, for the co-efficients of x, x^2, x^3, x^4, x^5, and the following powers of x in the terms of this series) $1 + B'''x + C'''x^2 + D''x^3 + E'''x^4 - F'''x^5 + \&c$, or $1 + \frac{r}{n}x + \frac{r-n}{2n}B'''x^2 + \frac{r-2n}{3n}C'''x^3 + \frac{r-3n}{4n}D'''x^4 - \overline{\frac{4n-r}{5n}}\Big|\, E'''x^5 + \&c$. Q. E. D.

127. And, if s be $= r + n$, or $q + 2n$, or $p + 3n$, or $m + 4n$, we shall have $\overline{1+x}\big|^{\frac{s}{n}} = \overline{1+x}\big|^{\frac{r+n}{n}} = \overline{1+x}\big|^{\frac{r}{n}+1} = \overline{1+x}\big|^{\frac{r}{n}} \times \overline{1+x}\big|^{1} = \overline{1+x}\big|^{\frac{r}{n}} \times \overline{1+x} =$ the series $1 + B'''x + C'''x^2 + D''x^3 + E'''x^4 - F'''x^5 + \&c \times \overline{1+x} =$ the compound series

$$\begin{array}{l} 1 + B'''x + C'''x^2 + D''x^3 + E'''x^4 - F'''x^5 + \&c \\ + \quad\; x + B'''x^2 + C'''x^3 + D''x^4 + E'''x^5 - \&c \end{array}$$

$= 1 + \overline{1 + B'''}\big| \times x + \overline{B''' + C'''}\big| \times x^2 + \overline{C''' + D''}\big| \times x^3 + \overline{D'' + E'''}\big| \times x^4 + \overline{E''' - F'''}\big| \times x^5 - \&c = 1 + \overline{1 + \frac{r}{n}}\Big| \times x + \overline{B''' + B''' \times \frac{r-n}{2n}}\Big| \times x^2 + \overline{C''' + C''' \times \frac{r-2n}{3n}}\Big| \times x^3 + \overline{D'' + D'' \times \frac{r-3n}{4n}}\Big| \times x^4 + \overline{E''' - E''' \times \frac{4n-r}{5n}}\Big| \times x^5 - \&c = 1 + \frac{n+r}{n}x + B''' \times \frac{2n+r-n}{2n}x^2 + C''' \times \frac{3n+r-2n}{3n}x^3 + D'' \times \frac{4n+r-3n}{4n}x^4 + E''' \times \frac{5n-4n+r}{5n}x^5 - \&c = 1 + \frac{n+r}{n}x + B''' \times \frac{n+r}{2n}x^2 + C''' \times \frac{n+r}{3n}x^3 + D'' \times \frac{n+r}{4n}x^4 + E''' \times \frac{n+r}{5n}x^5 - \&c = 1 + \frac{s}{n}x + B''' \times \frac{s}{2n}x^2 + C''' \times \frac{s}{3n}x^3 + D'' \times \frac{s}{4n}x^4 + E''' \times \frac{s}{5n}x^5 - \&c = 1 + \frac{s}{n}x + \frac{r}{n} \times \frac{s}{2n}x^2 + \frac{r}{n} \times \frac{r-n}{2n} \times \frac{s}{3n}x^3 + \frac{r}{n} \times \frac{r-n}{2n} \times \frac{r-2n}{3n} \times \frac{s}{4n}x^4 + \frac{r}{n} \times \frac{r-n}{2n} \times \frac{r-2n}{3n} \times \frac{r-3n}{4n} \times \frac{s}{5n}x^5 - \&c = 1 + \frac{s}{n}x + \frac{s-n}{n} \times \frac{s}{2n}x^2 + \frac{s-n}{n} \times \frac{s-2n}{2n} \times \frac{s}{3n}x^3 + \frac{s-n}{n} \times \frac{s-2n}{2n} \times \frac{s-3n}{3n} \times \frac{s}{4n}x^4 + \frac{s-n}{n} \times \frac{s-2n}{2n} \times \frac{s-3n}{3n} \times \frac{s-4n}{4n} \times \frac{s}{5n}x^5 - \&c = 1 + \frac{s}{n}x + \frac{s}{n} \times \frac{s-n}{2n}x^2 + \frac{s}{n} \times \frac{s-n}{2n} \times \frac{s-2n}{3n}x^3 + \frac{s}{n} \times \frac{s-n}{2n} \times \frac{s-2n}{3n} \times \frac{-3n}{4n}x^4 + \frac{s}{n} \times \frac{s-n}{2n} \times \frac{s-2n}{3n} \times \frac{s-3n}{4n} \times \frac{s-4n}{5n}x^5 - \&c$, or (putting B^{iv}, C^{iv}, D^{iv}, E^{iv}, F^{iv}, &c, for the several co-efficients of x, x^2, x^3, x^4, x^5, &c, in the

the terms of this series) $1 + B^{\text{iv}}x + C^{\text{iv}}x^2 + D^{\text{iv}}x^3 + E^{\text{iv}}x^4 + F^{\text{iv}}x^5 -$ &c, or $1 + \frac{s}{n}x + \frac{s-n}{2n}B^{\text{iv}}x^2 + \frac{s-2n}{3n}C^{\text{iv}}x^3 + \frac{s-3n}{4n}D^{\text{iv}}x^4 + \frac{s-4n}{5n}E^{\text{iv}}x^5 -$ &c. Q. E. D.

128. We have now shewn in the course of the foregoing articles, from art. 117 to art. 127, how from the series $1 + \frac{m}{n}Ax - \overline{\left|\frac{n-m}{2n}\right.}Bx^2 + \frac{2n-m}{3n}Cx^3 - \overline{\left|\frac{3n-m}{4n}\right.}Dx^4 + \frac{4n-m}{5n}Ex^5 -$ &c (which is equal to $\overline{1+x}|^{\frac{m}{n}}$ when m is less than n), we may derive the series which is equal to $\overline{1+x}|^{\frac{m}{n}+1}$, or $\overline{1+x}|^{\frac{m+n}{n}}$, or $\overline{1+x}|^{\frac{p}{n}}$, to wit, the series $1 + \frac{p}{n}x + \frac{p-n}{2n}B'x^2 - \overline{\left|\frac{2n-p}{3n}\right.}C'x^3 + \frac{3n-p}{4n}D'x^4 - \overline{\left|\frac{4n-p}{5n}\right.}E'x^5 +$ &c; and from this second series, which is equal to $\overline{1+x}|^{\frac{p}{n}}$, we may derive the series which is equal to $\overline{1+x}|^{\frac{m}{n}+2}$, or $\overline{1+x}|^{\frac{m+2n}{n}}$, or $\overline{1+x}|^{\frac{p+n}{n}}$, or $\overline{1+x}|^{\frac{q}{n}}$, to wit, the series $1 + \frac{q}{n}x + \frac{q-n}{2n}B''x^2 + \frac{q-2n}{3n}C''x^3 - \overline{\left|\frac{3n-q}{4n}\right.}D''x^4 + \frac{4n-q}{5n}E''x^5 -$ &c; and from this third series, which is equal to $\overline{1+x}|^{\frac{q}{n}}$, we may derive the series which is equal to $\overline{1+x}|^{\frac{m}{n}+3}$, or $\overline{1+x}|^{\frac{m+3n}{n}}$, or $\overline{1+x}|^{\frac{q+n}{n}}$, or $\overline{1+x}|^{\frac{r}{n}}$, to wit, the series $1 + \frac{r}{n}x + \frac{r-n}{2n}B'''x^2 + \frac{r-2n}{3n}C'''x^3 + \frac{r-3n}{4n}D'''x^4 - \overline{\left|\frac{4n-r}{5n}\right.}E'''x^5 +$ &c; and from this fourth series, which is equal to $\overline{1+x}|^{\frac{r}{n}}$, we may derive the series which is equal to $\overline{1+x}|^{\frac{m}{n}+4}$, or $\overline{1+x}|^{\frac{m+4n}{n}}$, or $\overline{1+x}|^{\frac{r+n}{n}}$, or $\overline{1+x}|^{\frac{s}{n}}$, to wit, the series $1 + \frac{s}{n}x + \frac{s-n}{2n}B^{\text{iv}}x^2 + \frac{s-2n}{3n}C^{\text{iv}}x^3 + \frac{s-3n}{4n}D^{\text{iv}}x^4 + \frac{s-4n}{5n}E^{\text{iv}}x^5 -$ &c. And in each of these serieses after the first (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), we may observe, 1st, that there is one more term marked with the sign $+$, or added to the first term 1, than in the series next before it; and, 2dly, that, after the several terms in the beginning of the series which are thus to be added to each other, the following terms are marked with the signs $-$ and $+$ alternately, or are to be alternately subtracted from, and added to, the first term; and 3dly, that the co-efficients B, C, D, E, F, &c, and B′, C′, D′, E′, F′, &c, and B″, C″, D″, E″, F″, &c, and B‴, C‴, D‴, E‴, F‴, &c, and B^{iv}, C^{iv}, D^{iv}, E^{iv}, F^{iv}, &c, in all these serieses, are derived

rived from the first term 1 by the same law, or by the continual multiplication of similar generating fractions, to wit, the generating fractions $\frac{m}{n}$, $\frac{n-m}{2n}$, $\frac{2n-m}{3n}$, $\frac{3n-m}{4n}$, $\frac{4n-m}{5n}$, &c, and $\frac{p}{n}$, $\frac{p-n}{2n}$, $\frac{2n-p}{3n}$, $\frac{3n-p}{4n}$, $\frac{4n-p}{5n}$, &c, and $\frac{q}{n}$, $\frac{q-n}{2n}$, $\frac{q-2n}{3n}$, $\frac{3n-q}{4n}$, $\frac{4n-q}{5n}$, &c, and $\frac{r}{n}$, $\frac{r-n}{2n}$, $\frac{r-2n}{3n}$, $\frac{r-3n}{4n}$, $\frac{4n-r}{5n}$, &c, and $\frac{s}{n}$, $\frac{s-n}{2n}$, $\frac{s-2n}{3n}$, $\frac{s-3n}{4n}$, $\frac{s-4n}{5n}$, &c; the denominators of all which generating fractions are n, $2n$, $3n$, $4n$, $5n$, &c, or n and its several successive multiples in their natural order; and the numerators of the first fractions in each set are m, p, q, r, s, or the indexes of the powers to which the nth root of $1+x$ is to be raised; and the numerators of the following fractions are the excesses of n, $2n$, $3n$, $4n$, &c, above the said indexes m, p, q, r, s, or of the said indexes above n, $2n$, $3n$, $4n$, &c, according as n, $2n$, $3n$, $4n$, &c, happen to be greater or less than the said indexes. And I apprehend that, from the manner in which those several serieses, which are equal to $\overline{1+x}|^{\frac{p}{n}}$, $\overline{1+x}|^{\frac{q}{n}}$, $\overline{1+x}|^{\frac{r}{n}}$, and $\overline{1+x}|^{\frac{s}{n}}$, or $\overline{1+x}|^{\frac{m+n}{n}}$, $\overline{1+x}|^{\frac{m+2n}{n}}$, $\overline{1+x}|^{\frac{m+3n}{n}}$, and $\overline{1+x}|^{\frac{m+4n}{n}}$, have been derived from the series which is equal to $\overline{1+x}|^{\frac{m}{n}}$, and from each other, it will be sufficiently evident that the following serieses, that will be equal to $\overline{1+x}|^{\frac{m+5n}{n}}$, $\overline{1+x}|^{\frac{m+6n}{n}}$, $\overline{1+x}|^{\frac{m+7n}{n}}$, $\overline{1+x}|^{\frac{m+8n}{n}}$, $\overline{1+x}|^{\frac{m+9n}{n}}$, &c, *ad infinitum*, will have the same three properties which have been found to belong to the four above-mentioned serieses which are equal to $\overline{1+x}|^{\frac{m+n}{n}}$, $\overline{1+x}|^{\frac{m+2n}{n}}$, $\overline{1+x}|^{\frac{m+3n}{n}}$, and $\overline{1+x}|^{\frac{m+4n}{n}}$; and consequently that, if Q be any whole number, how great soever, and M be $= m + Qn$, the quantity $\overline{1+x}|^{\frac{m+Qn}{n}}$, or $\overline{1+x}|^{\frac{M}{n}}$, will be equal to the series $1 + \frac{M}{n}x + \frac{M}{n} \times \frac{M-n}{2n} x^2 + \frac{M}{n} \times \frac{M-n}{2n} \times \frac{M-2n}{3n} x^3 + \frac{M}{n} \times \frac{M-n}{2n} \times \frac{M-2n}{3n} \times \frac{M-3n}{4n} x^4 + \frac{M}{n} \times \frac{M-n}{2n} \times \frac{M-2n}{3n} \times \frac{M-3n}{4n} \times \frac{M-4n}{5n} x^5 +$ &c, or (if we put B, C, D, E, F, &c, for the several co-efficients of x, x^2, x^3, x^4, x^5, &c, in the terms of this series) to the series $1 + \frac{M}{n}x + \frac{M-n}{2n}Bx^2 + \frac{M-2n}{3n}Cx^3 + \frac{M-3n}{4n}Dx^4 + \frac{M-4n}{5n}Ex^5 +$ &c, till we come to the term in which the multiple of n that enters the numerator of the generating fraction is greater than M, which multiple we will suppose to be $\overline{P+1}| \times n$, or $Pn + n$. The term in which this happens will be marked with the sign $-$, or will be subtracted from the first term 1, and all the following terms will be marked with the sign $+$ and the sign $-$ alternately, or will be alternately added to, and subtracted from, the said first terms; and the numerators of the following generating fractions will be $\overline{P+2}| \times n - M$,

$\overline{P+3}|\times n - M$, $\overline{P+4}|\times n - M$, $\overline{P+5}|\times n - M$, $\overline{P+6}|\times n - M$, &c, or $Pn + 2n - M$, $Pn + 3n - M$, $Pn + 4n - M$, $Pn + 5n - M$, $Pn + 6n - M$, &c *ad infinitum*. All which is agreeable to what has been ſhewn above in art. 105, 106, 107, &c . . . . 115, with only a ſmall variation in the notation, which the different methods of inveſtigation uſed in the two places had made neceſſary; the capital letters P and M being uſed in theſe latter articles inſtead of the ſmall letters p and m, reſpectively, which were employed in art. 105, 106, 107, &c . . . . 115. And it alſo agrees with what was ſtated and aſſerted above in the beginning of this diſcourſe, in art. 5 and 6.

Another Investigation of ſo many of the firſt terms of the ſeries $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, *&c (which is equal to* $\overline{1+x}|^{\frac{m}{n}}$*) as are to be added together before any terms are ſubtracted from the firſt term* 1, *when* m, *the numerator of the index* $\frac{m}{n}$, *is greater than* n, *its denominator; deduced from the binomial theorem in the caſe of integral powers.*

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129. But there is ſtill another method of diſcovering many of the firſt terms of the ſeries which is equal to $\overline{1+x}|^{\frac{m}{n}}$, in the caſe that has been laſt under conſideration, or when m, the numerator of the index $\frac{m}{n}$, is greater than n, its denominator, which the lovers of theſe ſubjects will, I imagine, be glad to ſee. It is founded on a ſuppoſition that the quantity $\overline{1+x}|^{\frac{m}{n}}$ will be equal to a ſeries of the following form, to wit, $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, of which 1 is the firſt term, and all the following terms conſiſt of the ſeveral ſucceſſive powers of x in their natural order, to wit, x, x^2, x^3, x^4, x^5, &c, multiplied into certain numeral co-efficients, which may be denoted by the capital letters B, C, D, E, F, &c, and that the ſecond term, Bx, is to be marked with the ſign +, or added to the firſt term 1, and all the following terms Cx^2, Dx^3, Ex^4, Fx^5, &c, are to be marked either with the ſign + or the ſign −, or to be connected with the ſaid firſt term 1, either by addition or ſubtraction, as ſhall be hereafter determined in the courſe of a proper inveſtigation of the ſubject. That this ſuppoſition is true, or "that the quantity $\overline{1+x}|^{\frac{m}{n}}$ will be equal to ſuch a ſeries," has been ſhewn in a pretty ample manner in the foregoing part of this diſcourſe, in art. 54, 55, 56, 57, and art. 82; to which I ſhall therefore now refer the reader.

130. Upon this ſuppoſition, "that $\overline{1+x}|^{\frac{m}{n}}$ is equal to the ſeries $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c," it was ſhewn above, in art. 58, 59, 60, and 61, and in art. 84, that, if

we put $Q = m \times \frac{m-1}{2}$,

and $R = m \times \frac{m-1}{2} \times \frac{m-2}{3} = \frac{m-2}{3} \times Q$,

and $S = m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} = \frac{m-3}{4} \times R$,

and $T = m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} = \frac{m-4}{5} \times S$,

and $q = n \times \frac{n-1}{2}$,

and $r = n \times \frac{n-1}{2} \times \frac{n-2}{3} = \frac{n-2}{3} \times q$,

and $s = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} = \frac{n-3}{4} \times r$,

and $t = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} = \frac{n-4}{5} \times s$,

we ſhall come to the following general and fundamental equation, to wit, the ſimple ſeries $m + Qx + Rxx + Sx^3 + Tx^4 +$ &c, is = the compound ſeries

$$\begin{array}{llllll}
nB, & nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\
 & +qB^2x, & 2qBCx^2, & 2qBDx^3, & 2qBEx^4, & \&c \\
 & & & qC^2x^3, & 2qCDx^4, & \&c \\
 & & +rB^3x^2, & 3rB^2Cx^3, & 3rB^2Dx^4, & \&c \\
 & & & & 3rBC^2x^4, & \&c \\
 & & & +sB^4x^3, & 4sB^3Cx^4, & \&c \\
 & & & & +tB^5x^4, & \&c
\end{array}$$

by the help of which we may determine both the ſigns which are to be prefixed to the ſeveral terms Cx^2, Dx^3, Ex^4, Fx^5, &c of the ſeries $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, and the values, or magnitudes, of the co-efficients B, C, D, E, F, &c.

131. And the manner in which theſe points are to be determined, is by deducing from the foregoing general equation ſeveral particular ſimple equations, involving the ſeveral co-efficients, B, C, D, E, F, &c, ſingly, or ſeparately from the others that have not yet been diſcovered, and reſolving the ſaid ſimple equations; which ſimple equations will, by art. 85, be as follows; to wit,

1ſt, $m = nB$;
2dly, $Q = nC + qB^2$;
3dly, $R = nD$, $2qBC + rB^3$;
4thly, $S = nE$, $2qBD$, qC^2, $3rB^2C + sB^4$;
and 5thly, $T = nF$, $2qBE$, $2qCD$, $3rB^2D$, $3rBC^2$, $4sB^3C + tB^5$.

And, by reſolving the firſt of theſe ſimple equations, to wit, $m = nB$, it was found, in art. 86, that B was $= \frac{m}{n}$; and by reſolving the ſecond of theſe equations,

equations, to wit, $Q = nC + qB^2$, it was found, in art. 87, that, if m was greater than n (as we have here supposed it to be), the term nC must have the sign + prefixed to it, or must be added to qB^2, and consequently that the term Cx^2, in the series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, (which is $= \overline{1+x}|^{\frac{m}{n}}$), must also have the sign + prefixed to it, or be added to the first term 1; and it was likewise found that the co-efficient C would be equal to $\frac{m-n}{2n} \times B$, or $\frac{m}{n} \times \frac{m-n}{2n}$.

And, in like manner, it would be possible, by resolving the third and fourth and fifth of these equations (if we would go through the labour of doing so), to determine which of the signs + and − was to be prefixed to each of the following terms, Dx^3, Ex^4, Fx^5, &c, in the series $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, &c, (which is equal to $\overline{1+x}|^{\frac{m}{n}}$), and what would be the values, or magnitudes, of the co-efficients D, E, and F, &c.

And, further, it is evident that all the values of the said co-efficients B, C, D, E, F, &c, which would be obtained by the resolutions of these simple equations, would be derived from m and n, the numerator and denominator of the index $\frac{m}{n}$, by various additions, subtractions, multiplications, and divisions; because all the known quantities that enter those equations, to wit, the quantities Q, R, S, T, &c, and q, r, s, t, &c, are only different multiples, or parts, or sums, or differences, or, in general, different combinations, of the said original quantities m and n. And therefore all the values of the co-efficients B, C, D, E, F, &c, must themselves also consist of certain combinations of the same quantities m and n; as has been found to be the case with the co-efficients B and C, which are equal to $\frac{m}{n}$ and $\frac{m}{n} \times \frac{m-n}{2n}$ respectively.

132. These things being premised, it will, I think, be evident, "that, if m "be greater than p times n, or pn, but less than $p+1$ times n, or $pn+n$ (p "being any whole number whatsoever), the several values of the co-efficients D, "E, F, G, H, &c, will consist of the very same combinations of the original "quantities m and n, when m is of any one magnitude greater than pn, but less "than $pn+n$, as when m is of any other magnitude less than its former magni-"tude, but yet greater than pn, and consequently as when the excess of m above "pn becomes equal to o, or m is exactly equal to pn." And consequently, if we can discover to what combinations of m and n the said co-efficients, D, E, F, G, H, &c, will be equal when m is exactly equal to pn, we may conclude that the said co-efficients will be equal to the same combinations of m and n when m is of any other magnitude greater than pn, but less than $pn+n$. This is the principle of the present investigation, which may be easily deduced from it, by the help of the binomial theorem in the case of integral powers, in the manner following.

133. Let

133. Let us then suppose m to be exactly equal to p times n, or pn, and try to discover what will be the values of the co-efficients B, C, D, E, F, G, H, &c, upon this supposition.

Now upon this supposition $\frac{m}{n}$ will be $= \frac{pn}{n} = p$; and consequently $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$, will be equal to $\overline{1+x}|^{p}$, or that integral power of the same binomial quantity $1 + x$, of which the whole number p is the index. But, by the binomial theorem in the case of integral powers (which has been demonstrated above in the tract contained in pages 153, 154, 155, 156, &c .... 169) $\overline{1+x}|^{p}$ is equal the series $1 + \frac{p}{1}x + \frac{p}{1} \times \frac{p-1}{2}x^2 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}x^3 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}x^4 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}x^5 +$ &c, continued to $p + 1$ terms. Therefore the quantity $\overline{1+x}|^{\frac{m}{n}}$ will, when m is $= pn$, be equal to the series $1 + \frac{p}{1}x + \frac{p}{1} \times \frac{p-1}{2}x^2 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}x^3 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}x^4 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}x^5 +$ &c, continued to $p + 1$ terms, and consequently (if we substitute $\frac{m}{n}$ in these terms instead of p, to which it is equal), to the series $1 + \frac{\frac{m}{n}}{1} \times x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 +$ &c, continued to $p + 1$ terms, or the series $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2}x^2 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3}x^3 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3} \times \frac{\frac{m-3n}{n}}{4}x^4 + \frac{m}{n} \times \frac{\frac{m-n}{n}}{2} \times \frac{\frac{m-2n}{n}}{3} \times \frac{\frac{m-3n}{n}}{4} \times \frac{\frac{m-4n}{n}}{5}x^5 +$ &c, continued to $p + 1$ terms, or the series $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 +$ &c, continued to $p + 1$ terms. Therefore by the principle laid down in the last article, when m is of any magnitude greater than pn, but less than $\overline{p+1}| \times n$, or $pn + n$, it will also be true that the first $p + 1$ terms of the series that is equal to $\overline{1+x}|^{\frac{m}{n}}$ will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}$

$\times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5$ continued to $p+1$ terms, or $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4 + \frac{m-4n}{5n} E x^5 +$ &c, continued to $p+1$ terms. Q. E. I.

Thus, for example, if n be 7, and m be 373, which is greater than 371, or 53 times 7, but lefs than 378, or 54 times 7, the firft 54 terms of the feries that is equal to $\overline{1+x}|^{\frac{m}{n}}$, or $\overline{1+x}|^{\frac{373}{7}}$ or to the 373d power of the 7th root of the binomial quantity $1+x$, will be $1 + \frac{373}{7} A x + \frac{373-7}{14} B x^2 + \frac{373-14}{21} C x^3 + \frac{373-21}{28} D x^4 + \frac{373-28}{35} E x^5 + \frac{373-35}{42} F x^6 + \frac{373-42}{49} G x^7 + \frac{373-49}{56} H x^8 + \frac{373-56}{63} I x^9 + \frac{373-63}{70} K x^{10} +$ &c, continued to 54 terms.

134. This inveftigation of the values of the co-efficients B, C, D, E, F, G, H, &c, of x, x^2, x^3, x^4, x^5, x^6, x^7, and the following powers of x in the feries $1 + Bx$, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c, which is equal to $\overline{1+x}|^{\frac{m}{n}}$, is fhorter and eafier than the foregoing ones. But it relates only to the $p+1$ firft terms of the faid feries, or thofe terms of it which are all to be added together. For to fo many terms only will the feries that is equal to $\overline{1+x}|^{p}$ (from which feries the feries that is equal to $\overline{1+x}|^{\frac{m}{n}}$ has been here derived) extend. We cannot therefore difcover by it what will be the co-efficients of the terms of the feries in queftion that come after the $\overline{p+1}|$th term (and of which the number will be infinite), nor whether the faid terms are to be added to, or fubtracted from, the firft term 1 of the feries. But, when p is a very great number, or m (which is greater than p times n, or pn) is very much greater than n (as in the foregoing example, in which m is fuppofed to be 373 and n to be 7), the firft $p+1$ terms of the feries $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4 + \frac{m-4n}{5n} E x^5 +$ &c, (which is equal to $1 + x^{\frac{m}{n}}$), will be very nearly equal to the whole feries, and confequently very nearly equal to $\overline{1+x}|^{\frac{m}{n}}$; unlefs when x is very nearly equal to 1, in which cafe it would be neceffary to take in more than the firft $p+1$ terms of the feries, and confequently to have recourfe to fome of the former inveftigations of it.

An

An example of the binomial theorem in raising the m*th power of the* n*th root of a binomial quantity, when* m *and* n *are very great whole numbers.*

135. Let m be $= 970{,}877$, and n be $= 10{,}000$. And let it be required to raise the binomial quantity $1 + \frac{24}{1000}$, or $1 + \frac{3}{125}$, to the $\frac{m}{n}$th, or $\frac{970{,}877}{10{,}000}$th, or 97.0877th power, or to find the 970,877th power of the 10,000th root of the said binomial quantity.

136. Now it has been shewn in several different ways in the course of the foregoing articles, that, if x be any quantity not greater than 1, and n be any whole number whatsoever, and m any other whole number greater than n, and m be greater than pn, or p times n, but less than $\overline{p+1}| \times n$, or $p + 1$ times n, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$, will be equal to the series $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 + \frac{m-5n}{6n}Fx^6 +$ &c, to be continued till we come to the term $\frac{m-pn}{\overline{p+1}|\times n} \times P'x^{p+1}$, in which P', or P with an accent placed over it, represents the co-efficient of the next preceeding term; the capital letter A being put for the first term 1, and the capital letters B, C, D, E, F, &c, for the numeral co-efficients of x, x^2, x^3, x^4, x^5, and the other following powers of x in the second, third, fourth, fifth, sixth, and other following terms of the said series. And after the term $\frac{m-pn}{\overline{p+1}|\times n} \times P'x^{p+1}$, the following terms of the series will be $-\frac{pn+n-m}{pn+2n} \times Q'x^{p+2} + \frac{pn+2n-m}{pn+3n} \times R'x^{p+3} - \frac{pn+3n-m}{pn+4n} \times S'x^{p+4} + \frac{pn+4n-m}{pn+5n} \times T'x^{p+5} - \frac{pn+5n-m}{pn+6n} \times V'x^{p+6}$ + &c, which are marked with the signs − and + alternately, or are alternately to be subtracted from, and added to, the first term 1.

137. This is in general the best method of expressing this series; more especially when we are considering the law of the generation of its terms, and the methods by which the said law may be investigated: which is the subject of the foregoing articles. But, when we have occasion actually to compute a considerable number of the first terms of the series, in order to find some high fractional power of the binomial quantity $1 + x$ in some particular example (as is the case at present), it will be more convenient to make the capital letters B, C, D, E, F, &c, stand for the *whole* second, third, fourth, fifth, and sixth, and other following

following terms, including the powers of x, instead of representing only the numeral co-efficients of the powers of x in the said terms. And then the foregoing theorem will be as follows; to wit, $\overline{1+x}|^{\frac{m}{n}} =$ the series $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx + \frac{m-2n}{3n}Cx + \frac{m-3n}{4n}Dx + \frac{m-4n}{5n}Ex + \frac{m-5n}{6n}Fx +$ &c. Now, if we substitute 970,877 in this equation instead of m, and 10,000 instead of n, and $\frac{24}{1000}$, or $\frac{3}{125}$, instead of x, we shall have $\overline{1+\frac{24}{1000}}|^{\frac{970,877}{10,000}} =$ the series

$$1 + \frac{970,877}{10,000}A \times \frac{24}{1000} + \frac{970,877 - 10,000}{2 \times 10,000}B \times \frac{24}{1000} + \frac{970,877 - 2 \times 10,000}{3 \times 10,000}C \times \frac{24}{1000} + \frac{970,877 - 3 \times 10,000}{4 \times 10,000}D \times \frac{24}{1000} + \frac{970,877 - 4 \times 10,000}{5 \times 10,000}E \times \frac{24}{1000} + \frac{970,877 - 5 \times 10,000}{6 \times 10,000}F \times \frac{24}{1000} + \text{\&c} =$$

the series

$$1 + 97.0877\,A \times \frac{24}{1000} + \frac{97.0877 - 1}{2}B \times \frac{24}{1000} + \frac{97.0877 - 2}{3}C \times \frac{24}{1000} + \frac{97.0877 - 3}{4}D \times \frac{24}{1000} + \frac{97.0877 - 4}{5}E \times \frac{24}{1000} + \frac{97.0877 - 5}{6}F \times \frac{24}{1000} + \text{\&c} =$$

the series

$$1 + 97.0877\,A \times \frac{24}{1000} + \frac{96.0877}{2}B \times \frac{24}{1000} + \frac{95.0877}{3}C \times \frac{24}{1000} + \frac{94.0877}{4}D \times \frac{24}{1000} + \frac{93.0877}{5}E \times \frac{24}{1000} + \frac{92.0877}{6}F \times \frac{24}{1000} + \frac{91.0877}{7}G \times \frac{24}{1000} + \frac{90.0877}{8}H \times \frac{24}{1000} + \frac{89.0877}{9}I \times \frac{24}{1000} + \frac{88.0877}{10}K \times \frac{24}{1000} + \frac{87.0877}{11}L \times \frac{24}{1000} + \frac{86.0877}{12}M \times \frac{24}{1000} + \frac{85.0877}{13}N \times \frac{24}{1000} + \frac{84.0877}{14}O \times \frac{24}{1000} + \frac{83.0877}{15}P \times \frac{24}{1000} + \frac{82.0877}{16}Q \times \frac{24}{1000} + \frac{81.0877}{17}R \times \frac{24}{1000} + \frac{80.0877}{18}S \times \frac{24}{1000} + \frac{79.0877}{19}T \times \frac{24}{1000} + \frac{78.0877}{20}V \times \frac{24}{1000} + \frac{77.0877}{21}W \times \frac{24}{1000} + \frac{76.0877}{22}X \times \frac{24}{1000} + \frac{75.0877}{23}Y \times \frac{24}{1000} + \frac{74.0877}{24}Z \times \frac{24}{1000} + \frac{73.0877}{25}A' \times \frac{24}{1000} + \frac{72.0877}{26}B' \times \frac{24}{1000} + \frac{71.0877}{27}C' \times \frac{24}{1000} + \frac{70.0877}{28}D' \times \frac{24}{1000} + \frac{69.0877}{29}E' \times \frac{24}{1000} + \frac{68.0877}{30}F' \times \frac{24}{1000} + \text{\&c} =$$

1.000,000,000,000,000,000,000,	A,
+ 2.330,104,800,000,000,000,000,	B,
+ 2.686,732,931,891,520,000,000,	C,
+ 2.043,802,040,062,570,290,432,	D,
+ 1.153,779,799,228,770,568,290,	E,
+ 0.515,532,997,520,006,524,943,	F,
+ 0.189,896,992,062,892,419,468,	G,
+ 0.059,304,960,836,321,574,297,	H,
+ 0.016,027,942,561,002,861,266,	I,
+ 0.003,807,713,435,978,278,943,	K,
+ 0.000,804,990,525,202,617,220,	L,
+ 0.000,152,955,869,152,773,747,	M,

+ 0.000,

$+\ 0.000,026,335,237,953,726,481,\quad N,$
$+\ 0.000,004,136,870,448,803,617,\quad O,$
$+\ 0.000,000,596,331,293,550,623,\quad P,$
$+\ 0.000,000,079,276,472,990,633,\quad Q,$
$+\ 0.000,000,009,761,435,007,869,\quad R,$
$+\ 0.000,000,001,117,457,383,747,\quad S,$
$+\ 0.000,000,000,119,326,122,283,\quad T,$
$+\ 0.000,000,000,011,920,709,761,\quad V,$
$+\ 0.000,000,000,001,117,032,969,\quad W,$
$+\ 0.000,000,000,000,098,410,859,\quad X,$
$+\ 0.000,000,000,000,008,168,570,\quad Y,$
$+\ 0.000,000,000,000,000,640,026,\quad Z,$
$+\ 0.000,000,000,000,000,047,741,\quad A',$
$+\ 0.000,000,000,000,000,003,327,\quad B',$
$+\ 0.000,000,000,000,000,000,221,\quad C',$
$+\ 0.000,000,000,000,000,000,013,\quad D',$
$+\ 0.000,000,000,000,000,000,000,\quad E',$

$+$ &c $= 9.999,979,282,720,950,507,346,$ &c. Therefore (if no miſtake has been made in computing theſe numbers) the quantity $\overline{1 + \frac{24}{1000}}\Big|^{\frac{970,877}{10000}}$, or the 970,877th power of the 10,000th root of the binomial quantity $1 + \frac{24}{1000}$, will be equal to the mixt number 9.999,979,282,720,950,507,346, &c.

Q. E. I.

137. By the help of this number we may find the logarithm of 2, or of the ratio of 2 to 1, in Briggs's Syſtem of Logarithms, to a conſiderable degree of exactneſs, in the manner above explained, in the tract intitled, "*An Appendix to Dr. Halley's Diſcourſe concerning Logarithms,*" in pages 123, 124, 125, &c, to 152, without having recourſe to the doctrine of fluxions, or infinite ſerieſes, or the arithmetic of infinites in any of its modifications. This may be done in the manner following.

An application of the foregoing example to the inveſtigation of the Logarithm of the ratio of 2 to 1, in Brigg's Syſtem of Logarithms.

138. The tenth power of 2 is 1024. Therefore the ratio of 2 to 1 is one tenth part of the ratio of 1024 to 1. But the ratio of 1024 to 1 is equal to the ratio of 1024 to 1000, together with the ratio of 1000 to 1. Therefore the ratio of 2 to 1 is equal to one tenth part of the ratio of 1024 to 1000, together with one tenth part of the ratio of 1000 to 1. Therefore the logarithm of the ratio of 2 to 1 is equal to one tenth part of the logarithm of the ratio of 1024 to 1000, together with one tenth part of the logarithm of the ratio of 1000 to 1. But the logarithm of the ratio of 1000 to 1 in Briggs's Syſtem of Logarithms is 3. Therefore

Therefore the logarithm of the ratio of 2 to 1 in Briggs's Syftem of Logarithms is equal to one tenth part of the logarithm of the ratio of 1024 to 1000, together with one tenth part of 3, or with $\frac{3}{10}$, or 0.300,000,000,000,000,000,000, &c. Therefore, if we can difcover the logarithm of the ratio of 1024 to 1000, we may eafily deduce from it the logarithm of the ratio of 2 to 1, by firft dividing it by 10, and then adding the quotient to 0.300,000,000,000,000,000,000, &c, which is the tenth part of the logarithm of the ratio of 1000 to 1. We muft therefore endeavour to find the logarithm of the ratio of 1024 to 1000. This may be done in the manner following.

139. The logarithm of the ratio of 1024 to 1000 is the fame with that of the ratio of $\frac{1024}{1000}$ to $\frac{1000}{1000}$, or of $1 + \frac{24}{1000}$ to 1. We muft therefore endeavour to find the logarithm of the ratio of $1 + \frac{24}{1000}$ to 1, or its proportion to the logarithm of the greater ratio of 10 to 1, or (which comes to the fame thing) the proportion of the ratio of $1 + \frac{24}{1000}$ to 1 to the greater ratio of 10 to 1.

140. Let this proportion be that of 1 to x. Then, fince the ratio of $1 + \frac{24}{1000}$ to 1 is to the ratio of $\overline{1 + \frac{24}{1000}}\Big|^{x}$ to 1 as 1 is to x; and the faid ratio of $1 + \frac{24}{1000}$ to 1 is to the ratio of 10 to 1 in the fame proportion of 1 to x, it follows that the ratio of $\overline{1 + \frac{24}{1000}}\Big|^{x}$ to 1 muft be equal to the ratio of 10 to 1, and confequently that $\overline{1 + \frac{24}{1000}}\Big|^{x}$ will be equal to 10. We muft therefore endeavour to refolve the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$.

141. By the binomial theorem the quantity $\overline{1 + \frac{24}{1000}}\Big|^{x}$ is equal to the feries $1 + \frac{x}{1} \times \frac{24}{1000} + \frac{x}{1} \times \frac{x-1}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \overline{\frac{24}{1000}}\Big|^{3} + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{x-4}{5} \times \overline{\frac{24}{1000}}\Big|^{5} + \&c = 1 + x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{x^3-3xx+2x}{6} \times \overline{\frac{24}{1000}}\Big|^{3} + \frac{x^4-6x^3+11xx-6x}{24} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{x^5-10x^4+35x^3-50x^2+24x}{120} \times \overline{\frac{24}{1000}}\Big|^{5} + \&c.$ Therefore the feries $1 + x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{x^3-3xx+2x}{6} \times \overline{\frac{24}{1000}}\Big|^{3} + \frac{x^4-6x^3+11xx-6x}{24} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{x^5-10x^4+35x^3-50x^2+24x}{120} \times \overline{\frac{24}{1000}}\Big|^{5} + \&c$, will be $= 10$, and confequently (fubtracting 1 from both fides) the feries $x \times \frac{24}{1000}$

$+ \frac{xx-x}{2}$

$+ \frac{xx-x}{2} \times \overline{\frac{24}{1000}}\Big|^2 + \frac{x^3-3x^2+2x}{6} \times \overline{\frac{24}{1000}}\Big|^3 + \frac{x^4-6x^3+11xx-6x}{24} \times \overline{\frac{24}{1000}}\Big|^4 + \frac{x^5-10x^4+35x^3-50x^2+24x}{120} \times \overline{\frac{24}{1000}}\Big|^5 +$ &c, will be $= 9$.

142. Now, if we take any finite number of terms of this ſeries, and ſuppoſe them to be equal to the whole ſeries, and conſequently to the abſolute term 9, and reſolve the equation reſulting from ſuch ſuppoſition, it is evident that the value of x, that will be obtained by ſuch reſolution, will always be greater than the true value of x in the foregoing equation between the whole of the ſaid ſeries and the abſolute term 9; but that, the more terms of the ſaid ſeries are retained in ſuch finite equation, the nearer will the root of ſuch finite equation approach to the true value of x in the ſaid infinite equation between the ſaid infinite ſeries and the abſolute term 9, or to the value of x in the original equation $1 + \overline{\frac{24}{1000}}\Big|^x = 10$. In order therefore to approach gradually to the true value of x in the ſaid original equation $1 + \overline{\frac{24}{1000}}\Big|^x = 10$, we will, firſt, ſuppoſe one term of the foregoing ſeries to be equal to 9; and, 2dly, two terms of it to be equal to the ſame quantity; and, 3dly, three terms of it, and, 4thly, four terms of it, to be equal to the ſame quantity, and will reſolve the ſeveral equations reſulting from theſe ſuppoſitions. This may be done in the manner following.

143. If we ſuppoſe the firſt term $x \times \frac{24}{1000}$ of the foregoing ſeries to be equal to the whole ſeries, and conſequently to the abſolute term 9, we ſhall have $x \times 24 = 9000$, and $x = \frac{9000}{24} = 375$. Therefore 375 is the firſt approximation to the value of x in the equation $1 + \overline{\frac{24}{1000}}\Big|^x = 10$. But this approximation is very much too great, the true value of x being (as we ſhall preſently ſee) ſomewhat greater than 97, but leſs than 98.

144. In the ſecond place, we will ſuppoſe the two firſt terms, $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\frac{24}{1000}}\Big|^2$, of the foregoing ſeries, to be equal to the whole ſeries, and conſequently to the abſolute term 9. And we ſhall then have $\frac{24x}{1000} \times \frac{2 \times 1000}{2 \times 1000} + \frac{\overline{xx-x}\big| \times \overline{24}\big|^2}{2 \times 1000 \times 1000} = 9$, or $\frac{48{,}000x}{2 \times 1000{,}000} + \frac{\overline{xx-x}\big| \times 576}{2 \times 1000{,}000} = 9$, or $\frac{48{,}000 + 576xx - 576x}{2 \times 1000{,}000} = 9$, or $\frac{47{,}424x + 576xx}{2 \times 1000{,}000} = 9$, or $\frac{23{,}712x + 288xx}{1000{,}000} = 9$, and conſequently $23{,}712x + 288xx = 9000{,}000$, and (dividing all the terms by 288), $xx + 82.333{,}333$, &c $x = 31{,}250$. This quadratick equation we muſt now reſolve.

Add

Add $\overline{41.166,666, \&c}|^2$ to both sides; and we shall have $xx + 82.333,333$, &c $x + \overline{41.166,666, \&c}|^2$ ($= 31,250 + \overline{41.166,666, \&c}|^2 = 31250 + 1694.1456$) $= 32,944.1456$, and consequently $x + 41.166,666$, &c ($= \sqrt{32,944.1456}$) $= 181.50$, and x ($= 181.50 - 41.16$) $= 140.34$. Therefore 140.34, or (dropping the decimal fraction $.34$) 140, is a second approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}|^x = 10$.

Q. E. I.

This approximation is much nearer than the first approximation, 375, to the true value of x in this equation, but yet is much too great.

145. In the third place let us suppose the three first terms, $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\frac{24}{1000}}|^2 + \frac{x^3-3xx+2x}{6} \times \overline{\frac{24}{1000}}|^3$, of the foregoing series, to be equal to the whole series, and consequently to the absolute term 9.

Then, since the two first terms of this series have been shewn to be equal to $\frac{47,424x+576xx}{2 \times 1000,000}$, or $\frac{23,712x+288xx}{1000,000}$, it follows that $\frac{23,712x+288xx}{1000,000} + \frac{x^3-3xx+2x}{6} \times \overline{\frac{24}{1000}}|^3$ will, upon this supposition, be $= 9$, or that $\frac{23,712x+288xx}{1000,000} + \frac{x^3-3xx+2x}{6} \times \frac{\overline{24}|^3}{\overline{1000}|^3}$ will be $= 9$, or that $\frac{23,712x+288xx}{1000,000} + \frac{x^3-3xx+2x}{6} \times \frac{13824}{\overline{1000}|^3}$ will be $= 9$, or that $\frac{23,712x+288xx}{1000,000} + \overline{x^3-3xx+2x}| \times \frac{2304}{\overline{1000}|^3}$ will be $= 9$, or that $\frac{23,712x+288xx}{1000,000} + \frac{2304x^3-6912xx+4608x}{1000,000,000}$ will be $= 9$, or that $\frac{23,712,000x+288,000xx}{1000,000,000} + \frac{2304x^3-6912xx+4608x}{1000,000,000}$ will be $= 9$, or that $\frac{2304x^3+281,088xx+23,716,608x}{1000,000,000}$ will be $= 9$, and consequently that $2304x^3 + 281,088xx + 23,716,608x$ will be $= 9000,000,000$, and (dividing all the terms by 2304) that $x^3 + 122xx + 10293.666$ &c x will be $= 3,906,250$. This cubick equation we must now endeavour to resolve.

Now, since we know that the value of x in this equation must be less than the root of the foregoing quadratick equation, and that root was found to be nearly equal to 140, it seems reasonable to conjecture that the root of the present cubick equation will be pretty nearly equal to 100. We will therefore suppose x to be $= 100$, and will substitute 100 instead of it in the compound quantity $x^3 + 122xx + 10293.666$ &c x, in order to discover whether the result will be nearly equal to the absolute term 3,906,250, and which of them is greater than the other.

Now, if we suppose x to be $= 100$, we shall have $xx = 10,000$, and $x^3 = 1000,000$, and consequently $122xx$ ($= 122 \times 10,000$) $= 1,220,000$, and $10,293.666x$ ($= 10,293.666 \times 100$) $= 1,029,366.6$, and $x^3 + 122xx + 10,293.666x$ ($= 1000,000 + 1,220,000 + 1,029,366.6$) $= 3,249,366.6$; which

which is lefs than the abfolute term 3,906,250. Therefore 100 is lefs than the true value of x in the faid cubick equation $x^3 + 122xx + 10,293.666x = 3,906,250$.

We will therefore form a fecond conjecture concerning the root of this cubick equation, and will fuppofe it to be $= 110$.

Now, if x is $= 110$, we fhall have $xx = 12,100$, and $x^3 = 1,331,000$, and $122xx$ $(= 122 \times 12,100) = 1,476,200$, and $10,293.666x$ $(= 10,293.666 \times 110) = 1,132,303.26$, and confequently $x^3 + 122xx + 10,293.666x$ $(= 1000,000 + 1,476,200 + 1,132,303.26) = 3,939,503.26$; which is but little greater than the abfolute term 3,906,250. We may therefore conclude that the root of the cubick equation $x^3 + 122xx + 10,293.666x = 3,906,250$ is fomewhat, but not much, lefs than 110, and confequently that 110 will be a third approximation to the value of the index x in the equation $\left.\overline{1 + \frac{24}{1000}}\right|^{x} = 10$. Q. E. I.

146. In the 4th place let us fuppofe the four firft terms of the foregoing feries, to wit, the terms $x \times \frac{24}{1000} + \frac{xx - x}{2} \times \left.\overline{\frac{24}{1000}}\right|^{2} + \frac{x^3 - 3xx + 2x}{6} \times \left.\overline{\frac{24}{1000}}\right|^{3} + \frac{x^4 - 6x^3 + 11x^2 - 6x}{24} \times \left.\overline{\frac{24}{1000}}\right|^{4}$, to be equal to the whole feries, and confequently to the abfolute term 9.

Then, fince it has been feen in the laft article, that the three firft terms of this feries are equal to $\frac{2304x^3 + 281,088xx + 23,716,608x}{1000,000,000}$ it follows that the four firft terms of this feries will be $= \frac{2304x^3 + 281,088xx + 23,716,608x}{1000,000,000} + \frac{x^4 - 6x^3 + 11xx - 6x}{24} \times \left.\overline{\frac{24}{1000}}\right|^{4} = \frac{2304x^3 + 281,088xx + 23,716,608x}{1000,000,000} + \frac{x^4 - 6x^3 + 11xx - 6x}{24} \times \frac{\overline{24}|^4}{\overline{1000}|^4} =$
$\frac{2304x^3 + 281,088xx + 23,716,608x}{1000,000,000} + \left.\overline{x^4 - 6x^3 + 11xx - 6x}\right| \times \frac{\overline{24}|^3}{\overline{1000}|^4} =$
$\frac{2304x^3 + 281,088xx + 23,716,608x}{1000,000,000} + \left.\overline{x^4 - 6x^3 + 11xx - 6x}\right| \times \frac{13824}{\overline{1000}|^4} =$
$\frac{2304x^3 + 281,088xx + 23,716,608x}{1000,000,000} + \frac{13,824x^4 - 82,944x^3 + 152,064xx - 82,944x}{1000,000,000,000} =$
$\frac{2,304,000x^3 + 281.088,000xx + 23,716,608,000x}{1000,000,000,000} + \frac{13,824x^4 - 82,944x^3 + 152,064x^2 - 82,944x}{1000,000,000,000}$
$= \frac{13,824x^4 + 2,221,056x^3 + 281,240,064xx + 23,716,525,056x}{1000,000,000,000}$. Therefore $\frac{13,824x^4 + 2,221,056x^3 + 281,240,064xx + 23,716,525,056x}{1000,000,000,000}$ will be $= 9$, and confequently $13,824x^4 + 2,221,056x^3 + 281,240,064xx + 23,716,525,056x$ will be $= 9000,000,000,000$, and (dividing all the terms by 13,824) $x^4 + 160.666,666$, &c $x^3 + 20,344.333,333$, &c $xx + 1,715,605.111,111$, &c x will be $= 651,041,666.666,666$, &c. This biquadratick equation we muft now endeavour to refolve.

Now,

Now, ſince we know that the root of this biquadratick equation muſt be leſs than the root of the foregoing cubick equation, and that root was found to be nearly equal to 110, we may reaſonably conjecture that the root of this biquadratick equation will be nearly equal to 100. And ſo, upon trial, we ſhall find it to be. For, if x be $= 100$, we ſhall have $xx = 10{,}000$, and $x^3 = 1{,}000{,}000$, and $x^4 = 100{,}000{,}000$, and conſequently $160.666{,}666$, &c $\times x^3$ ($= 160.666{,}666$, &c $\times 1{,}000{,}000 =$) $160{,}666{,}666.666$, &c, and $20{,}344.333{,}333$, &c $\times xx$ ($= 20{,}344.333{,}333$, &c $\times 10{,}000$) $= 203{,}443{,}333.333$, &c, and $1{,}715{,}605.111{,}111$, &c $\times x$ ($= 1{,}715{,}605.111{,}111$, &c $\times 100$) $= 171{,}560{,}511.111{,}111$, &c, and $x^4 + 160.666{,}666$, &c $\times x^3 + 20{,}344{,}333{,}333$, &c $\times x^2 + 1{,}715{,}605.111{,}111$, &c $\times x$ ($= 100{,}000{,}000 + 160{,}666{,}666.666$, &c $+ 203{,}443{,}333.333$, &c $+ 171{,}560{,}511.111$, &c) $= 635{,}670{,}511.110$, &c; which is not a great deal leſs than the abſolute term $651{,}041{,}666.666$, &c. And conſequently 100 is not a great deal leſs than the true value of x in the aforeſaid biquadratick equation $x^4 + 160.666{,}666$, &c $\times x^3 + 20{,}344.333{,}333$, &c $\times xx + 1{,}715{,}605{,}111{,}111$, &c $\times x = 651{,}041{,}666.666{,}666$, &c.

We may therefore conſider the number 100 as a fourth approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$. Q. E. I.

147. By the reſolution of theſe four equations we have obtained the four numbers 375, 140, 110, and 100 for ſo many ſucceſſive approximations to the true value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$. And, as the difference between the ſecond and third of theſe values, to wit, 140 and 110, is 30, and the difference between the third and fourth, to wit, 110 and 100, is only 10, (which is but a third part of 30) it ſeems reaſonable to conjecture that the difference between 100, the fourth, or laſt, approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$, and the ſaid true value will be leſs than a third part of the former difference 10, or will be nearly equal to 3, and conſequently that the ſaid true value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$ will be nearly equal to $100 - 3$, or 97. We will therefore ſuppoſe the true value of x in this equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$ to be equal to 97, and will now proceed to try whether it is ſo, or not, and whether it is greater, or leſs, than the ſaid true value, by raiſing the binomial quantity $1 + \frac{24}{1000}$ to the 97th power. This may be done moſt conveniently by means of the binomial theorem in the manner following.

148. By the binomial theorem we ſhall have $\overline{1 + \frac{24}{1000}}\Big|^{97} =$ the ſeries $1 + 97\,A \times \frac{24}{1000} + \frac{96}{2}\,B \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{95}{3}\,C \times \overline{\frac{24}{1000}}\Big|^{3} + \frac{94}{4}\,D \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{93}{5}\,E \times \overline{\frac{24}{1000}}\Big|^{5}$

$$\overline{\tfrac{24}{1000}}\Big|^{5} + \tfrac{92}{6} F \times \overline{\tfrac{24}{1000}}\Big|^{6} \times \tfrac{91}{7} G \times \overline{\tfrac{24}{1000}}\Big|^{7} + \tfrac{90}{8} H \times \overline{\tfrac{24}{1000}}\Big|^{8} + \tfrac{89}{9} I \times \overline{\tfrac{24}{1000}}\Big|^{9}$$
$$+ \tfrac{88}{10} K \times \overline{\tfrac{24}{1000}}\Big|^{10} + \tfrac{87}{11} L \times \overline{\tfrac{24}{1000}}\Big|^{11} + \tfrac{86}{12} M \times \overline{\tfrac{24}{1000}}\Big|^{12} + \tfrac{85}{13} N \times \overline{\tfrac{24}{1000}}\Big|^{13} +$$
$$\tfrac{84}{14} O \times \overline{\tfrac{24}{1000}}\Big|^{14} + \tfrac{83}{15} P \times \overline{\tfrac{24}{1000}}\Big|^{15} + \tfrac{82}{16} Q \times \overline{\tfrac{24}{1000}}\Big|^{16} + \tfrac{81}{17} R \times \overline{\tfrac{24}{1000}}\Big|^{17} + \tfrac{80}{18}$$
$$S \times \overline{\tfrac{24}{1000}}\Big|^{18} + \tfrac{79}{19} T \times \overline{\tfrac{24}{1000}}\Big|^{19} + \tfrac{78}{20} V \times \overline{\tfrac{24}{1000}}\Big|^{20} + \tfrac{77}{21} W \times \overline{\tfrac{24}{1000}}\Big|^{21} + \tfrac{76}{22} X$$
$$+ \overline{\tfrac{24}{1000}}\Big|^{22} + \tfrac{75}{23} Y \times \overline{\tfrac{24}{1000}}\Big|^{23} + \tfrac{74}{24} Z \times \overline{\tfrac{24}{1000}}\Big|^{24} + \tfrac{73}{25} A' \times \overline{\tfrac{24}{1000}}\Big|^{25} + \tfrac{72}{26} B' \times$$
$$\overline{\tfrac{24}{1000}}\Big|^{26} + \tfrac{71}{27} C' \times \overline{\tfrac{24}{1000}}\Big|^{27} + \tfrac{70}{28} D' \times \overline{\tfrac{24}{1000}}\Big|^{28} + \tfrac{69}{29} E' \times \overline{\tfrac{24}{1000}}\Big|^{29} + \&c;$$

in which ſeries the capital letter A ſtands for the firſt term 1, and the following letters B, C, D, E, F, &c, ſtand for the co-efficients of the powers of $\frac{24}{1000}$ in the ſecond, third, fourth, fifth, ſixth, and other following terms of the ſeries. But, for the purpoſe of computing theſe ſeveral terms, it will be more convenient to make the letters B, C, D, E, F, &c, ſtand for the whole ſecond, third, fourth, fifth, ſixth, and other following terms of the ſeries reſpectively, including the powers of $\frac{24}{1000}$; and then we ſhall have $\overline{1 + \frac{24}{1000}}\Big|^{97}$ = the ſeries $1 + 97\,A \times \frac{24}{1000}$

$$+ \tfrac{96}{2} B \times \tfrac{24}{1000} + \tfrac{95}{3} C \times \tfrac{24}{1000} + \tfrac{94}{4} D \times \tfrac{24}{1000} + \tfrac{93}{5} E \times \tfrac{24}{1000} + \tfrac{92}{6} F \times$$
$$\tfrac{24}{1000} + \tfrac{91}{7} G \times \tfrac{24}{1000} + \tfrac{90}{8} H \times \tfrac{24}{1000} + \tfrac{89}{9} I \times \tfrac{24}{1000} + \tfrac{88}{10} K \times \tfrac{24}{1000} + \tfrac{87}{11} L$$
$$\times \tfrac{24}{1000} + \tfrac{86}{12} M \times \tfrac{24}{1000} + \tfrac{85}{13} N \times \tfrac{24}{1000} + \tfrac{84}{14} O \times \tfrac{24}{1000} + \tfrac{83}{15} P \times \tfrac{24}{1000} + \tfrac{82}{16}$$
$$Q \times \tfrac{24}{1000} + \tfrac{81}{17} R \times \tfrac{24}{1000} + \tfrac{80}{18} S \times \tfrac{24}{1000} + \tfrac{79}{19} T \times \tfrac{24}{1000} + \tfrac{78}{20} V \times \tfrac{24}{1000} +$$
$$\tfrac{77}{21} W \times \tfrac{24}{1000} + \tfrac{76}{22} X \times \tfrac{24}{1000} + \tfrac{75}{23} Y \times \tfrac{24}{1000} + \tfrac{74}{24} Z \times \tfrac{24}{1000} + \tfrac{73}{25} A' \times \tfrac{24}{1000}$$
$$+ \tfrac{72}{26} B' \times \tfrac{24}{1000} + \tfrac{71}{27} C' \times \tfrac{24}{1000} + \tfrac{70}{28} D' \times \tfrac{24}{1000} + \tfrac{69}{29} E' \times \tfrac{24}{1000} + \&c =$$

1.000,000,000,000,000,000,
+ 2.328,000,000,000,000,000,
+ 2.681,856,000,000,000,000,
+ 2.038,210,560,000,000,000,
+ 1.149,550,755,840,000,000,
+ 0.513,159,457,406,976,000,
+ 0.188,842,680,325,767,168,
+ 0.058,918,916,261,639,356,
+ 0.015,908,107,390,642,626,
+ 0.003,775,524,154,045,849,
+ 0.000,797,390,701,334,483,
+ 0.000,151,359,253,126,036,
+ 0.000,026,033,791,537,678,

+ 0,000,

+ 0.000,004,085,302,672,066,
+ 0.000,000,588,283,584,777,
+ 0.000,000,078,124,060,058,
+ 0.000,000,009,609,259,387,
+ 0.000,000,001,098,847,073,
+ 0.000,000,000,117,210,354,
+ 0.000,000,000,011,696,359,
+ 0.000,000,000,001,094,779,
+ 0.000,000,000,000,096,340,
+ 0.000,000,000,000,007,987,
+ 0.000,000,000,000,000,625,
+ 0.000,000,000,000,000,046,
+ 0.000,000,000,000,000,003,
+ 0.000,000,000,000,000,000,
+ &c

= 9.979,201,547,673,599,050, &c; which is somewhat less than 10. Therefore 97 is somewhat less than the true value of the index x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$. But it differs from it by less than an unit. For, if we multiply the number just now found for the value of $\overline{1 + \frac{24}{1000}}|^{97}$, to wit, 9.979,201,547,673,599,050, into $1 + \frac{24}{1000}$, or 1.024, we shall find the product to be = 10.218,702,384,817,765,427,200, which is therefore the value of $\overline{1 + \frac{24}{1000}}|^{98}$. Therefore $\overline{1 + \frac{24}{1000}}|^{98}$ is greater than 10, or than $\overline{1 + \frac{24}{1000}}|^{x}$; and $\overline{1 + \frac{24}{1000}}|^{97}$ is less than 10, or than $\overline{1 + \frac{24}{1000}}|^{x}$; and consequently x must be less than 98, but greater than 97, or the difference between 97 and the true value of x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$ must be less than 1. We therefore now know with certainty that the two first figures of the true value of x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$ are 97; which is a considerable step towards a more accurate determination of it.

149. Let the excess of the true value of x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$ above 97 be called z. And we shall then have $\overline{1 + \frac{24}{1000}}|^{97+z}$ $(= \overline{1 + \frac{24}{1000}}|^{x})$ $= 10$. But $\overline{1 + \frac{24}{1000}}|^{97+z}$ is $= \overline{1 + \frac{24}{1000}}|^{97} \times \overline{1 + \frac{24}{1000}}|^{z} = 9.979,201,547,673,599,050, \times \overline{1 + \frac{24}{1000}}|^{z}$. Therefore $9.979,201,547,673,599,050 \times \overline{1 + \frac{24}{1000}}|^{z}$ is

is = 10; and consequently $\overline{1 + \frac{24}{1000}}\big|^{z}$ is $= \frac{10.000,000,000,000,000,000}{9.979,201,547,673,599,050} = 1.002,$ 084,180,004,486,389. But, by the binomial theorem, $\overline{1 + \frac{24}{1000}}\big|^{z}$ is = the series $1 + z \times \frac{24}{1000} + z \times \frac{z-1}{2} \times \overline{\frac{24}{1000}}\big|^{2} + z \times \frac{z-1}{2} \times \frac{z-2}{3} \times \overline{\frac{24}{1000}}\big|^{3} + z \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \overline{\frac{24}{1000}}\big|^{4} + \&c = 1 + z \times \frac{24}{1000} + \frac{zz-z}{2} \times \overline{\frac{24}{1000}}\big|^{2} + \frac{z^3-3zz+2z}{6} \times \overline{\frac{24}{1000}}\big|^{3} + \frac{z^4-6z^3+11zz-6z}{24} \times \overline{\frac{24}{1000}}\big|^{4} + \&c =$ (because z is less than 1, and consequently zz is less than z, and z^3 than zz, and z^4 than z^3) $1 + z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \overline{\frac{24}{1000}}\big|^{2} + \frac{2z-3zz+z^3}{6} \times \overline{\frac{24}{1000}}\big|^{3} - \overline{\frac{6z-11zz+6z^3-z^4}{24}} \times \overline{\frac{24}{1000}}\big|^{4} + \&c.$ Therefore the said series $1 + z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \overline{\frac{24}{1000}}\big|^{2} + \frac{2z-3zz+z^3}{6} \times \overline{\frac{24}{1000}}\big|^{3} - \overline{\frac{6z-11zz+6z^3-z^4}{24}} \times \overline{\frac{24}{1000}}\big|^{4} + \&c$, will be = 1.002,084, 180,004,486,389, and consequently (subtracting 1 from both sides) the series $z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \overline{\frac{24}{1000}}\big|^{2} + \frac{2z-3zz+z^3}{6} \times \overline{\frac{24}{1000}}\big|^{3} - \overline{\frac{6z-11zz+6z^3-z^4}{24}} \times \overline{\frac{24}{1000}}\big|^{4} + \&c$, will be = 0.002,084,180,004,486,389, &c. This equation we must now endeavour to resolve.

150. To find the value of z in this equation we will, first, suppose the first term $z \times \frac{24}{1000}$ of the last-mentioned series to be, alone, equal to the whole series, and consequently to the absolute term 0.002,084,180,004,486,389, &c; and we shall then have $z \times 24 = 1000 \times 0.002,084,180,004,486,389$, &c $= 2.084,180,004,486,389$, &c, and consequently $z = \frac{2.084,180,004,486,389, \&c,}{24}$ = 0.086,840, &c.

Q. E. I.

Therefore x, or $97 + z$, will be = 97 + 0.086,840, &c, or 97.086,840, &c; of which number the first four figures 97.08 are exact, the more accurate value of x being 97.087,787,353,856,001,437, as I have found by dividing 1 by 0.010,299,956,639,811,952,13, which is the logarithm of the ratio of 1024 to 1000. And of these four figures, 97.08, which are exact, the two last, to wit, .08, have been obtained by the resolution of this very easy simple equation.

151. This first value of z (which is obtained by resolving the simple equation $z \times \frac{24}{1000} = 0.002,084,180,004,486,389$) must be less than the true value of z in the infinite equation set down in the latter part of art. 149; because the second and other following terms of the infinite series contained in that equation are marked alternately with the sign — and the sign +, or are alternately sub-

4 tracted

tracted from, and added to, the firſt term of it, the conſequence of which is (as the terms form a decreaſing progreſſion) that the firſt term alone muſt be greater than the whole ſeries, and the firſt and ſecond together, that is, the exceſs of the firſt above the ſecond, muſt be leſs than the whole ſeries, and, in like manner, that the three firſt terms muſt be greater, and the four firſt terms muſt be leſs, and every following odd number of terms muſt be greater, and every following even number of terms muſt be leſs, than the whole ſeries. Therefore, when the firſt term is ſuppoſed to be equal to the whole ſeries, or to the abſolute term 0.002,084,180,004,486,389, it is ſuppoſed to be leſs than it really is, and the value of it derived from ſuch ſuppoſition will be leſs than its true value. Therefore 0.086,840 is leſs than the true value of z. Q. E. D.

152. In the next place we will ſuppoſe the two firſt terms, $z \times \frac{24}{1000} - \left(\frac{z-zz}{2} \times \frac{24}{1000}\right)^2$, of the foregoing ſeries to be equal to the whole ſeries, and conſequently to the abſolute term 0.002,084,180,004,486,389, and will reſolve the quadratick equation reſulting from ſuch ſuppoſition, which (from what has been ſhewn in the foregoing article) will give us a value of z ſomewhat greater than the truth, but which will come nearer to it than the laſt value 0.086,840.

Now theſe two terms $z \times \frac{24}{1000} - \left(\frac{z-zz}{2} \times \frac{24}{1000}\right)^2$ are $= z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \frac{\overline{24}|^2}{\overline{1000}|^2} = z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \frac{576}{1000,000} = z \times \frac{24}{1000} - \frac{576z-576zz}{2 \times 1000,000} = z \times \frac{24}{1000} \times \frac{2 \times 1000}{2 \times 1000} - \frac{576z-576zz}{2 \times 1000,000} = \frac{48,000z}{2 \times 1000,000} - \frac{576z-576zz}{2 \times 1000,000} = \frac{48,000z-576z+576zz}{2 \times 1000,000} = \frac{47,424z+576zz}{2 \times 1000,000} = \frac{23,712z+288zz}{1000,000}$. Therefore $\frac{23,712z+288zz}{1000,000}$ will be $=$ 0.002,084,180,004,486,389, and conſequently $23,712z + 288zz$ will be ($=$ 1000,000 $\times$ 0.002,084,180,004,486,389) $=$ 2,084.180,004,486,389, and (dividing all the terms by 288) $zz +$ 82.333,333, &c, $\times z$ will be $=$ 7.236,736,126,688. Therefore (adding $\overline{41.166,666, \&c}|^2$ to both ſides) we ſhall have $zz +$ 82.333,333, &c, $\times z + \overline{41.166,666, \&c}|^2$ ($=$ 7.236,736,126,688 $+ \overline{41.166,666, \&c}|^2 =$ 7.236,736,126,688 $+$ 1694.694,389,555,556) $=$ 1701.931,125,682,244, and conſequently $z +$ 41.166,666, &c ($= \sqrt{}$ 1701.931,125,682,244) $=$ 41.254,467, and z ($=$ 41.254,467 $-$ 41.166,666) $=$ 0.087,801. Therefore 0.087,801 is a ſecond approximation to the true value of z in the infinite equation ſet down at the end of art. 149, or in the equation $\overline{1 + \frac{24}{1000}}|^z =$ 1.002,084,180,004,486,389. Q. E. I.

153. The arithmetical mean between 0.086,840, and 0.087,801 is 0.087,320. But the true value of z is much nearer to 0.087,801 than to 0.086,840. Therefore it muſt be greater than the ſaid arithmetical mean between them, to wit, 0.087,320, and probably not much leſs than 0.087,801, the latter and greater of the two foregoing approximations to it. It ſeems reaſonable there-

fore to conjecture that it is equal to 0.087,7. We will therefore ſuppoſe it to be equal to 0.087,7, and conſequently ſuppoſe x, or $97 + z$, to be equal to 97.0877, and ſhall proceed to make a trial whether this number is greater or leſs than the true value of x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$, by raiſing the binomial quantity $1 + \frac{24}{1000}$ to the 97.0877th, or $\frac{970,877}{10,000}$th power by means of the binomial theorem.

154. Now this has been done above in art. 136, and it has appeared that the ſaid power of the binomial quantity $1 + \frac{24}{1000}$ is equal to the mixt number 9.999,979,282,720,950,507,346, &c, which is very little leſs than 10. Therefore the number 97.0877 muſt be very little leſs than the true value of the index x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$.

Let y be the exceſs whereby the true value of x exceeds 97.0877. Then will $97.0877 + y$ be $= x$, and $\overline{1 + \frac{24}{1000}}|^{x}$ will be $= \overline{1 + \frac{24}{1000}}|^{97.0877+y}$ and conſequently $\overline{1 + \frac{24}{1000}}|^{97.0877+y}$ will be $= 10$. But $\overline{1 + \frac{24}{1000}}|^{97.0877+y}$ is $= \overline{1 + \frac{24}{1000}}|^{97.0877} \times \overline{1 + \frac{24}{1000}}|^{y} = 9.999,979,282,720,950,507,346, \&c \times \overline{1 + \frac{24}{1000}}|^{y}$. Therefore $9.999,979,282,720,950,507,346, \&c \times \overline{1 + \frac{24}{1000}}|^{y}$ will be $= 10$, and conſequently $\overline{1 + \frac{24}{1000}}|^{y}$ will be $\left(= \frac{10.000,000,000,000,000,000,000,}{9.999,979,282,720,950,507,346,}\right) = 1.000,002,071,732,197,014, \&c.$ But, by the binomial theorem, $\overline{1 + \frac{24}{1000}}|^{y}$ is = the infinite ſeries $1 + y \times \frac{24}{1000} + y \times \frac{y-1}{2} \times \overline{\frac{24}{1000}}|^{2} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \overline{\frac{24}{1000}}|^{3} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \overline{\frac{24}{1000}}|^{4} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \frac{y-4}{5} \times \overline{\frac{24}{1000}}|^{5} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \frac{y-4}{5} \times \frac{y-5}{6} \times \overline{\frac{24}{1000}}|^{6} + \&c = 1 + y \times \frac{24}{1000} + \frac{yy-y}{2} \times \overline{\frac{24}{1000}}|^{2} + \frac{y^3-3yy+2y}{6} \times \overline{\frac{24}{1000}}|^{3} + \frac{y^4-6y^3+11yy-6y}{24} \times \overline{\frac{24}{1000}}|^{4} + \frac{y^5-10y^4+35y^3-50yy+24y}{120} \times \overline{\frac{24}{1000}}|^{5} + \frac{y^6-15y^5+85y^4-235y^3+274yy-120y}{720} \times \overline{\frac{24}{1000}}|^{6} + \&c =$ (becauſe y is leſs than 1, and conſequently yy is leſs than y, and y^3 than yy, and y^4 than y^3, and every following power of y leſs than the next preceeding power of it), $1 + y \times \frac{24}{1000} - \frac{y-yy}{2} \times \overline{\frac{24}{1000}}|^{2} + \frac{2y-3yy+y^3}{6} \times \overline{\frac{24}{1000}}|^{3} - \frac{6y-11yy+6y^3-y^4}{24} \times \overline{\frac{24}{1000}}|^{4} + \frac{24y-50yy+35y^3-10y^4+y^5}{120} \times \overline{\frac{24}{1000}}|^{5} -$

4 120y—

$\left(\frac{120y - 274yy + 235y^3 - 85y^4 + 15y^5 - y^6}{720} \times \frac{24}{1000}\right)^6$ + &c. Therefore this series $1 + y \times \frac{24}{1000} - \left(\frac{y-yy}{2} \times \frac{24}{1000}\right)^2 + \frac{2y - 3yy + y^3}{6} \times \left(\frac{24}{1000}\right)^3 - \left(\frac{6y - 11yy + 6y^3 - y^4}{24} \times \frac{24}{1000}\right)^4$ + &c will be = 1.000,002,071,732,197,014; and consequently (subtracting 1 from both sides) the series $y \times \frac{24}{1000} - \left(\frac{y-yy}{2} \times \frac{24}{1000}\right)^2 + \frac{2y - 3yy + y^3}{6} \times \left(\frac{24}{1000}\right)^3 - \left(\frac{6y - 11yy + 6y^3 - y^4}{24} \times \frac{24}{1000}\right)^4$ + &c will be = 0.000,002,071,732,197,014. This equation we must now endeavour to resolve.

155. Now, if we suppose the first term, $y \times \frac{24}{1000}$, of this series to be equal to the whole series, and consequently to the absolute term 0.000,002,071,732, 197,014, we shall have $y \times 24$ (= 1000 × 0.000,002,071,732,197,014) = 0.002,071,732,197,014, and consequently y $\left(= \frac{0.002,071,732,197,014}{24}\right)$ = 0.000, 086,322,1. Therefore x, or 97.0877 + y, will be = 97.087,786,322,1, of which the first seven figures 97.087,78 are exact, the more accurate value of x being (as we before observed in art. 150) 97.087,787,353,856,001,437.

156. In the next place we will suppose the two first terms, $y \times \frac{24}{1000} - \left(\frac{y-yy}{2} \times \frac{24}{1000}\right)^2$, of the foregoing series, to be equal to the whole series, and consequently to the absolute term 0.000,002,071,732,197,014, and will resolve the quadratick equation thence resulting.

Now $y \times \frac{24}{1000} - \left(\frac{y-yy}{2} \times \frac{24}{1000}\right)^2$ is $= y \times \frac{24}{1000} \times \frac{2 \times 1000}{2 \times 1000} - \frac{y-yy}{2} \times \frac{576}{1000,000} = \frac{48,000y}{2 \times 1000,000} - \frac{576y - 576yy}{2 \times 1000,000} = \frac{48,000y - 576y + 576yy}{2 \times 1000,000} = \frac{47,424y + 576yy}{2 \times 1000,000} = \frac{23,712y + 288yy}{1000,000}$. Therefore $\frac{23,712y + 288yy}{1000,000}$ will be = 0.000,002,071,732,197,014, and consequently $23,712y + 288yy$ will be (= 1000,000 × 0.000,002,071, 732,197,014) = 2.071,732,197,014, and (dividing all the terms by 288) 82. 333,333, &c × y + yy will be = 0.007,193,514,572. Therefore, if we add $\overline{41.166,666, \&c}|^2$, or $\left(41 + \frac{1}{6}\right)^2$, to both sides, we shall have $\overline{41.166,666, \&c}|^2$ + 82.333,333, &c × y + yy (= 0.007,193,514,572 + $\left(41 + \frac{1}{6}\right)^2$ = 0.007, 193,514,572 + $\overline{41}|^2$ + 2 × 41 × $\frac{1}{6}$ + $\frac{1}{6} \times \frac{1}{6}$ = 0.007,193,514,572 + 1681 + 41 × $\frac{1}{3}$ + 0.166,666,666,666, &c × $\frac{1}{6}$ = 0.007,193,514,572 + 1681 + 13.666,666,666,666, + $\frac{0.166,666,666,666}{6}$ = 0.007,193,514,572 + 1694.666,666,666,666, + 0.027,777,777,777 = 0.007,193,514,572 + 1694.694,444,444,444) = 1694.701,637,959,015. Therefore 41.166,666, 666,

666, &c $+ y$ will be ($= \sqrt{1694.701,637,959,015}$) $= 41.166,754,037$, and y will be ($= 41.166,754,037 - 41.166,666,666$) $= 0.000,087,371$, and consequently x, or $97.0877 + y$, will be $= 97.087,787,371$; of which number the first nine figures 97.087,787,3 are exact, the more accurate value of x being 97.087,787,353,856,001,437. This is a very considerable degree of exactness.

157. In order to obtain the value of y to a still greater degree of exactness, it will be convenient to set down the infinite equation that is obtained in art. 154, in a different manner, as follows; to wit,

$$\begin{aligned} y \times \frac{24}{1000} - \frac{y}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{2y}{6} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{6y}{24} \times \overline{\frac{24}{1000}}\Big|^{4} &+ \&c \\ + \frac{yy}{2} \times \overline{\frac{24}{1000}}\Big|^{2} - \frac{3yy}{6} \times \overline{\frac{24}{1000}}\Big|^{3} + \frac{11yy}{24} \times \overline{\frac{24}{1000}}\Big|^{4} &- \&c \\ + \frac{y^3}{6} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{6y^3}{24} \times \overline{\frac{24}{1000}}\Big|^{4} &+ \&c \\ + \frac{y^4}{24} \times \overline{\frac{24}{1000}}\Big|^{4} &- \&c \end{aligned}$$

$= 0.000,002,071,732,197,014.$

Now, since y is a very small quantity (being less than 0.000,088, and, *à fortiori*, less than 0.0001, or $\frac{1}{10,000}$), it follows that yy and y^3 and y^4 and all the following powers of y will be very small in comparison of y; and consequently, if we omit all the terms in the foregoing equation which involve yy, or any higher power of y, the series which forms the left-hand side of the said equation will not be much affected thereby, but the remaining terms of it (which are those that involve only the simple power of y) will be very nearly equal to the absolute term 0.000,002,071,732,197,014; that is, the upper line of terms alone, to wit, $y \times \frac{24}{1000} - \frac{y}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{2y}{6} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{6y}{24} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{24y}{120} \times \overline{\frac{24}{1000}}\Big|^{5} - \frac{120y}{720} \times \overline{\frac{24}{1000}}\Big|^{6} + \&c$, will be very nearly equal to 0.000,002,071,732,197,014. But this line of terms is $= y \times \frac{24}{1000} - \frac{y}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{y}{3} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{y}{4} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{y}{5} \times \overline{\frac{24}{1000}}\Big|^{5} - \frac{y}{6} \times \overline{\frac{24}{1000}}\Big|^{6} + \&c = y \times$ the series $\frac{24}{1000} - \frac{1}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{1}{3} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{1}{4} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{1}{5} \times \overline{\frac{24}{1000}}\Big|^{5} - \frac{1}{6} \times \overline{\frac{24}{1000}}\Big|^{6} + \&c =$ (if we put A for the first term $\frac{24}{1000}$ of the said series, and B for its second term $\frac{1}{2} \times \overline{\frac{24}{1000}}\Big|^{2}$, and C for its third term $\frac{1}{3} \times \overline{\frac{24}{1000}}\Big|^{3}$, and D, E, F, G, H, &c, for its fourth, fifth, sixth, seventh, eighth, and other following terms respectively) $y \times$ the series $\frac{24}{1000} - \frac{1}{2}A \times \frac{24}{1000} + \frac{2}{3}B \times \frac{24}{1000} - \frac{3}{4}C \times \frac{24}{1000} + \frac{4}{5}D \times \frac{24}{1000} - \frac{5}{6}E \times \frac{24}{1000} + \frac{6}{7}F \times \frac{24}{1000} - \frac{7}{8}G \times \frac{24}{1000} + \frac{8}{9}H \times \frac{24}{1000} - \frac{9}{10}I \times \frac{24}{1000} + \frac{10}{11}K \times \frac{24}{1000} - \frac{11}{12}L \times \frac{24}{1000} + \&c = y \times$ the series

$$\begin{array}{rll} & 0.024,000,000,000,000,000,000, & A, \\ - & 0.000,288,000,000,000,000,000, & B, \\ + & 0.000,004,608,000,000,000,000, & C, \\ - & 0.000,000,082,944,000,000,000, & D, \\ + & 0.000,000,001,592,524,800,000, & E, \\ - & 0.000,000,000,031,850,496,000, & F, \\ + & 0.000,000,000,000,655,210,203, & G, \\ - & 0.000,000,000,000,013,759,414, & H, \\ + & 0.000,000,000,000,000,293,534, & I, \\ - & 0.000,000,000,000,000,006,222, & K, \\ + & 0.000,000,000,000,000,000,135, & L, \\ - & 0.000,000,000,000,000,000,002, & M, \\ + & \&c = & \\ & 0.024,004,609,593,180,303,872 & \times y \\ - & 0.000,288,082,031,864,261,638 & \times y \\ = & 0.023,716,527,561,316,042,234 & \times y. \end{array}$$

Therefore 0.023,716,527,561,316,042,234 $\times y$ will be very nearly equal to the absolute term 0.000,002,071,732,197,014; and consequently y will be very nearly equal to $\frac{0.000,002,071,732,197,014}{0.023,716,527,561,316,042,234}$, or 0.000,087,353,943,011. Therefore x, or 97.0877 $+ y$, will be $=$ 97.087,787,353,943,011; of which number the first eleven figures, 97.087,787,353, are exact, the more accurate value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$ being 97.087,787,353,856,001, 437.

158. We have now obtained the value of the index x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$ exact to eleven places of figures. And, if we wish to obtain it to a still greater degree of exactness, we need only retain a few of those terms of the compound infinite series set down in art. 157 which involve the square of y, and which constitute the second horizontal row of terms in the said compound series. Now the first five terms of the said second horizontal row are $+ \frac{yy}{2} \times \overline{\frac{24}{1000}}\Big|^{2} - \frac{3yy}{6} \times \overline{\frac{24}{1000}}\Big|^{3} + \frac{11yy}{24} \times \overline{\frac{24}{1000}}\Big|^{4} - \frac{50yy}{120} \times \overline{\frac{24}{1000}}\Big|^{5} + \frac{274yy}{720} \times \overline{\frac{24}{1000}}\Big|^{6}$, which are equal to $yy \times$ the series $\frac{1}{2} \times \overline{\frac{24}{1000}}\Big|^{2} - \frac{3}{6} \times \overline{\frac{24}{1000}}\Big|^{3} + \frac{11}{24} \times \overline{\frac{24}{1000}}\Big|^{4} - \frac{50}{120} \times \overline{\frac{24}{1000}}\Big|^{5} + \frac{274}{720} \times \overline{\frac{24}{1000}}\Big|^{6} = yy \times$ the series $\frac{1}{2} \times \overline{0.024}|^{2} - \frac{1}{2} \times \overline{0.024}|^{3} + \frac{11}{24} \times \overline{0.024}|^{4} - \frac{5}{12} \times \overline{0.024}|^{5} + \frac{137}{360} \times \overline{0.024}|^{6} = yy \times$ the series $\frac{1}{2} \times 0.000,576 - \frac{1}{2} \times 0.000,013,824 + \frac{11}{24} \times 0.000,000,331,776 - \frac{5}{12} \times 0.000,000,007,962,624 + \frac{137}{360} \times 0.000,000,000,031,850,496 = yy \times$ the series

0.000,288,

$$\left.\begin{array}{r} 0.000{,}288{,}000{,}000{,}000{,}000{,}000, \\ -\ 0.000{,}006{,}912{,}000{,}000{,}000{,}000, \\ +\ 0.000{,}000{,}152{,}064{,}000{,}000{,}000, \\ -\ 0.000{,}000{,}003{,}317{,}760{,}000{,}000, \\ +\ 0.000{,}000{,}000{,}012{,}120{,}883{,}200, \end{array}\right\} = yy \times$$

0.000,288,152,076,120,883,200 − yy × 0.000,006,915,317,760,000,000 = yy × 0.000,281,236,758,360,883,200. Therefore the two upper horizontal lines of the compound infinite ſeries ſet down in art. 157 are = 0.023,716,527, 561,316,042,234, × y + 0.000,281,236,758,360,883,200, × yy; and conſequently theſe two terms 0.023,716,527,561,316,042,234 × y + 0.000,281, 236,758,360,883,200 × yy will be very nearly equal to the abſolute term 0.000,002,071,732,197,014. This quadratick equation we muſt now endeavour to reſolve.

This equation may be moſt conveniently reſolved by approximation, by ſubſtituting, inſtead of y, in the quantity 0.000,281,236,758,360,883,200 × yy the value of y derived from the ſimple equation 0.023,716,527,561,316,042,234 × y = 0.000,002,071,732,197,014, which is = $\left(\frac{0.000,002,071,732,197,014}{0.023,716,527,561,316,042,234},\right.$ or) 0.000,087,353,943,011. We ſhall then have yy = $\overline{0.000,087,353,94}|^2$ = 0.000,000,007,630,710,833,523,6, or (dropping the laſt ſeven figures), 0.000,000,007,630,710, and conſequently 0.000,281,236,758,360,883,200 × yy (= 0.000,281,236,758,360,883,200 × 0.000,000,007,630,710) = 0.000, 000,000,002,146,036,144,391,975,043,072. Therefore 0.023,716,527,561, 316,042,234 × y + 0.000,000,000,002,146,036,144,391,975,043,072 will be (= 0.023,716,527,561,316,042,234 × y + 0.000,281,236,758,360,883, 200 × yy) = 0.000,002,071,732,197,014, and conſequently (dividing all the terms by 0.023,716,527,561,316,042,234) y + 0.000,000,000,090,487 will be = 0.000,087,353,943,011, and y will be =

$$\begin{array}{r} 0.000{,}087{,}353{,}943{,}011 \\ -\ 0.000{,}000{,}000{,}090{,}487 \end{array}$$

= 0.000,087,353,852,524. Therefore x, or 97.0877 + y will be = 97.087, 787,353,852,524; of which number the firſt thirteen figures 97.087,787,353, 85 are exact, the more accurate value of x being (as we have before obſerved) 97.087,787,353,856,001,437. We have therefore now obtained the value of x, or the index of the power of the binomial quantity $1 + \frac{24}{1000}$ in the original equation $\overline{1 + \frac{24}{1000}}|^x = 10$, exact to thirteen places of figures.

Q. E. I.

159. The ratio of $\overline{1 + \frac{24}{1000}}|^x$, or 10, to 1 is to the ratio of $1 + \frac{24}{1000}$ to 1 as x is to 1, and conſequently as 1 is to the fraction $\frac{1}{x}$; ſo that, if 1 be taken for the repreſentative of the ratio of 10 to 1 (as it is in Briggs's Syſtem of Logarithms), the fraction $\frac{1}{x}$ will be the repreſentative of the ratio of $1 + \frac{24}{1000}$ to 1, that

that is, in other words, the fraction $\frac{1}{x}$ will be Briggs's logarithm of the ratio of $1 + \frac{24}{1000}$ to 1. Therefore Briggs's logarithm of the ratio of $1 + \frac{24}{1000}$ to 1, or of the ratio of 1024 to 1000, will be ($= \frac{1.000,000,000,000,000,000}{97.087,787,353,852,524}$) $=$ 0.010, 299,956,639,812. Therefore the logarithm of the ratio of 1024 to 1 will be ($=$ log. $\frac{1024}{1000}$ + log. $\frac{1000}{1}$ = 0.010,299,956,639,812 + log. $\frac{1000}{1}$ = 0.010,299, 956,639,812 + 3) = 3.010,299,956,639,812; and consequently the logarithm of the ratio of 2 (which is the 10th root of 1024) to 1 will be ($= \frac{3.010,299,956,639,812}{10}$) = 0.301,029,995,663,981,2; or, according to the common mode of expression, the logarithm of the number 2 will be = 0.301,029,995, 663,981,2. Q. E. I.

This value of the logarithm of 2 is exact to fifteen places of figures, and errs only in the 16th, or last, figure, which should be an unit instead of a 2, the more accurate value of this logarithm (according to Mr. Abraham Sharp's computation) being 0.301,029,995,663,981,195,213,738,894,724,493.

160. As the solution of the foregoing problem consists of a great number of steps, which, for the ease of the reader, have been set forth distinctly and at considerable length, it will not be amiss to take a short review of all the foregoing processes, and of the several gradual approximations, to the value of x, (or of the index of the power of the binomial quantity $1 + \frac{24}{1000}$ in the equation $\overline{1 + \frac{24}{1000}}|^x = 10$) which have been obtained by means of them.

A review of the several steps of the foregoing resolution of the equation $\overline{1 + \frac{24}{1000}}|^x = 10$, and computation of the logarithm of 2.

161. The first step towards finding the value of the index x in the equation $\overline{1 + \frac{24}{1000}}|^x = 10$ was to expand the quantity $\overline{1 + \frac{24}{1000}}|^x$ into an infinite series by means of Sir Isaac Newton's binomial theorem, by which we obtained the equation $1 + x \times \frac{24}{1000} + \frac{xx - x}{2} \times \overline{\frac{24}{1000}}|^2 + \frac{x^3 - 3xx + 2x}{6} \times \overline{\frac{24}{1000}}|^3 + \frac{x^4 - 6x^3 + 11x^2 - 6x}{24} \times \overline{\frac{24}{1000}}|^4 + \frac{x^5 - 10x^4 + 35x^3 - 50x^2 + 24x}{120} \times \overline{\frac{24}{1000}}|^5 + \&c = 10$, and consequently (by subtracting 1 from both sides) the equation $x \times \frac{24}{1000} +$

$\frac{xx - x}{2}$

$$\frac{xx-x}{2} \times \overline{\left.\frac{24}{1000}\right|}^{2} + \frac{x^3-3xx+2x}{6} \times \overline{\left.\frac{24}{1000}\right|}^{3} + \frac{x^4-6x^3+11x^2-6x}{24} \times \overline{\left.\frac{24}{1000}\right|}^{4} + \frac{x^5-10x^4+35x^3-50xx+24x}{120} \times \overline{\left.\frac{24}{1000}\right|}^{5} + \&c = 9.$$

162. We then fuppofed, firft, one term, then two terms, then three terms, and laftly, four terms, of the feries which forms the left-hand fide of this laft equation, to be equal to the whole feries, and confequently to the abfolute term 9, and we refolved the feveral equations refulting from thofe fuppofitions.

By refolving the fimple equation refulting from the firft of thefe fuppofitions, to wit, the fimple equation $x \times \frac{24}{1000} = 9$, we found x to be equal to 375; which was therefore our firft approximation to the value of x in the original equation $1 + \overline{\left.\frac{24}{1000}\right|}^{x} = 10$. This approximation is very wide of the true value of x, being more than three times as great.

From the fecond fuppofition, to wit, that the two terms $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\left.\frac{24}{1000}\right|}^{2}$ were equal to the whole feries, and confequently to 9, there refulted the quadratick equation $xx + 82.333,333$, &c $\times x = 31,250$; by the refolution of which x appeared to be $= 140.34$; which was therefore our fecond approximation to the value of x in the original equation $1 + \overline{\left.\frac{24}{1000}\right|}^{x} = 10$. This approximation is much nearer to the truth than the former, but yet is confiderably too large.

From the third fuppofition, to wit, that the three terms $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\left.\frac{24}{1000}\right|}^{2} + \frac{x^3-3xx+2x}{6} \times \overline{\left.\frac{24}{1000}\right|}^{3}$ were equal to the whole feries, and confequently to 9, there refulted the cubick equation $x^3 + 122xx + 10,293.666,666$, &c $\times x = 3,906,250$; by the refolution of which in a grofs manner, by conjecturing x to be equal, firft, to 100, and then (upon finding it to be greater than 100), to 110, we found it to be fomewhat lefs than 110; which was therefore our third approximation to the value of x in the original equation $1 + \overline{\left.\frac{24}{1000}\right|}^{x} = 10$.

And from the fourth fuppofition, to wit, that the four terms $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\left.\frac{24}{1000}\right|}^{2} + \frac{x^3-3xx+2x}{6} \times \overline{\left.\frac{24}{1000}\right|}^{3} + \frac{x^4-6x^3+11xx-6x}{24} \times \overline{\left.\frac{24}{1000}\right|}^{4}$ are equal to the whole feries, and confequently to 9, there refulted the biquadratick equation $x^4 + 160.666,666$, &c $\times x^3 + 20,344.333,333$, &c $\times xx + 1,715,605.111,111$, &c $\times x = 651,041,666.666,666$, &c; by the refolution of which in a grofs manner, by a conjecture and trial, we found x to be fomewhat greater than 100; which was therefore our fourth approximation to the value of x in the original equation $1 + \overline{\left.\frac{24}{1000}\right|}^{x} = 10$.

163. Having

163. Having thus obtained the numbers 375, 140, 110, and 100 for our four first approximations to the value of x in the equation $\overline{1 + \frac{24}{1000}}\big|^{x} = 10$, we observed that the difference between the second and third approximations, to wit, 140 and 110, was 30, and that the difference between the third and fourth approximations, to wit, 110 and 100, was only 10, or one third part of the preceeding difference; and we were thereby led to conjecture that the difference between the fourth approximation 100 and the true value of x in the equation $\overline{1 + \frac{24}{1000}}\big|^{x} = 10$ would probably be less than a third part of the last difference 10, or would be nearly equal to 3, and consequently that the true value of x in the equation $\overline{1 + \frac{24}{1000}}\big|^{x} = 10$ would be nearly equal to $100 - 3$, or 97, or that 97 would be a fifth approximation to the said true value of x. And we then tried whether the said true value of x was greater or less than 97, by raising the binomial quantity $1 + \frac{24}{1000}$ to the 97th power by means of the binomial theorem; and we found the said 97th power of $1 + \frac{24}{1000}$ to be equal to 9.979,201,547,673,599,050, which is somewhat less than 10; whence it followed that 97 must be somewhat less than the true value of x in the equation $\overline{1 + \frac{24}{1000}}\big|^{x} = 10$. We then multiplied the number 9.979,201,547,673,599,050 (which is equal to the 97th power of $1 + \frac{24}{1000}$) into $1 + \frac{24}{1000}$, or 1.024, and found the product to be $=$ 10.218,702,384,817,765,427,200, which is greater than 10, and we therefore concluded that the 98th power of $1 + \frac{24}{1000}$ was greater than 10, and consequently that the true value of x in the equation $\overline{1 + \frac{24}{1000}}\big|^{x} = 10$ was greater than 97, but less than 98.

164. We then put z for the unknown difference by which the true value of x in the equation $\overline{1 + \frac{24}{1000}}\big|^{x} = 10$ exceeds 97, so that x was $= 97 + z$. And we thereby had $\overline{1 + \frac{24}{1000}}\big|^{97+z} \left(= \overline{1 + \frac{24}{1000}}\big|^{x}\right) = 10$.

But $\overline{1 + \frac{24}{1000}}\big|^{97+z}$ is $= \overline{1 + \frac{24}{1000}}\big|^{97} \times \overline{1 + \frac{24}{1000}}\big|^{z}$; and $\overline{1 + \frac{24}{1000}}\big|^{97}$ had been found to be $=$ 9.979,201,547,673,599,050, &c. Therefore 9.979,201,547,673,599,050 $\times \overline{1 + \frac{24}{1000}}\big|^{z}$ is $\left(= \overline{1 + \frac{24}{1000}}\big|^{97} \times \overline{1 + \frac{24}{1000}}\big|^{z} = \overline{1 + \frac{24}{1000}}\big|^{97+z}\right) = 10$, and consequently $\overline{1 + \frac{24}{1000}}\big|^{z}$ is $\left(= \frac{10.000,000,000,000,000,000,000}{9.979,201,547,673,599,050, \text{ \&c.}}\right) = 1.002,084,180,004,486,389$. And thus we obtained a new equation, in which the unknown index z of the power of the bi-

 nomial

nomial quantity $1 + \frac{24}{1000}$ is lefs than an unit, inftead of the original equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$, in which the index x of the power of the fame quantity is greater than 97. In confequence of this change of equations the fubfequent approximations to the true value of x, or $97 + z$, became much fwifter than they were before.

165. We then expanded the quantity $\overline{1 + \frac{24}{1000}}|^{z}$ into an infinite feries by the binomial theorem, and thereby obtained the equation $1 + z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \overline{\frac{24}{1000}}|^{2} + \frac{2z-3zz+z^3}{6} \times \overline{\frac{24}{1000}}|^{3} - \overline{\frac{6z-11zz+6z^3-z^4}{24}} \times \overline{\frac{24}{1000}}|^{4} + \&c = 1.002,084,180,004,486,389$, and (by fubtracting 1 from both fides) the equation $z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \overline{\frac{24}{1000}}|^{2} + \frac{2z-3zz+z^3}{6} \times \overline{\frac{24}{1000}}|^{3} - \overline{\frac{6z-11zz+6z^3-z^4}{24}} \times \overline{\frac{24}{1000}}|^{4} + \&c = 0.002,084,180,004,486,389$, &c.

166. We then proceeded to find approximations to the value of z in this new equation in the fame manner as we had before found approximations to the value of x in the former equation $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \overline{\frac{24}{1000}}|^{2} + \frac{x^3-3xx+2x}{6} \times \overline{\frac{24}{1000}}|^{3} + \frac{x^4-6x^3+11x^2-6x}{24} \times \overline{\frac{24}{1000}}|^{4} + \&c = 9$, by, firft, fuppofing the firft term, $z \times \frac{24}{1000}$, alone, of the feries which forms the left-hand fide of the equation, to be equal to the whole feries, and confequently to the abfolute term 0.002,084, 180,004,486,389, &c, and then fuppofing the two firft terms, $z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}} \times \overline{\frac{24}{1000}}|^{2}$, of the fame feries to be equal to the fame quantity, and by refolving the equations refulting from thofe fuppofitions.

From the firft of thofe fuppofitions we had the fimple equation $z \times 24 = 2.084,180,004,486,389$, &c; by the refolution of which we had $z = 0.086,840$, and confequently x, or $97 + z$, $= 97.086,840$; of which number the four firft figures, 97.08, are exact; the more accurate value of x in the original equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$ being 97.087,787,353,856,001,437, as I have found by dividing 1 by 0.010,299,956,639,811,952,13, which is the logarithm of the ratio of 1024 to 1000. And of thefe four figures, 97.08, which are exact, the two laft, to wit, .08, were obtained by the refolution of the foregoing very eafy fimple equation.

This number, 97.086,840, is therefore the fixth approximation to the true value of x in the original equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$.

From the fecond fuppofition there refulted the quadratick equation $zz + 82.333,333$,

333,333, &c $\times z = 7.236,736,126,688$; by the resolution of which we had $z = 0.087,801$, and consequently x, or $97 + z$, $= 97.087,801$, of which number the five first figures, 97.087, are exact. Therefore 97.087,801 was a seventh approximation to the value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$.

167. We then found the arithmetical mean between the two last values of z, to wit, 0.086,840, and 0.087,801, which was 0.087,320; and we observed that this mean must be less than the truth, because the second value of z, to wit, 0.087,801, must be much nearer to its true value than the first value of it, to wit, 0.086,840. And therefore we conjectured that the true value of z (being greater than 0.087,320, and, probably not much less than 0.087,801) might be very nearly equal to 0.087,7, and consequently that the true value of x, or $97 + z$, might be very nearly equal to 97.087,7. And thus we obtained 97.0877 for an eighth approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$.

168. We then dropped all further consideration of the equation $z \times \frac{24}{1000} - \overline{\frac{z-zz}{2}}\Big| \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{2z-3zz+z^3}{6} \times \overline{\frac{24}{1000}}\Big|^{3} - \overline{\frac{6z-11zz+6z^3-z^4}{24}}\Big| \times \overline{\frac{24}{1000}}\Big|^{4} + \&c = 0.002,084,180,004,486,389$, &c, and made a trial of the exactness of the last value of x obtained by the foregoing processes, to wit, 97.0877, by raising the binomial quantity $1 + \frac{24}{1000}$ to the power of which 97.0877 is the index; which was done by the help of Sir Isaac Newton's binomial theorem. And we found that the said power of $1 + \frac{24}{1000}$ was $= 9.999,979,282,720,950,507,346$, which is very little less than 10; and we thence concluded that 97.0877 must be a very little less than the true value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$.

169. We then supposed x to be $= 97.0877 + y$, and consequently $\overline{1 + \frac{24}{1000}}\Big|^{97.0877+y}$ to be $= 10$.

Then, since $\overline{1 + \frac{24}{1000}}\Big|^{97.0877+y}$ is $= \overline{1 + \frac{24}{1000}}\Big|^{97.0877} \times \overline{1 + \frac{24}{1000}}\Big|^{y}$, we had $\overline{1 + \frac{24}{1000}}\Big|^{97.0877} \times \overline{1 + \frac{24}{1000}}\Big|^{y} = 10$, and consequently (because $\overline{1 + \frac{24}{1000}}\Big|^{97.0877}$ has been found to be $= 9.999,979,282,720,950,507,346$) $9.999,979,282,720,950,507,346 \times \overline{1 + \frac{24}{1000}}\Big|^{y} = 10$, and $\overline{1 + \frac{24}{1000}}\Big|^{y} = \frac{10.000,000,000,000,000,000,000,}{9.999,979,282,720,950,507,346}$. $= 1.000,002,071,732,197,014$, &c.

170. Having thus obtained a third equation $\overline{1 + \frac{24}{1000}}\Big|^{y} = 1.000,002,071,$ 732,197,014, &c, in which y is much smaller than z in the former equation, we

we proceeded to expand $\overline{1 + \frac{24}{1000}}|^{y}$ into an infinite series by means of Sir Isaac Newton's binomial theorem, and thereby obtained the equation $1 + y \times \frac{24}{1000} - \overline{\frac{y-yy}{2}} \times \overline{\frac{24}{1000}}|^{2} + \frac{2y-3yy+y^3}{6} \times \overline{\frac{24}{1000}}|^{3} - \overline{\frac{6y-11yy+6y^3-y^4}{24}} \times \overline{\frac{24}{1000}}|^{4} + \frac{24y-50yy+35y^3-10y^4+y^5}{120} \times \overline{\frac{24}{1000}}|^{5} - \overline{\frac{120y-274yy+235y^3-85y^4+15y^5-y^6}{720}} \times \overline{\frac{24}{1000}}|^{6}$ + &c $= 1.000,002,071,732,197,014$, &c, and (by subtracting 1 from both sides) the equation $y \times \frac{24}{1000} - \overline{\frac{y-yy}{2}} \times \overline{\frac{24}{1000}}|^{2} + \frac{2y-3yy+y^3}{6} \times \overline{\frac{24}{1000}}|^{3} - \overline{\frac{6y-11yy+6y^3-y^4}{24}} \times \overline{\frac{24}{1000}}|^{4} + \frac{24y-50yy+35y^3-10y^4+y^5}{120} \times \overline{\frac{24}{1000}}|^{5} - \overline{\frac{120y-274yy+235y^3-85y^4+15y^5-y^6}{720}} \times \overline{\frac{24}{1000}}|^{6}$ + &c $= 0.000,002,071,732,197,014$, &c.

171. We then proceeded to approximate to the value of y in this equation, by first supposing the first term, $y \times \frac{24}{1000}$, alone, of the series $y \times \frac{24}{1000} - \overline{\frac{y-yy}{2}} \times \overline{\frac{24}{1000}}|^{2}$ + &c (which forms the left-hand side of this equation), to be equal to the whole series, and consequently to the absolute term 0.000, 002,071,732,197,014, &c; and, secondly, by supposing the two first terms, $y \times \frac{24}{1000} - \overline{\frac{y-yy}{2}} \times \overline{\frac{24}{1000}}|^{2}$, of the said series to be equal to the same quantity, and resolving the equations resulting from these suppositions.

From the first of these suppositions we had the simple equation $y \times 24 = 0.002,071,732,197,014$, &c; by the resolution of which we had $y = 0.000,086,322,1$, and consequently x ($= 97.0877 + y$) $= 97.087,786,322,1$; of which number the first seven figures 97.087,78 are exact, the more accurate value of x being (as we have before observed) 97.087,787,353,856,001,437.

Therefore the number 97.087,786,322,1 is a ninth approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$.

And from the second of these suppositions there resulted the quadratick equation $yy + 82.333,333$, &c $\times y = 0.007,193,514,572$; by the resolution of which we had $y = 0.000,087,371$, and consequently x ($= 97.0877 + y$) $= 97.087,787,371$; of which number the first nine figures, 97.087,787,3, are exact.

This number 97.087,787,371, is therefore a tenth approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}|^{x} = 10$.

172. We then, in order to obtain the value of y to a still greater degree of exactness, had recourse to a different method of resolving the equation $y \times \frac{24}{1000}$

$- \overline{\frac{y-yy}{2}} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{2y-3yy+y^{3}}{6} \times \overline{\frac{24}{1000}}\Big|^{3} - \overline{\frac{6y-11yy+6y^{3}-y^{4}}{24}} \times \overline{\frac{24}{1000}}\Big|^{4} + \&c =$ 0.000,002,071,732,197,014, &c; which was grounded on the omiſſion of all the members in each term of the ſeries that involved either yy, or y^{3}, or y^{4}, or any other power of y, except the ſimple power, or y itſelf: by which means the ſaid equation was changed into the following ſimple equation, to wit, $y \times \frac{24}{1000} - \frac{y}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{2y}{6} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{6y}{24} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{24y}{120} \times \overline{\frac{24}{1000}}\Big|^{5} - \frac{120y}{720} \times \overline{\frac{24}{1000}}\Big|^{6} + \&c =$ 0.000,002,071,732,197,014, &c, or $y \times \frac{24}{1000} - \frac{y}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{y}{3} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{y}{4} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{y}{5} \times \overline{\frac{24}{1000}}\Big|^{5} - \frac{y}{6} \times \overline{\frac{24}{1000}}\Big|^{6} + \&c =$ 0.000,002,071, 732,197,014, &c, or $y \times$ the infinite ſeries $1 \times \frac{24}{1000} - \frac{1}{2} \times \overline{\frac{24}{1000}}\Big|^{2} + \frac{1}{3} \times \overline{\frac{24}{1000}}\Big|^{3} - \frac{1}{4} \times \overline{\frac{24}{1000}}\Big|^{4} + \frac{1}{5} \times \overline{\frac{24}{1000}}\Big|^{5} - \frac{1}{6} \times \overline{\frac{24}{1000}}\Big|^{6} + \&c =$ 0.000,002,071, 732,197,014, &c, or (if we put A for the firſt term $1 \times \frac{24}{1000}$ of this ſeries, and B for its ſecond term $\frac{1}{2} \times \overline{\frac{24}{1000}}\Big|^{2}$, and C, D, E, F, &c for its third, fourth, fifth, ſixth, and other following terms, reſpectively), $y \times$ the ſeries $1 \times \frac{24}{1000} - \frac{1}{2} A \times \frac{24}{1000} + \frac{2}{3} B \times \frac{24}{1000} - \frac{3}{4} C \times \frac{24}{1000} + \frac{4}{5} D \times \frac{24}{1000} - \frac{5}{6} E \times \frac{24}{1000} + \frac{6}{7} F \times \frac{24}{1000} - \frac{7}{8} G \times \frac{24}{1000} + \frac{8}{9} H \times \frac{24}{1000} - \frac{9}{10} I \times \frac{24}{1000} + \frac{10}{11} K \times \frac{24}{1000} - \frac{11}{12} L \times \frac{24}{1000} + \&c =$ 0.000,002,071,732,197,014, &c. We then computed the value of the ſaid infinite ſeries $1 \times \frac{24}{1000} - \frac{1}{2} A \times \frac{24}{1000} + \frac{2}{3} B \times \frac{24}{1000} - \frac{3}{4} C \times \frac{24}{1000} + \frac{4}{5} D \times \frac{24}{1000} - \frac{5}{6} E \times \frac{24}{1000} + \&c$, and found it to be = 0.023,716,527,561,316,042,234; which gave us the ſimple equation $y \times$ 0.023,716,527,561,316,042,234 = 0.000,002,071,732,197,014, &c, by the reſolution of which we had y ($= \frac{0.000,002,071,732,197,014}{0.023,716,527,561,316,042,234}$) = 0.000,087,353,943,011, and conſequently x (= 97.0877 + y) = 97.087, 787,353,943,011; of which number the firſt eleven figures, 97.087,787,353, are exact, the more accurate value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$ being (as we have before obſerved) 97.087,787,353,856,001,437.

This number 97.087,787,353,943,011, is therefore the eleventh approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}\Big|^{x} = 10$.

173. And, laſtly, to obtain the value of y to a ſtill greater degree of exactneſs, we retained in art. 158 the five firſt terms of the infinite ſeries ſet down in art.

art. 157 and 170, that involved the ſquare of y, to wit, the five terms $\frac{yy}{2} \times \overline{\frac{24}{1000}}\big|^2 - \frac{3yy}{6} \times \overline{\frac{24}{1000}}\big|^3 + \frac{11yy}{24} \times \overline{\frac{24}{1000}}\big|^4 - \frac{50yy}{120} \times \overline{\frac{24}{1000}}\big|^5 + \frac{274yy}{720} \times \overline{\frac{24}{1000}}\big|^6$, which are equal to $yy \times$ the ſeries $\frac{1}{2} \times \overline{\frac{24}{1000}}\big|^2 - \frac{3}{6} \times \overline{\frac{24}{1000}}\big|^3 + \frac{11}{24} \times \overline{\frac{24}{1000}}\big|^4 - \frac{50}{120} \times \overline{\frac{24}{1000}}\big|^5 + \frac{274}{720} \times \overline{\frac{24}{1000}}\big|^6 = yy \times 0.000,281,236,758,360,883,200$. And hence we obtained the quadratick equation $0.023,716,527,561,316,042,234 \times y + 0.000,281,236,758,360,883,200 \times yy = 0.000,002,071,732,197,014$, &c; which (being reſolved by approximation by ſubſtituting, inſtead of y, in the quantity $0.000,281,236,758,360,883,200 \times yy$ the value of y before obtained by the reſolution of the ſimple equation, to wit, $0.000,087,353,943,011$) gave us $y = 0.000,087,353,852,524$, and conſequently x $(= 97.0877 + y)$ $97.087,787,353,852,524$; of which number the firſt thirteen figures, $97.087,787,353,85$, are exact.

This number $97.087,787,353,852,524$ is therefore the twelfth approximation to the true value of x in the equation $\overline{1 + \frac{24}{1000}}\big|^x = 10$.

174. We then divided 1 by this laſt, or twelfth, near value of x, to wit, $97.087,787,353,852,524$, in order to obtain the value of $\frac{1}{x}$, or the logarithm of the ratio of $1 + \frac{24}{1000}$ to 1 in Briggs's ſyſtem; and we found the quotient to be $= 0.010,299,956,639,812$. And hence it followed that the logarithm of 2, or of the ratio of 2 to 1 (being $= \frac{1}{10} \log. \frac{1024}{1} = \frac{1}{10} \log. \frac{1024}{1000} + \frac{1}{10} \log. \frac{1000}{1} = \frac{1}{10} \log. \frac{1024}{1000} + \frac{1}{10} \times 3 = \frac{1}{10} \log. \frac{1024}{1000} + 0.3 = \frac{1}{10} \log. \frac{1.024}{1} + 0.3 = \frac{1}{10} \log. \frac{1 + \frac{24}{1000}}{1} + 0.3 = \frac{1}{10} \times 0.010,299,956,639,812 + 0.3$) would be $= 0.3 + 0.001,029,995,663,981,2$, or $0.301,029,995,663,981,2$; which is exact to 15 places of figures, the more accurate value of that logarithm (according to Mr. Abraham Sharp's computations) being $0.301,029,995,663,981,195,213,738,894,724,493$.

End of the review of the ſeveral ſteps of the foregoing reſolution of the equation $\overline{1 + \frac{24}{1000}}\big|^x = 10$, *and computation of the logarithm of* 2.

A SCHOLIUM.

175. The foregoing method of computing Briggs's logarithm of 2 is certainly ſomewhat laborious, but much leſs ſo than the methods uſed for the ſame purpoſe by Mr. Briggs himſelf, which required many very long extractions of the ſquare-

ſquare-root. For the difficulty of performing the operations that were neceſſary in thoſe methods was ſo great, that (according to what Mr. *Euclid Speidall* informs us (ſee above, page 73) he had been told) *it was the work of eight perſons for a whole year to compute the logarithm of 2 by thoſe methods exact to 15 places of figures*, or to the degree of exactneſs to which it has been obtained in the foregoing articles. This aſſertion of Mr. Speidall ſeems, I confeſs, a little ſtrange. Yet, as he publiſhed his tract on logarithms (which has been printed above in the former part of this volume) ſo long ago as in the year 1688, it ſeems probable that in his youth (perhaps, about the year 1660) he might have converſed with ſome old men who had been acquainted with Mr. Briggs himſelf, who publiſhed his *Arithmetica Logarithmica* in the year 1624, which was leſs than 40 years before that time; and this ſeems the more likely to have been the caſe, as his father, Mr. John Speidall, was an eminent mathematician, and had very much cultivated the, at that time, new invention of logarithms; which muſt have given both him and his ſon an opportunity of hearing many remarkable particulars relating to them.

We may further obſerve that the foregoing method of computing logarithms by the help of the binomial theorem, and Mr. Briggs's methods of computing them by repeated extractions of the ſquare-root, are equally founded on the pure and genuine principles of arithmetick, without any reference to the hyperbola, or the logarithmick curve, or any other geometrical figure, and alſo without any recourſe to the doctrine of infiniteſimals, or of fluxions, or of the limits of ratios, or in general, of the arithmetick of infinites in any of its modifications; which is, in Dr. Halley's opinion, the proper way of treating this ſubject, and the way in which he boaſts (though without being ſufficiently authorized in his pretenſions), that he himſelf has treated it in the foregoing diſcourſe reprinted above in this volume in pages 84, 85, 86, 87, 88, 90, and 91.

CONCLUSION.

176. I have now completed the inveſtigation of the famous binomial theorem in all the caſes of fractional powers; which was the propoſed ſubject of this diſcourſe. This, however, is but a part of that important and moſt comprehenſive propoſition. For it is found to be true likewiſe in the caſes of negative powers, both integral and fractional, that is, in the caſes of $\overline{1+x}|^{-m}$, and $\overline{1+x}|^{-\frac{m}{n}}$, or of $\frac{1}{\overline{1+x}|^{m}}$ and $\frac{1}{\overline{1+x}|^{\frac{m}{n}}}$. But, as the knowledge of theſe caſes is not neceſſary to the underſtanding any of the foregoing methods of computing logarithms, I ſhall not on this occaſion enter into any inquiries concerning them. And therefore I here conclude what I meant to offer to the reader's conſideration concerning the binomial theorem *properly ſo called*. But, as the theorem concerning the fractional powers of *a reſidual quantity*, ſuch as

$1-x$,

$1 - x$, is very nearly related to the foregoing theorem concerning the fractional powers of the binomial quantity $1 + x$, insomuch that it is usually considered as a branch of it;—and, as the said *residual theorem*, in the first case of it, or the case of the nth root of the residual quantity $1 - x$, is made use of (as well as the binomial theorem) in the investigations of some of the foregoing methods of computing logarithms;—I shall now proceed to shew how we may derive from the theorems above demonstrated concerning the roots, and the powers of the roots, of the binomial quantity $1 + x$, the like theorems concerning the roots, and the powers of the roots, of the residual quantity $1 - x$. But this shall be the subject of a separate tract. And therefore I here conclude this discourse concerning the roots, and the powers of the roots, of the binomial quantity $1 + x$.

End of the Discourse concerning the Binomial Theorem in the case of Fractional Powers.

A DISCOURSE

A

D I S C O U R S E

CONCERNING

Sir ISAAC NEWTON's RESIDUAL THEOREM,

OR THEOREM FOR RAISING THE POWERS OF THE RESIDUAL QUANTITY $1-x$, IN THE CASE OF FRACTIONAL POWERS, OR POWERS OF WHICH THE INDEXES ARE FRACTIONS.

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Art. 1. IN the foregoing diſcourſe we inveſtigated the ſerieſes that were equal to the nth root of the binomial quantity $1+x$ and to the mth power of its nth root. In the preſent diſcourſe we are to inveſtigate the ſerieſes that are equal to the nth root of the reſidual quantity $1-x$, and to the mth power of its nth root. Now theſe ſerieſes may be inveſtigated by the ſame methods which were employed in the foregoing diſcourſe to inveſtigate the ſerieſes which are equal to $\overline{1+x}\vert^{\frac{1}{n}}$ and $\overline{1+x}\vert^{\frac{m}{n}}$: but they may likewiſe be derived from thoſe former ſerieſes (obtained in the foregoing diſcourſe), by a juſt and legitimate train of reaſoning, with much leſs trouble than would be neceſſary to the diſcovery of them by a new application of all the methods of inveſtigation uſed in the foregoing tract. And therefore I ſhall, for brevity's ſake, have recourſe to this derivative method of obtaining them, rather than to the methods employed in the foregoing diſcourſe. And, firſt, I ſhall conſider the nth root of the reſidual quantity $1-x$, and endeavour to ſhew that it is equal to an infinite ſeries conſiſting of the very ſame terms as the ſeries which is equal to the nth root of the binomial quantity $1+x$, but with the ſign $-$ prefixed to all the terms after the firſt term 1, inſtead of only the third, and fifth, and ſeventh, and other following odd terms of the ſeries, as in the ſeries which is equal to $\sqrt[n]{1+x}$, or $\overline{1+x}\vert^{\frac{1}{n}}$.

2. Now it has been ſhewn in the foregoing tract, that, if n be any whole number whatſoever, the quantity $\overline{1+x}\vert^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, will be equal to the

infinite series $1 + \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.}Dx^4 + \frac{4n-1}{5n}Ex^5 - \overline{\left|\frac{5n-1}{6n}\right.}Fx^6 +$ &c *ad infinitum*. We will therefore now proceed to shew that the quantity $\sqrt[n]{1-x}$, or $\overline{1-x}\big|^{\frac{1}{n}}$, will be equal to the infinite series $1 - \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.}Bx^2 - \overline{\left|\frac{2n-1}{3n}\right.}Cx^3 - \overline{\left|\frac{2n-1}{4n}\right.}Dx^4 - \overline{\left|\frac{4n-1}{5n}\right.}Ex^5 - \overline{\left|\frac{5n-1}{6n}\right.}Fx^6 -$ &c *ad infinitum*, which consists of the very same terms as the former series (which is equal to $\overline{1+x}\big|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$,) but with the sign $-$ prefixed to every term after the first term, instead of every other term. This may be shewn in the manner following.

3. Since the series $1 + \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c is equal to $\overline{1+x}\big|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, it follows that, if we raise the said series to the nth power, or multiply it $n-1$ times into itself, the product of these multiplications will be equal to $1+x$; that is, the product of the series $1 + Bx - Cxx + Dx^3 - Ex^4 + Fx^5 -$ &c *ad infinitum* multiplied $n-1$ times into itself (upon the supposition that the co-efficient B is $= \frac{1}{n}A = \frac{1}{n} \times 1$, or $\frac{1}{n}$, and that the co-efficient C is $= \frac{n-1}{2n}B$, and that the co-efficients D, E, F, &c, are equal, respectively, to $\frac{2n-1}{3n}C$, $\frac{3n-1}{4n}D$, $\frac{4n-1}{5n}E$, &c) will be equal to $1+x$; and consequently the co-efficient of x will be equal to 1, and the compound quantities which will be the co-efficients of x^2, x^3, x^4, x^5, and of all the following powers of x in the terms of the said product, will be, each of them, equal to 0, or will consist of several members, of which some will be marked with the sign $+$, and others with the sign $-$, and which will be of such magnitudes that the sum of the terms, or members, which are marked with the sign $-$ will be equal to the sum of the terms, or members, which are marked with the sign $+$. Some of these multiplications will be as follow.

$$\begin{array}{llllll}
1 + & Bx - & Cx^2 & + Dx^3 & - \&c & = \overline{1+x}\big|^{\frac{1}{n}}. \\
1 + & Bx - & Cx^2 & + Dx^3 & - \&c & = \overline{1+x}\big|^{\frac{1}{n}}. \\
\hline
1 + & Bx - & Cx^2 & + Dx^3 & - \&c & \\
+ & Bx + & B^2x^2 & - BCx^3 & + \&c & \\
 & - & Cx^2 & - BCx^3 & + \&c & \\
 & & + & Dx^3 & + \&c & \\
\hline
1 + & 2Bx - & 2Cx^2 & + 2Dx^3 & \&c & \multirow{2}{*}{$\left.\right\} = \overline{1+x}\big|^{\frac{2}{n}}.$} \\
 & + & B^2x^2 & - 2BCx^3 & + \&c & \\
\hline
\end{array}$$

$1 + Bx$

$$\begin{array}{rrrrl}
1 & +\ Bx & -\ Cx^2 & +\ Dx^3 & -\ \&c \quad = \overline{1+x}|^{\frac{1}{n}}. \\
\hline
1 & +\ 2Bx & -\ 2Cx^2 & +\ 2Dx^3 & \&c \\
 & & +\ B^2x^2 & -\ 2BCx^3 & \&c \\
 & +\ Bx & +\ 2B^2x^2 & -\ 2BCx^3 & +\ \&c \\
 & & & +\ B^3x^3 & -\ \&c \\
 & & -\ Cx^2 & -\ 2BCx^3 & +\ \&c \\
 & & & +\ Dx^3 & +\ \&c \\
\hline
1 & +\ 3Bx & -\ 3Cx^2 & +\ 3Dx^3 & \&c \\
 & & +\ 3B^2x^2 & -\ 6BCx^3 & \&c \\
 & & & +\ B^3x^3 & \&c
\end{array}
\left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\} = \overline{1+x}|^{\frac{3}{n}}.$$

$$\begin{array}{rrrrl}
1 & +\ Bx & -\ Cx^2 & +\ Dx^3 & -\ \&c \quad = \overline{1+x}|^{\frac{1}{n}}. \\
\hline
1 & +\ 3Bx & -\ 3Cx^2 & +\ 3Dx^3 & \&c \\
 & & +\ 3B^2x^2 & -\ 6BCx^3 & \&c \\
 & & & +\ B^3x^3 & \&c \\
 & +\ Bx & +\ 3B^2x^2 & -\ 3BCx^3 & +\ \&c \\
 & & & +\ 3B^3x^3 & -\ \&c \\
 & & -\ Cx^2 & -\ 3BCx^3 & \&c \\
 & & & +\ Dx^3 & +\ \&c \\
\hline
1 & +\ 4Bx & -\ 4Cx^2 & +\ 4Dx^3 & \&c \\
 & & +\ 6B^2x^2 & -\ 12BCx^3 & \&c \\
 & & & +\ 4B^3x^3 & \&c
\end{array}
\left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\} = \overline{1+x}|^{\frac{4}{n}}.$$

$$\begin{array}{rrrrl}
1 & +\ Bx & -\ Cx^2 & +\ Dx^3 & -\ \&c \quad = \overline{1+x}|^{\frac{1}{n}}. \\
\hline
1 & +\ 4Bx & -\ 4Cx^2 & +\ 4Dx^3 & \&c \\
 & & +\ 6B^2x^2 & -\ 12BCx^3 & \&c \\
 & & & +\ 4B^3x^3 & \&c \\
 & +\ Bx & +\ 4B^2x^2 & -\ 4BCx^3 & +\ \&c \\
 & & & +\ 6B^3x^3 & -\ \&c \\
 & & -\ Cx^2 & -\ 4BCx^3 & +\ \&c \\
 & & & +\ Dx^3 & +\ \&c \\
\hline
1 & +\ 5Bx & -\ 5Cx^2 & +\ 5Dx^3 & \&c \\
 & & +\ 10B^2x^2 & -\ 20BCx^3 & \&c \\
 & & & +\ 10B^3x^3 & \&c
\end{array}
\left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\} = \overline{1+x}|^{\frac{5}{n}}.$$

4. By these multiplications it appears that the square of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (carried only to the third power of x) is equal to the compound series

$$\begin{array}{rrrl}
1 + 2Bx & -\ 2Cx^2 & +\ 2Dx^3 & \&c \\
 & +\ B^2x^2 & -\ 2BCx^3 & \&c;
\end{array}$$

and that its cube is equal to the compound series

$$\begin{array}{rrrl}
1 + 3Bx & -\ 3Cx^2 & +\ 3Dx^3 & -\ \&c \\
 & +\ 3B^2x^2 & -\ 6BCx^3 & \&c \\
 & & +\ B^3x^3 & \&c;
\end{array}$$

 and

and that its fourth power is equal to the compound series

$$\begin{array}{rrrl} 1 + 4Bx & - 4Cx^2 & + 4Dx^3 & \&c \\ & + 6B^2x^2 & - 12BCx^3 & \&c \\ & & + 4B^3x^3 & \&c; \end{array}$$

and that its fifth power is equal to the compound series

$$\begin{array}{rrrl} 1 + 5Bx & - 5Cx^2 & + 5Dx^3 & \&c \\ & + 10B^2x^2 & - 20BCx^3 & \&c \\ & & + 10B^3x^3 & \&c. \end{array}$$

5. Now, when n is $= 2$, the first of these compound serieses (which is equal to the square of the series $1 + Bx - Cx^2 + Dx^3 - \&c$), to wit, the series

$$\begin{array}{rrrl} 1 + 2Bx & - 2Cx^2 & + 2Dx^3 & \&c \\ & + B^2x^2 & - 2BCx^3 & \&c, \end{array}$$

must be equal to $1 + x$, and consequently $2B$ (the co-efficient of x) must be $= 1$, and $B^2 - 2C$ (the compound co-efficient of x^2) must be $= 0$, and $2D - 2BC$ (the compound co-efficient of x^3) must also be $= 0$, and every following co-efficient of one of the powers of x in the said series must, in like manner, be $= 0$, if the co-efficients of the terms of the series $1 + Bx - Cx^2 + Dx^3 - \&c$, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \&c$, have been rightly assigned. For otherwise the said compound series cannot be equal to $1 + x$, as it ought to be upon the present supposition that n is $= 2$, because upon this supposition the series $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \&c$ (which is universally equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{(1+x)}$), will be $= \overline{1+x}|^{\frac{1}{2}}$, or, $\sqrt[2]{\overline{1+x}}$, or the square-root of $1 + x$, and consequently the square of the said series must be equal to $1 + x$.

And accordingly we shall find that, if n be supposed to be $= 2$, the co-efficient $2B$, of the second term Bx, of this compound series, will be $= 1$, and $B^2 - 2C$ and $2D - 2BC$ (the compound co-efficients of the two following powers of x, to wit, x^2 and x^3 in the said series) will each of them be equal to 0. For, if n is $= 2$, we shall have $B\ (= \frac{1}{n} \times A = \frac{1}{2} \times A = \frac{1}{2} \times 1) = \frac{1}{2}$, and $C\ (= \frac{n-1}{2n}B = \frac{2-1}{2\times 2} \times \frac{1}{2} = \frac{1}{4} \times \frac{1}{2}) = \frac{1}{8}$, and $D\ (= \frac{2n-1}{3n}C = \frac{2\times 2-1}{3\times 2} \times \frac{1}{8} = \frac{4-1}{6} \times \frac{1}{8} = \frac{3}{6} \times \frac{1}{8} = \frac{1}{2} \times \frac{1}{8}) = \frac{1}{16}$; and consequently $2B\ (= 2 \times \frac{1}{2}) = 1$, and $B^2 - 2C\ (= \frac{1}{4} - 2 \times \frac{1}{8} = \frac{1}{4} - \frac{2}{8} = \frac{1}{4} - \frac{1}{4}) = 0$, and $2D - 2BC\ (= 2 \times \frac{1}{16} - 2 \times \frac{1}{2} \times \frac{1}{8} = \frac{2}{16} - \frac{2}{16}) = 0$.

6. In like manner, when n is $= 3$, the second of these compound serieses (which is equal to the cube of the series $1 + Bx - Cx^2 + Dx^3 - \&c$), to wit, the compound series

$1 + 3Bx$

$$\begin{array}{l} 1 + 3\,Bx - 3\,Cx^2 + 3\,Dx^3 \;\&c \\ \qquad\qquad + 3\,B^2x^2 - 6\,BCx^3 \;\&c \\ \qquad\qquad\qquad\qquad + B^3x^3 \;\&c \end{array}$$

muſt be $=$ to $1 + x$, and conſequently $3\,B$ muſt be $= 1$, and $3\,B^2 - 3\,C$ muſt be $= 0$, and $3\,D - 6\,BC + B^3$ muſt be $= 0$, or $3\,C$ muſt be $= 3\,B^2$, and $6\,BC$ muſt be $= B^3 + 3D$.

And ſo we ſhall find theſe ſeveral quantities to be. For when n is $= 3$, we ſhall have $B\,(= \frac{1}{n}\,A = \frac{1}{3} \times A = \frac{1}{3} \times 1) = \frac{1}{3}$, and $C\,(= \frac{n-1}{2n}\,B = \frac{3-1}{2\times3} \times \frac{1}{3} = \frac{2}{2\times3} \times \frac{1}{3} = \frac{1}{3} \times \frac{1}{3}) = \frac{1}{9}$, and $D\,(= \frac{2n-1}{3n}\,C = \frac{2\times3-1}{3\times3} \times \frac{1}{9} = \frac{6-1}{9} \times \frac{1}{9} = \frac{5}{9} \times \frac{1}{9}) = \frac{5}{81}$; and conſequently $3\,B\,(= 3 \times \frac{1}{3}) = 1$, and $3\,B^2 - 3\,C\,(= 3 \times \frac{1}{9} - 3 \times \frac{1}{9}) = 0$, and $3\,D - 6\,BC + B^3\,(= 3 \times \frac{5}{81} - 6 \times \frac{1}{3} \times \frac{1}{9} + \frac{1}{27} = \frac{5}{27} - \frac{6}{27} + \frac{1}{27} = \frac{6}{27} - \frac{6}{27}) = 0$.

7. And, when n is $= 4$, the third of the foregoing compound ſerieſes (which is equal to the fourth power of the ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$), to wit, the compound ſeries

$$\begin{array}{l} 1 + 4\,Bx - 4\,Cx^2 + 4\,Dx^3 \;\&c \\ \qquad\qquad + 6\,B^2x^2 - 12\,BCx^3 \;\&c \\ \qquad\qquad\qquad\qquad + 4\,B^3x^3 \;\&c, \end{array}$$

muſt be equal to $1 + x$; and conſequently $4\,B$ muſt be $= 1$, and $6\,B^2 - 4\,C$ muſt be $= 0$, and $4\,D - 12\,BC + 4\,B^3$ muſt likewiſe be $= 0$.

And ſo we ſhall find theſe quantities to be. For, if n is $= 4$, we ſhall have $B\,(= \frac{1}{n}) = \frac{1}{4}$, and $C\,(= \frac{n-1}{2n}\,B = \frac{4-1}{2\times4} \times \frac{1}{4} = \frac{3}{8} \times \frac{1}{4}) = \frac{3}{32}$, and $D\,(= \frac{2n-1}{3n}\,C = \frac{2\times4-1}{3\times4} \times \frac{3}{32} = \frac{8-1}{3\times4} \times \frac{3}{32} = \frac{7}{4} \times \frac{1}{32}) = \frac{7}{128}$; and conſequently $4\,B\,(= 4 \times \frac{1}{4}) = 1$, and $6\,B^2 - 4\,C\,(= 6 \times \frac{1}{16} - 4 \times \frac{3}{32} = \frac{3}{8} - \frac{3}{8}) = 0$, and $4\,D - 12\,BC + 4\,B^3\,(= 4 \times \frac{7}{128} - 12 \times \frac{1}{4} \times \frac{3}{32} + 4 \times \frac{1}{64} = \frac{7}{32} - \frac{9}{32} + \frac{2}{32} = \frac{9}{32} - \frac{9}{32}) = 0$.

8. And, when n is $= 5$, the laſt of the foregoing compound ſerieſes (which is equal to the fifth power of the ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$), to wit, the compound ſeries

$$\begin{array}{l} 1 + 5\,Bx - 5\,Cx^2 + 5\,Dx^3 \;\&c \\ \qquad\qquad + 10\,B^2x^2 - 20\,BCx^3 \;\&c \\ \qquad\qquad\qquad\qquad + 10\,B^3x^3 \;\&c, \end{array}$$

muſt be equal to $1 + x$, and conſequently $5\,B$ muſt be $= 1$, and $10\,B^2 - 5\,C$ muſt be $= 0$, and $5\,D - 20\,BC + 10\,B^3$ muſt likewiſe be $= 0$.

And

And so we shall find these quantities to be. For, if n is $= 5$, we shall have $B\ (= \frac{1}{n} A = \frac{1}{5} \times A = \frac{1}{5} \times 1) = \frac{1}{5}$, and $C\ (= \frac{n-1}{2n} B = \frac{5-1}{2 \times 5} \times \frac{1}{5} = \frac{4}{10} \times \frac{1}{5} = \frac{2}{5} \times \frac{1}{5}) = \frac{2}{25}$, and $D\ (= \frac{2n-1}{3n} \times C = \frac{2 \times 5 - 1}{3 \times 5} \times \frac{2}{25} = \frac{10-1}{15} \times \frac{2}{25} = \frac{9}{15} \times \frac{2}{25} = \frac{3}{5} \times \frac{2}{25}) = \frac{6}{125}$; and consequently $5B\ (= 5 \times \frac{1}{5}) = 1$, and $10B^2 - 5C\ (= 10 \times \frac{1}{5} \times \frac{1}{5} - 5 \times \frac{2}{25} = \frac{10}{25} - \frac{10}{25}) = 0$, and $5D - 20BC + 10B^3\ (= 5 \times \frac{6}{125} - 20 \times \frac{1}{5} \times \frac{2}{25} + 10 \times \frac{1}{125} = \frac{30}{125} - \frac{40}{125} + \frac{10}{125} = \frac{40}{125} - \frac{40}{125}) = 0$.

9. Thus we see in all these instances, that, when the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (which is equal to $\overline{1+x}\big|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$) is raised to the nth power, or multiplied into itself $n - 1$ times, the co-efficient of x in the compound series obtained by such multiplication is always equal to 1, and the co-efficients of x^2 and x^3 (which are compound quantities, or quantities consisting of more than 1 term) are, each of them, equal to 0, the term, or terms, which are marked with the sign $-$, or subtracted from the other terms, being equal to the term, or terms, which are marked with the sign $+$, and from which they are to be subtracted. And the same thing must take place in the co-efficients of all the following terms of the compound series obtained by such multiplication. For otherwise the said series could not be equal to $1 + x$. We may therefore lay it down universally, as an undoubted truth, resulting from the nature of powers and roots, "that, if the series which is equal to $\overline{1+x}\big|^{\frac{1}{n}}$, or "$\sqrt[n]{1+x}$, to wit, the series $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 -$ "$\frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, or the series $1 + Bx - Cx^2 + Dx^3 -$ "$Ex^4 + Fx^5 - \&c$ (in which latter series the co-efficients B, C, D, E, F, "&c, are respectively equal to the former co-efficients $\frac{1}{n}$ A, or $\frac{1}{n}$, and $\frac{n-1}{2n}$ B, "$\frac{2n-1}{3n}$ C, $\frac{3n-1}{4n}$ D, $\frac{4n-1}{5n}$ E, &c), be raised to the nth power, or multiplied into "itself $n-1$ times, the co-efficient of x in the compound series produced by such "multiplication will be $= 1$, and the co-efficients of x^2, x^3, x^4, x^5, and all the "following powers of x in the said series, will be, each of them, equal to 0; the "said co-efficients being compound quantities consisting of simple terms, or members, connected with each other, by the signs $+$ and $-$, and the sum of the "members of each of the said co-efficients that are marked with the sign $-$, or "subtracted from the other members of them, which are marked with the sign $+$, "being always equal to the sum of the said other members from which they are "subtracted." This fundamental proposition being well understood, we may derive from it a proof of the proposition asserted above in art. 2, concerning the quantity

quantity $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{\overline{1-x}}$, or the nth root of the residual quantity $1-x$, to wit, "that it will be equal to the series $1 - \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2$
" $- \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 - \frac{4n-1}{5n}Ex^5 -$ &c, or $1 - Bx - Cx^2 -$
" $Dx^3 - Ex^4 - Fx^5 -$ &c; which consists of the very same terms as the
" former series which is equal to the nth root of the binomial quantity $1+x$,
" but with the sign $-$ prefixed to every term after the first term 1, instead of
" every other term."

10. To render the proof of this proposition as easy as possible, it will be convenient to multiply each of the two serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, into itself, and then to compare together the squares, or products thence arising. These multiplications will be as follow.

$$
\begin{array}{lllllll}
1 & +Bx & -Cx^2 & +Dx^3 & -Ex^4 & +Fx^5 & -\&c \\
1 & +Bx & -Cx^2 & +Dx^3 & -Ex^4 & +Fx^5 & -\&c \\
\hline
1 & +Bx & -Cx^2 & +Dx^3 & -Ex^4 & +Fx^5 & -\&c \\
 & +Bx & +B^2x^2 & -BCx^3 & +BDx^4 & -BEx^5 & +\&c \\
 & & -Cx^2 & -BCx^3 & +C^2x^4 & -CDx^5 & +\&c \\
 & & & +Dx^3 & +BDx^4 & -CDx^5 & +\&c \\
 & & & & -Ex^4 & -BEx^5 & +\&c \\
 & & & & & +Fx^5 & +\&c \\
\hline
1 & +2Bx & -2Cx^2 & +2Dx^3 & -2Ex^4 & +2Fx^5 & \&c \\
 & & +B^2x^2 & -2BCx^3 & +2BDx^4 & -2BEx^5 & \&c \\
 & & & & +C^2x^4 & -2CDx^5 & \&c \\
\hline
\end{array}
$$

$$
\begin{array}{lllllll}
1 & -Bx & -Cx^2 & -Dx^3 & -Ex^4 & -Fx^5 & -\&c \\
1 & -Bx & -Cx^2 & -Dx^3 & -Ex^4 & -Fx^5 & -\&c \\
\hline
1 & -Bx & -Cx^2 & -Dx^3 & -Ex^4 & -Fx^5 & -\&c \\
 & -Bx & +B^2x^2 & +BCx^3 & +BDx^4 & +BEx^5 & +\&c \\
 & & -Cx^2 & +BCx^3 & +C^2x^4 & +CDx^5 & +\&c \\
 & & & -Dx^3 & +BDx^4 & +CDx^5 & +\&c \\
 & & & & -Ex^4 & +BEx^5 & +\&c \\
 & & & & & -Fx^5 & +\&c \\
\hline
1 & -2Bx & -2Cx^2 & -2Dx^3 & -2Ex^4 & -2Fx^5 & \&c \\
 & & +B^2x^2 & +2BCx^3 & +2BDx^4 & +2BEx^5 & \&c \\
 & & & & +C^2x^4 & +2CDx^5 & \&c.
\end{array}
$$

11. Now, if we compare this last compound series (which is equal to the square of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c) with the former compound series (which is equal to the square of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c), we shall find the following observations to be true concerning them.

In

In the first place, all the terms of the second of these compound serieses are the very same with the corresponding terms of the former of them, but are differently connected with each other by the signs + and −. And this, it is evident, must be the case with all the following terms of both these compound serieses (to whatever number of terms these serieses may be continued) as well as with the few terms here computed. For, since the terms of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, are the same with the terms of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, though differently connected with each other by the signs + and −, it follows that the products of the multiplication of the terms of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, into each other must be equal to the products of the multiplication of the correspondent terms of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, into each other, that is, the several members of the square of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, must be equal to the corresponding members of the square of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, though differently connected with each other by the signs + and −.

And secondly, we may observe that the first term in both these compound serieses is 1.

And, thirdly, we may observe that the third and fifth terms in the second of these products, or compound serieses, which is equal to the square of the second simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ (in which all the terms after the first term 1 are marked with the sign −) to wit, the terms $-2Cx^2 + B^2x^2$ and $-2Ex^4 + 2BDx^4 + C^2x^4$, have the same signs + and − prefixed to their several members $2Cx^2$, B^2x^2, $2Ex^4$, $2BDx^4$, and C^2x^4, respectively, as are prefixed to the same members of the third and fifth terms of the former of these products, or compound serieses, which is equal to the square of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (in which the second and other following terms are marked with the signs + and − alternately), to wit, the terms $-2Cx^2 + B^2x^2$ and $-2Ex^4 + 2BDx^4 + C^2x^4$. And the same observation is true likewise of the seventh and ninth, and eleventh, and all the following odd terms of these two compound serieses (to whatever number of terms the said serieses may be continued), to wit, that the signs + and − that are to be prefixed to the several members of any of the said odd terms in the latter of these compound serieses will be the same that are to be prefixed to the same members of the same odd terms in the former of these compound serieses; as will appear from an attentive consideration of the two foregoing operations of multiplication set down in the foregoing article.

And, 4thly, we may observe that the second, fourth, and sixth terms of the second of these products, or compound serieses (which is equal to the square of the second simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$), to wit, the terms $-2Bx$, $-2Dx^3 + 2BCx^3$, and $-2Fx^5 + 2BEx^5 + 2CDx^5$, have their several members $2Bx$, $2Dx^3$, $2BCx^3$, $2Fx^5$, $2BEx^5$, and $2CDx^5$, marked with contrary signs to those which are prefixed to the same members of the second, fourth, and sixth, terms of the former

mer of those compound serieses (which is equal to the square of the first simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c), to wit, the terms $+ 2Bx$, $+ 2Dx^3 - 2BCx^3$, and $+ 2Fx^5 - 2BEx^5 - 2CDx^5$. And the same observation is true likewise of the eighth, and tenth, and twelfth, and other following even terms of these two compound serieses (to whatever number of terms they may be continued), to wit, that the signs + and — that are to be prefixed to the several members of any of the said even terms in the latter of these compound serieses will be, respectively, contrary to those which are to be prefixed to the same members of the same even terms in the former of these compound serieses; as will appear from an attentive consideration of the two foregoing operations of multiplication set down in the foregoing article.

12. And if we were to repeat the foregoing multiplications of the two serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c into themselves any number of times whatsoever, so as to obtain the cubes, and the fourth powers, and the fifth powers, or any higher powers of the said serieses, the foregoing observations would be true of all the compound serieses, or of all the powers of the said two simple serieses, which would be thereby obtained; to wit, 1st, That the terms of every power of the one series would be the very same with the corresponding terms of the same power of the other series; and, 2dly, That the first term of every power of both serieses would be 1; and, 3dly, That the signs + and —, which would be prefixed to the several members of the third, fifth, seventh, ninth, eleventh, and other following odd terms of any power of one of these serieses, would be the very same that were to be prefixed to the same members of the same odd terms, respectively, of the same power of the other series; and, 4thly, That the signs + and —, which would be prefixed to the second term, and to the several members of the fourth, sixth, eighth, tenth, twelfth, and other following even terms of any power of one of these two serieses would be, respectively, contrary to the signs that would be prefixed to the second term, and to the same members of the same following even terms of the same power of the other series.

13. Of the truth of these observations in the case of higher powers of the two foregoing simple serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c than the square, I shall here give one example, by raising the said two serieses to the cube, or third power.

We have already seen in art. 10 that the square of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c is the compound series

$$\begin{array}{llllll} 1 + 2Bx & - 2Cx^2 & + 2Dx^3 & - 2Ex^4 & + 2Fx^5 & - \&c \\ & + B^2x^2 & - 2BCx^3 & + 2BDx^4 & - 2BEx^5 & + \&c \\ & & & + C^2x^4 & - 2CDx^5 & + \&c. \end{array}$$

Therefore to find its cube, we must multiply this compound series into the original series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c; which may be done as follows.

$$\begin{array}{lllllll}
1 & +2Bx & -2Cx^2 & +2Dx^3 & -2Ex^4 & +2Fx^5 & -\&c \\
 & & +B^2x^2 & -2BCx^3 & +2BDx^4 & -2BEx^5 & +\&c \\
 & & & & +C^2x^4 & -2CDx^5. & +\&c \\
 & & & & & & \\
1 & +Bx & -Cx^2 & +Dx^3 & -Ex^4 & +Fx^5 & -\&c \\
\hline
1 & +2Bx & -2Cx^2 & +2Dx^3 & -2Ex^4 & +2Fx^5 & -\&c \\
 & & +B^2x^2 & -2BCx^3 & +2BDx^4 & -2BEx^5 & +\&c \\
 & & & & +C^2x^4 & -2CDx^5 & +\&c \\
 & +Bx & +2B^2x^2 & -2BCx^3 & +2BDx^4 & -2BEx^5 & +\&c \\
 & & & +B^3x^3 & -2B^2Cx^4 & +2B^2Dx^5 & -\&c \\
 & & & & & +BC^2x^5 & -\&c \\
 & & -Cx^2 & -2BCx^3 & +2C^2x^4 & -2CDx^5 & +\&c \\
 & & & & -B^2Cx^4 & +2BC^2x^5 & -\&c \\
 & & & +Dx^3 & +2BDx^4 & -2CDx^5 & +\&c \\
 & & & & & +B^2Dx^5. & -\&c \\
 & & & & -Ex^4 & -2BEx^5 & +\&c \\
 & & & & & +Fx^5 & -\&c \\
\hline
1 & +3Bx & -3Cx^2 & +3Dx^3 & -3Ex^4 & +3Fx^5 & -\&c \\
 & & +3B^2x^2 & -6BCx^3 & +6BDx^4 & -6BEx^5 & +\&c \\
 & & & +B^3x^3 & +3C^2x^4 & -6CDx^5 & +\&c \\
 & & & & -3B^2Cx^4 & +3B^2Dx^5 & -\&c \\
 & & & & & +3BC^2x^5 & -\&c.
\end{array}$$

Therefore the cube of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ is the compound series

$$\begin{array}{lllllll}
1 & +3Bx & -3Cx^2 & +3Dx^3 & -3Ex^4 & +3Fx^5 & -\&c \\
 & & +3B^2x^2 & -6BCx^3 & +6BDx^4 & -6BEx^5 & +\&c \\
 & & & +B^3x^3 & +3C^2x^4 & -6CDx^5 & +\&c \\
 & & & & -3B^2Cx^4 & +3B^2Dx^5 & -\&c \\
 & & & & & +3BC^2x^5 & -\&c.
\end{array}$$

And we have already seen in art. 10 that the square of the second series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ is the compound series

$$\begin{array}{lllllll}
1 & -2Bx & -2Cx^2 & -2Dx^3 & -2Ex^4 & -2Fx^5 & -\&c \\
 & & +B^2x^2 & +2BCx^3 & +2BDx^4 & +2BEx^5 & +\&c \\
 & & & & +C^2x^4 & +2CDx^5 & +\&c.
\end{array}$$

Therefore, to find the cube of the said second series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, we must multiply this last compound series into the said second series itself; which may be done as follows.

$1 - 2Bx$

$$
\begin{array}{lllllll}
1 - 2Bx & -\ 2Cx^2 & -\ 2Dx^3 & -\ 2Ex^4 & -\ 2Fx^5 & -\ \&c \\
 & +\ B^2x^2 & +\ 2BCx^3 & +\ 2BDx^4 & +\ 2BEx^5 & +\ \&c \\
 & & & +\ C^2x^4 & +\ 2CDx^5 & +\ \&c \\
1 - Bx & -\ Cx^2 & -\ Dx^3 & -\ Ex^4 & -\ Fx^5 & -\ \&c \\
\hline
1 - 2Bx & -\ 2Cx^2 & -\ 2Dx^3 & -\ 2Ex^4 & -\ 2Fx^5 & -\ \&c \\
 & +\ B^2x^2 & +\ 2BCx^3 & +\ 2BDx^4 & +\ 2BEx^5 & +\ \&c \\
 & & & +\ C^2x^4 & +\ 2CDx^5 & +\ \&c \\
-\ Bx & +\ 2B^2x^2 & +\ 2BCx^3 & +\ 2BDx^4 & +\ 2BEx^5 & +\ \&c \\
 & & -\ B^3x^3 & -\ 2B^2Cx^4 & -\ 2B^2Dx^5 & -\ \&c \\
 & & & & -\ BC^2x^5 & -\ \&c \\
 & -\ Cx^2 & +\ 2BCx^3 & +\ 2C^2x^4 & +\ 2CDx^5 & +\ \&c \\
 & & & -\ B^2Cx^4 & -\ 2BC^2x^5 & -\ \&c \\
 & & -\ Dx^3 & +\ 2BDx^4 & +\ 2CDx^5 & +\ \&c \\
 & & & & -\ B^2Dx^5 & +\ \&c \\
 & & & -\ Ex^4 & +\ 2BEx^5 & +\ \&c \\
 & & & & -\ Fx^5 & +\ \&c \\
\hline
1 - 3Bx & -\ 3Cx^2 & -\ 3Dx^3 & -\ 3Ex^4 & -\ 3Fx^5 & \&c \\
 & +\ 3B^2x^2 & +\ 6BCx^3 & +\ 6BDx^4 & +\ 6BEx^5 & +\ \&c \\
 & & -\ B^3x^3 & +\ 3C^2x^4 & +\ 6CDx^5 & +\ \&c \\
 & & & -\ 3B^2Cx^4 & -\ 3B^2Dx^5 & \&c \\
 & & & & -\ 3BC^2x^5 & -\ \&c
\end{array}
$$

Therefore the cube of the said second series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ is the compound series

$$
\begin{array}{lllllll}
1 - 3Bx & -\ 3Cx^2 & -\ 3Dx^3 & -\ 3Ex^4 & -\ 3Fx^5 & -\ \&c \\
 & +\ 3B^2x^2 & +\ 6BCx^3 & +\ 6BDx^4 & +\ 6BEx^5 & +\ \&c \\
 & & -\ B^3x^3 & +\ 3C^2x^4 & +\ 6CDx^5 & +\ \&c \\
 & & & -\ 3B^2Cx^4 & -\ 3B^2Dx^5 & \&c \\
 & & & & -\ 3BC^2x^5 & -\ \&c.
\end{array}
$$

14. The four observations made in art. 11 and 12 are evidently true of the two compound serieses obtained in the foregoing article 13, and which are equal to the cubes of the two simple serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$. For, in the 1st place, the several terms of the latter compound series are exactly the same with the corresponding terms of the former compound series; and 2dly, the first term in both these compound serieses is 1; and, 3dly, the signs + and − that are prefixed to the several members of the third and fifth terms, $-3Cx^2 + 3B^2x^2$ and $-3Ex^4 + 6BDx^4 + 3C^2x^4 - 3B^2Cx^4$ of the latter compound series (which is equal to the cube of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$) are the same with those which are prefixed to the same members of the third and fifth terms, $-3Cx^2 + 3B^2x^2$ and $-3Ex^4 + 6BDx^4 + 3C^2x^4 - 3B^2Cx^4$, of the former compound series, which is equal to the cube of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$; and, 4thly, the signs + and − which are prefixed to the second term, $-3Bx$, and to the several members of the fourth and sixth terms, $-3Dx^3 + 6BCx^3 - B^3x^3$ and $-3Fx^5 + 6BEx^5 + 6CDx^5 - 3B^2Dx^5 - 3BC^2x^5$,

$3\,BC^2x^5$, of the latter compound series (which is equal to the cube of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c) are, respectively, contrary to those which are prefixed to the second term, $+ 3\,Bx$, and to the several members of the fourth and sixth terms, $+ 3\,Dx^3 - 6\,BCx^3 + B^3x^3$ and $+ 3\,Fx^5 - 6\,BEx^5 - 6\,CDx^5 + 3\,B^2Dx^5 + 3\,BC^2x^5$, of the former compound series, which is equal to the cube of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c.

And the same observations will be found to be true of the several members of all the following terms of the said two compound serieses after the sixth terms (to whatever number of terms the said serieses may be continued) and of the signs + and — that are to be prefixed to the said members. And they will be true also of the terms of the several compound serieses that are equal to the fourth powers, and to the fifth powers, and to all higher powers, of the said two simple serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c.

15. This relation between the signs that are to be prefixed to the same members of the several terms of the said compound serieses which are equal to the squares and the cubes and other higher powers of the two simple serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c is the consequence of the known rules of Algebräick multiplication; according to which the product of the multiplication of two quantities which are both marked with the sign — is to be marked with the sign +, as well as the product of the multiplication of two quantities which are both marked with the sign +; and the product of the multiplication of two quantities, of which one is marked with the sign +, and the other with the sign —, is to be marked with the sign —. For it will follow from hence that in the compound series which is equal to the square, or cube, or fourth power, or fifth power, or other higher power, of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, all the terms that will involve the odd powers of x, to wit, x, x^3, x^5, x^7, x^9, x^{11}, &c, will be marked with contrary signs, respectively, to those which are to be prefixed to the same terms in the compound series which is equal to the square, or cube, or fourth power, or fifth power, or other corresponding higher power, of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c; and all the terms that will involve the even powers of x, to wit, x^2, x^4, x^6, x^8, x^{10}, x^{12}, &c, in the former compound series, which is equal to the square, or cube, or other higher power, of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, will be marked with the same signs + and —, respectively, as are prefixed to the same terms in the latter compound series, which is equal to the square, or cube, or other corresponding higher power, of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c.

16. To make this more evident, let the capital letter P be put $= Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} +$ &c, or the sum of all the terms in the first series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - Gx^6 + Hx^7 - Ix^8 + Kx^9 - Lx^{10} + Mx^{11} -$ &c, which involve the odd powers of x, and which in this series have, all of them, the sign + prefixed to them; and let

let the capital letter Q be put $= Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} +$ &c, or the ſum of all the terms in the ſame ſeries which involve the even powers of x, and which in this ſeries, as well as in the ſecond ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - Gx^6 - Hx^7 - Ix^8 - Kx^9 - Lx^{10} - Mx^{11} -$ &c, are all marked with the ſign $-$.

Then will the ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c be $= 1 + P - Q$, and the ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c will be $= 1 - P - Q$; and conſequently the ſquare of the former ſeries will be ($= \overline{1 + P - Q}|^2 = \overline{1 + \overline{P - Q}}|^2 = 1 + 2 \times 1 \times \overline{P - Q} + \overline{P - Q}|^2$) $= 1 + 2P - 2Q + P^2 - 2PQ + Q^2$, and the ſquare of the latter ſeries will be ($= \overline{1 - P - Q}|^2 = \overline{1 - \overline{P + Q}}|^2 = 1 - 2 \times 1 \times \overline{P + Q} + \overline{P + Q}|^2$) $= 1 - 2P - 2Q + P^2 + 2PQ + Q^2$.

17. The terms of this latter quantity (which is equal to the ſquare of the latter ſeries) are exactly the ſame with the terms of the former quantity $1 + 2P - 2Q + P^2 - 2PQ + Q^2$, (which is equal to the ſquare of the former ſeries) but are not connected with each other by the ſigns $+$ and $-$ in the ſame manner as the terms of the ſaid former quantity. And the difference between the terms of theſe two quantities in this reſpect is as follows. The third term $2Q$, and the fourth term P^2, and the ſixth, or laſt, term Q^2, in both the quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$ are marked with the ſame ſigns $+$ and $-$, being $- 2Q + P^2 + Q^2$ in both quantities; but the ſecond term $2P$, and the fifth term $2PQ$, have different ſigns in the two quantities, being $+ 2P$ and $- 2PQ$ in the former quantity, and $- 2P$ and $+ 2PQ$ in the latter quantity.

18. Now the three terms $2Q$, P^2, and Q^2, which are marked with the ſame ſigns in both theſe quantities, will be equal to ſerieſes that will involve only the even powers of x.

For, in the firſt place, the ſecond term $2Q$ is $= 2 \times$ the ſeries $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} +$ &c $=$ the ſeries $2Cx^2 + 2Ex^4 + 2Gx^6 + 2Ix^8 + 2Lx^{10} +$ &c, which contains only the even powers of x.

And, in the ſecond place, the ſixth term Q^2 (being equal to the ſquare of the ſeries $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} +$ &c) will evidently be equal to a ſeries that will contain only x^4, x^6, x^8, x^{10}, and the following even powers of x.

And, laſtly, the fourth term P^2 (being equal to the ſquare of the ſeries $Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} +$ &c, which contains only the odd powers of x) will alſo be equal to a ſeries that will contain only the even powers of x; becauſe its terms will be the products of the multiplication of the terms of the ſeries $Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} +$ &c (which involve only the odd powers of x) into each other: it being evident that the multiplication of two odd powers of x into each other muſt always produce an even power of it.

Therefore the three terms $2Q$, P^2, and Q^2, which are marked with the ſame ſigns $+$ and $-$ in both the quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and

and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$, will be equal to feriefes which will involve only the even powers of x. Q. E. D.

19. And the fecond term $2P$, and the fifth term $2PQ$, of the faid quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$, which are marked with contrary figns in the latter of thefe two quantities to thofe with which they are marked in the former quantity, will be equal to feriefes which will contain only the odd powers of x.

For, in the 1ft place, $2P$ is $= 2 \times$ the feries $Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} + \&c =$ the feries $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c$, which contains only the odd powers of x.

And, 2dly, $2PQ$ is $= Q \times 2P = Q \times$ the feries $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c =$ the feries $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} + \&c$ (which involves the even powers of x) $\times$ the feries $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c$ (which involves the odd powers of x) $=$ a feries confifting of terms which will involve only x^3, x^5, x^7, x^9, x^{11}, and the following odd powers of x; becaufe all the terms of it will be the products of the multiplication of fome of the terms of the feries $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} + \&c$ (which involve only the even powers of x) into fome of the terms of the feries $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c$, which involve only the odd powers of x; it being evident that the product of the multiplication of an even power of x into an odd power of it muft always be an odd power of it.

Therefore the two terms $2P$ and $2PQ$ of the two quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$, which are marked with contrary figns + and − in thofe two quantities, will be equal to feriefes which will contain only the odd powers of x. Q. E. D.

20. It appears, therefore, from the two preceeding articles, that, to whatever number of terms the two feriefes $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, and the fquares of thefe two feriefes, may be continued, the terms involving the even powers of x in the fquares of both feriefes, will be marked with the fame figns + and −, and the terms involving the odd powers of x in the fquare of the latter feries will be marked with the contrary figns to thofe with which the fame terms are marked in the fquare of the former feries.

21. The reafonings ufed in the fix preceeding articles to prove, "that, "in the two compound feriefes which are equal to the fquares of the two fimple "feriefes $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - $ "$Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, the feveral terms, and members of "terms, which involve the even powers of x will be marked with the fame "figns + and − refpectively, and that the feveral terms, and members of "terms, which involve the odd powers of x in the latter of the faid compound fe-"riefes will be marked with the contrary figns to thofe which are to be prefixed "to the fame terms, and members of terms, in the former of the faid com-"pound

" pound seriefes," may be extended to the more complicated compound seriefes which are equal to the cubes of the said simple seriefes $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, and to the fourth powers of the said seriefes, and to the fifth powers of the said seriefes, and to all higher powers of the said seriefes, so as to prove, " that in " all these compound seriefes the several quantities which involve the even " powers of x will be marked with the same signs in every two corresponding " seriefes, and that in the compound series which is equal to the cube, or the " fourth power, or the fifth power, or any higher power, of the series $1 - Bx -$ " $Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, the quantities which involve the odd " powers of x will be marked with the contrary signs to those which are prefixed " to the same quantities in the compound series which is equal to the cube, or " the fourth power, or the fifth power, or other corresponding higher power of " the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c." And therefore I think, we may now consider the four observations set forth above in art. 11 and 12 as sufficiently established.

22. Now, if these observations are admitted to be true, it will follow that, whenever the co-efficients of the third, fourth, fifth, sixth, and other following terms of the compound series which is equal to the square, or the cube, or the fourth power, or the fifth power, or any higher power, of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c are all equal to 0, or those of the members of the said co-efficients which are marked with the sign $-$, taken together, are equal to the other members of the said co-efficients which are marked with the sign $+$, and from which the former members marked with the sign $-$, are to be subtracted, the same thing will also take place in the compound series which is equal to the square, or the cube, or the fourth power, or the fifth power, or the other corresponding higher power, of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c; to wit, that the co-efficients of the third, fourth, fifth, sixth, and other following terms of the said compound series will likewise all be equal to 0, or that those members of the said co-efficients which are marked with the sign $-$, or are subtracted from the other members of them which are marked with the sign $+$, will be equal to the said other members.

For in the said correspondent compound seriefes the members of the several corresponding terms, or terms involving the same powers of x, will be the very same quantities, by the 1st observation in art. 11 and 12. And in the third, fifth, seventh, and other following odd terms of the said compound seriefes (which will involve the even powers of x) the signs $+$ and $-$, to be prefixed to the several members of the said terms, will be the same in both seriefes; and in the second terms and the several members of the fourth, sixth, eighth, and other following even terms of the said compound seriefes (which will involve the odd powers of x) the signs $+$ and $-$, to be prefixed to the said second terms, and to the several members of the fourth, sixth, eighth, and other following even terms of the said compound seriefes, will be, respectively, contrary in one of those compound seriefes to what they are in the other; as is evident from the third and fourth observations in art. 11 and 12. Therefore, if in the third,

 fourth,

fourth, fifth, sixth, and other following terms of the compound series which is equal to any power of the first simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, the sum of the members of each of the said terms that are marked with the sign $-$ is equal to the sum of the members of the same term which are marked with the sign $+$, the same thing will also take place in the compound series which is equal to the same power of the second simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, or the sum of the members which are marked with the sign $-$ in the third, and the fourth, and the fifth, and the sixth, and every other following term of the said second compound series, will be equal to the sum of the members of the same term which are marked with the sign $+$; that is, in other words, when the third, fourth, fifth, sixth, and other following terms of the compound series which is equal to any power of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, become, all, equal to 0, and the whole series consists of only the two first terms, the third, fourth, fifth, sixth, and other following terms of the compound series which is equal to the same power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c will also be, all, equal to 0, and the whole series will consist of only the two first terms. Q. E. D.

23. And hence we may derive a proof of the proposition asserted above, in art. 2, concerning the quantity $\overline{1-x}|^{\frac{1}{n}}$, or $\sqrt[n]{1-x}$, or the nth root of the residual quantity $1 - x$; to wit, that, if A be $= 1$, and B be $= \frac{1}{n}$ A, and C be $= \frac{n-1}{2n}$ B, and D be $= \frac{2n-1}{3n}$ C, and E, F, G, H, &c be equal to $\frac{3n-1}{4n}$ D, $\frac{4n-1}{5n}$ E, $\frac{5n-1}{6n}$ F, $\frac{6n-1}{7n}$ G, &c, respectively, the said quantity $\overline{1-x}|^{\frac{1}{n}}$, or $\sqrt[n]{1-x}$, will be equal to the infinite series $1 - \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 - \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 - \frac{4n-1}{5n}Ex^5 -$ &c, or $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, in which all the terms after the first term 1 are marked with the sign $-$, or are subtracted from the said first term.

For, since the series $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c, or $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, has been shewn in art. 47 of the preceeding discourse concerning the Binomial Theorem, page 237, to be equal to $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, it follows that, if the said series were to be raised to the nth power (n being any whole number whatsoever) or to be multiplied $n - 1$ times into itself, the compound series thence arising would be equal to $1 + x$, which would be the two first terms of it, and consequently the co-efficients of x^2, x^3, x^4, x^5, &c *ad infinitum* in the following terms of the said compound series would be, each of them, equal to 0, or those members of the said co-efficients which would be marked with

with the ſign $-$, or would be to be ſubtracted from the others which would be marked with the ſign $+$, would be equal to the ſaid other members, from which they would be to be ſubtracted; of which we have given ſome examples above in art. 5, 6, 7, and 8. But, by the foregoing article 22, whenever the co-efficients of the third, fourth, fifth, ſixth, and other following terms of the compound ſeries which is equal to any power of the ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, are all equal to 0, or the ſubtracted members of each of the ſaid co-efficients are equal, taken together, to the members from which they are ſubtracted, the co-efficients of the third, fourth, fifth, ſixth, and other following terms of the compound ſeries, which is equal to the ſame power of the ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, will alſo be, all of them, equal to 0, or the ſubtracted members of each of the ſaid co-efficients will be equal, taken together, to the members from which they are ſubtracted. Therefore, if the ſeries $1 - \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 - \overline{\frac{2n-1}{3n}}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 - \overline{\frac{4n-1}{5n}}Ex^5 -$ &c, or $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c (in which B is $= \frac{1}{n}A$, or $\frac{1}{n} \times 1$, or $\frac{1}{n}$, and C is $= \frac{n-1}{2n}B$, and D is $= \frac{2n-1}{3n}C$, and E is $= \frac{3n-1}{4n}D$, and F, G, H, I, &c, are equal to $\frac{4n-1}{5n}E$, $\frac{5n-1}{6n}F$, $\frac{6n-1}{7n}G$, and $\frac{7n-1}{8n}H$, &c, reſpectively), be raiſed to the nth power, or multiplied $n-1$ times into itſelf, the co-efficients of the third, fourth, fifth, ſixth, and other following terms of the compound ſeries which will be produced by ſuch multiplication, and which will be equal to the ſaid nth power of the ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, will, each of them, be equal to 0, or the ſubtracted members of each of the ſaid co-efficients will be, all taken together, equal to the other members of them, from which they are to be ſubtracted; and conſequently the whole of the ſaid compound ſeries will be equal to its two firſt terms; which will be $1 - x$, becauſe the two firſt terms of the compound ſeries which is equal to the nth power of the ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c (to which the compound ſeries that is equal to the nth power of the ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, has been ſhewn to be analogous in the manner above-deſcribed), are $1 + x$.

And, ſince the nth power of the ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, or $1 - \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 - \overline{\frac{2n-1}{3n}}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 - \overline{\frac{4n-1}{5n}}Ex^5 -$ &c, will be equal to $1 - x$, it follows that the ſaid ſeries itſelf will be equal to $\overline{1-x}|^{\frac{1}{n}}$, or $\sqrt[n]{1-x}$, or the nth root of the reſidual quantity $1 - x$. Q. E. D.

24. We must now proceed to consider the quantity $\overline{1-x}|^{\frac{m}{n}}$, or the mth power of the nth root of the residual quantity $1-x$.

Of the series which is equal to the quantity $\overline{1-x}|^{\frac{m}{n}}$, or the m*th power of the* n*th root of the residual quantity $1-x$, when* m *and* n *are any whole numbers whatsoever.*

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25. It has been shewn in the foregoing discourse concerning the binomial theorem, art. 77, p. 269, that, if m be any whole number whatsoever, and n any other whole number greater than m, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n}Ax - \overline{\frac{n-m}{2n}}Bx^2 + \frac{2n-m}{3n}Cx^3 - \overline{\frac{3n-m}{4n}}Dx^4 + \frac{4n-m}{5n}Ex^5 -$ &c *ad infinitum*; in which series the second, and third, and fourth, and other following terms, are alternately marked with the sign + and the sign −. And it has been shewn in art. 104 of the said discourse, page 289, that, when m is greater than n, but less than $2n$, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \overline{\frac{2n-m}{3n}}Cx^3 + \frac{3n-m}{4n}Dx^4 - \overline{\frac{4n-m}{5n}}Ex^5 +$ &c; in which the three first terms of the series are added together, and the fourth and fifth, and other following terms of it are marked with the sign − and the sign + alternately, or are alternately subtracted from, and added to, the said three first terms. And it has been shewn in art. 115 of the said discourse, page 299, that, if m be of any magnitude greater than $2n$, and pn is the greatest multiple of n that is less than m, so that m is greater than pn, but less than $\overline{p+1}| \times n$, or $pn+n$, the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 + \frac{m-5n}{6n}Fx^6 +$ &c, $- \overline{\frac{pn+n-m}{pn+2n}}C'x^{p+2} + \frac{pn+2n-m}{pn+3n}D'x^{p+3} - \overline{\frac{pn+3n-m}{pn+4n}}E'x^{p+4} + \frac{pn+4n-m}{pn+5n}F'x^{p+5} - \overline{\frac{pn+5n-m}{pn+6n}}G'x^{p+6}$ + &c *ad infinitum*; in which series all the terms after the first term 1 are to be added to the said first term, till we come to the term $\frac{pn+n-m}{pn+2n}C'x^{p+2}$, which

is to be subtracted from it; and all the terms after the said term $\frac{pn+n-m}{pn+2n}$ $C'x^{p+2}$ are to be added to, and subtracted from, the said first term alternately.

Now the quantity $\overline{1-x}|^{\frac{m}{n}}$, or the mth power of the nth root of the residual quantity $1-x$, will always be equal to a series consisting of the very same terms as the series that is equal to the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the same power of the same root of the binomial quantity $1+x$, but with the signs $+$ and $-$ changed into their contraries in all the terms that involve x, x^3, x^5, x^7, x^9, x^{11}, and the other following odd powers of x, or in the second, and fourth, and sixth, and eighth, and tenth, and other following even terms of the series. This I shall now endeavour to demonstrate.

26. It has been shewn above in art. 23, that $\overline{1-x}|^{\frac{1}{n}}$, or the nth root of the residual quantity $1-x$, is equal to the infinite series $1 - \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.}Bx^2 - \overline{\left|\frac{2n-1}{3n}\right.}Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.}Dx^4 - \overline{\left|\frac{4n-1}{5n}\right.}Ex^5 -$ &c, which consists of the very same terms, only differently connected together by the signs $+$ and $-$, as the series $1 + \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, or the nth root of the binomial quantity $1+x$. And the difference between these two serieses with respect to the signs $+$ and $-$, that are to be prefixed to their second and other following terms, is that in the series which is equal to $\overline{1-x}|^{\frac{1}{n}}$ the sign $-$ is to be prefixed to all the terms after the first term, whereas in the series which is equal to $\overline{1-x}|^{\frac{1}{n}}$ the sign $-$ is to be prefixed only to its third, and fifth, and seventh, and other following odd terms, and the sign $+$ is to be prefixed to its second, and fourth, and sixth, and other following even terms.

Now, since $\overline{1-x}|^{\frac{1}{n}}$ is equal to the infinite series $1 - \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.}Bx^2 - \overline{\left|\frac{2n-1}{3n}\right.}Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.}Dx^4 - \overline{\left|\frac{4n-1}{5n}\right.}Ex^5 -$ &c, it follows that the quantity $\overline{1-x}|^{\frac{m}{n}}$, or the mth power of $\overline{1-x}|^{\frac{1}{n}}$ will be equal to the mth power of the said infinite series $1 - \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.}Bx^2 - \overline{\left|\frac{2n-1}{3n}\right.}Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.}Dx^4 - \overline{\left|\frac{4n-1}{5n}\right.}Ex^5 -$ &c, or (as it will be convenient to denote it for the sake of brevity), of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c *ad infinitum*. We must therefore inquire what will be the terms of the series that is

 equal

equal to the mth power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, and what will be the signs $+$ and $-$ that will be to be prefixed to them, or what will be the terms of the series that will be produced by multiplying the said series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c $m - 1$ times into itself, and which of the signs $+$ and $-$ will be to be prefixed to each of them: and we must compare the series so produced with the series which is produced by the multiplication of the series $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c, or $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c (which is equal to $\overline{1+x}|^{\frac{1}{n}}$) $m - 1$ times into itself, and which is therefore equal to the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$.

27. Now, from what has been shewn above in art. 11, 12, ——— 21, it is evident, in the first place, that if we raise both the serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, to the mth power (m being any whole number whatsoever), or multiply each of them $m - 1$ times into itself, the products of the said multiplications will be two compound serieses consisting of exactly the same terms, of which the first term will be 1; and, 2dly, that the signs $+$ and $-$, that will be prefixed to the members of the several terms of the said compound serieses, will be the same in all the terms which will involve the even powers of x in both serieses, but will be different in those terms of the said two serieses which will involve the odd powers of x. Therefore the compound series which is equal to the mth power of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, or $1 - \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 - \overline{\frac{2n-1}{3n}}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 - \overline{\frac{4n-1}{5n}}Ex^5 -$ &c, may be derived from the compound series which is equal to the mth power of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c, by changing the signs of all those terms of the said latter compound series which involve the odd powers of x, into their contraries. But the compound series which is equal to the mth power of the said simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c, is equal to the mth power of $\overline{1+x}|^{\frac{1}{n}}$, or to the quantity $\overline{1+x}|^{\frac{m}{n}}$, and consequently (by what is shewn in art. 115 of the foregoing discourse, page 299) is also equal to the simple series $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 +$ &c $- \overline{\frac{pn+n-m}{pn+2n}}C'x^{p+2}$

$\overline{\frac{pn+n-m}{pn+2n}} C'x^{p+2} + \frac{pn+2n-m}{pn+3n} D'x^{p+3} - \&c.$ Therefore, if the ſigns $+$ and $-$ of thoſe terms in this laſt ſimple ſeries which involve the odd powers of x, that is, of the ſecond, fourth, ſixth, and other following even terms of it, be changed into their contraries, the ſimple ſeries thereby produced will be equal to the compound ſeries produced by changing the ſigns of thoſe terms of the compound ſeries that is equal to the mth power of the ſimple ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \overline{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, which involve the odd powers of x, or the ſigns of its ſecond, fourth, ſixth, and other following even terms, into their contraries. But it has been juſt now ſhewn, that the compound ſeries produced by changing the ſigns of thoſe terms of the compound ſeries that is equal to the mth power of the ſimple ſeries $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or $1 + \frac{1}{n}Ax - \overline{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \overline{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, which involve the odd powers of x, into their contraries, is equal to the compound ſeries, or rather is the compound ſeries, which is equal to the mth power of the ſimple ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or $1 - \frac{1}{n}Ax - \overline{\frac{n-1}{2n}} Bx^2 - \overline{\frac{2n-1}{3n}} Cx^3 - \overline{\frac{3n-1}{4n}} Dx^4 - \overline{\frac{4n-1}{5n}} Ex^5 - \&c.$ Therefore the ſimple ſeries that is produced by changing the ſigns of thoſe terms of the ſimple ſeries $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 + \&c, - \overline{\frac{pn+n-m}{pn+2n}} C'x^{p+2} + \frac{pn+2n-m}{pn+3n} D'x^{p+3} - \overline{\frac{pn+3n-m}{pn+4n}} E'x^{p+4} + \&c$, which involve the odd powers of x, or the ſigns of the ſecond, fourth, ſixth, and other following even terms of it, into their contraries, will be equal to the compound ſeries which is equal to the mth power of the ſimple ſeries $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or $1 - \frac{1}{n}Ax - \overline{\frac{n-1}{2n}} Bx^2 - \overline{\frac{2n-1}{3n}} Cx^3 - \overline{\frac{3n-1}{4n}} Dx^4 - \overline{\frac{4n-1}{5n}} Ex^5 - \&c$, and conſequently will be equal to the mth power of $\overline{1-x}|^{\frac{1}{n}}$ (which is equal to the ſaid laſt-mentioned ſimple ſeries), or to the mth power of the nth root of the reſidual quantity $1 - x$.

Q. E. D.

28. The reaſonings in the foregoing article appear ſomewhat perplexed in conſequence of the multitude of words which have been made uſe of in deſcribing the ſeveral infinite ſerieſes mentioned in them. I will therefore, now repeat them in a conciſer manner, which will, I hope, remove all obſcurity from them; and for this purpoſe I ſhall denote all the ſerieſes that we ſhall have occaſion to conſider, by ſingle letters.

29. Let

29. Let the Greek capital letter Γ be put for the series $1 + \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.} Bx^2 + \frac{2n-1}{3n}Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.} Dx^4 + \frac{4n-1}{5n}Ex^5 -$ &c, *ad infinitum*, or $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c *ad infinitum*, which is equal to $\overline{1+x}|^{\frac{1}{n}}$, or the nth root of the binomial quantity $1 + x$; and let the Greek capital letter Δ be put for the series $1 - \frac{1}{n}Ax - \overline{\left|\frac{n-1}{2n}\right.} Bx^2 - \overline{\left|\frac{2n-1}{3n}\right.} Cx^3 - \overline{\left|\frac{3n-1}{4n}\right.} Dx^4 - \overline{\left|\frac{4n-1}{5n}\right.} Ex^5 -$ &c *ad infinitum*, or $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c *ad infinitum*, which is equal to $\overline{1-x}|^{\frac{1}{n}}$, or the nth root of the residual quantity $1 - x$. And let Γ^m stand for the compound series which is equal to the mth power (m being any whole number whatsoever) of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 -$ &c, or Γ, or the product which arises by multiplying the said simple series $m - 1$ times into itself; and let Δ^m stand for the compound series which is equal to the same, or the mth, power of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 -$ &c, or Δ, or the product which arises by multiplying it $m - 1$ times into itself. And let the Greek capital letter Λ denote the simple series $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 +$ &c $- \overline{\left|\frac{pn+n-m}{pn+2n}\right.} C'x^{p+2} + \frac{pn+2n-m}{pn+3n}D'x^{p+3} - \frac{pn+3n-m}{pn+4n}E'x^{p+4} + \frac{pn+4n-m}{pn+5n}F'x^{p+5} -$ &c *ad infinitum*, which is equal to the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the mth power of the nth root of the binomial quantity $1 + x$. And, lastly, let the Greek capital letter Π denote the simple series which is derived from the simple series Λ by changing the signs of those terms in it which involve the odd powers of x, that is, of the second, fourth, sixth, eighth, and other following even terms of it, into their contraries. With this notation the reasonings contained in the foregoing article 27 will be as follows.

30. From what has been shewn above, in art. 11, 12, &c, — — — 21, it is evident, in the 1st place, that the compound serieses Γ^m and Δ^m will consist of exactly the same terms, of which the first term will be 1; and, 2dly, that the signs + and — that will be prefixed to the members of the several terms of the said two compound serieses will be the same in all the quantities, or members of the terms of the said serieses, which will involve the even powers of x, that is, in the third, and fifth, and seventh, and other following odd terms of the said serieses; but will be different in those terms of the said two serieses which will involve the odd powers of x, that is, in the second, and fourth, and sixth, and other following even terms of the said serieses. Therefore the compound series Δ^m may be derived from the compound series Γ^m by changing the signs of all the members of those terms of the series Γ^m which involve in them the odd powers of

of x, that is, the signs of all the members of the second, fourth, sixth, and other following even terms of it, into their contraries. But, because the simple series Γ is equal to $\overline{1+x}|^{\frac{1}{n}}$, the compound series Γ^m will be equal to the mth power of $\overline{1+x}|^{\frac{1}{n}}$, or to the quantity $\overline{1+x}|^{\frac{m}{n}}$, and consequently (by what is shewn in art. 115 of the foregoing discourse, page 299) will be also equal to the simple series Λ. Therefore, if the signs + and − in those terms of this simple series Λ which involve the odd powers of x, that is, in the second, fourth, sixth, and other following even terms of it, be changed into their contraries, the simple series thereby produced will be equal to the compound series that is produced by changing the signs + and − of all the members of those terms of the compound series Γ^m, which also involve the odd powers of x, or the signs of all the members of the second, fourth, sixth, and other following even terms of the said compound series Γ^m, into their contraries. But it has been just now shewn, that the compound series which is produced by changing the signs + and − of all the members of those terms of the compound series Γ^m which involve the odd powers of x, or of all the members of the second, fourth, sixth, and other following even terms of it, is the compound series Δ^m. Therefore, if the signs + and − in those terms of the simple series Λ which involve the odd powers of x, or in the second, fourth, sixth, and other following even terms of it, are changed into their contraries, the simple series thereby produced will be equal to the compound series Δ^m, and consequently will be equal to the mth power of $\overline{1-x}|^{\frac{1}{n}}$, or of the nth root of the residual quantity $1-x$, which is equal to the simple series Δ; that is, the simple series Π will be equal to the mth power of $\overline{1-x}|^{\frac{1}{n}}$, or of the nth root of the residual quantity $1-x$, or to the quantity $\overline{1-x}|^{\frac{m}{n}}$. Q. E. D.

31. And, if still greater brevity be desir'd, this demonstration of the equality between the simple series Π and the quantity $\overline{1-x}|^{\frac{m}{n}}$ may be expressed in the manner following.

Since the simple series Γ is $= \overline{1+x}|^{\frac{1}{n}}$, it follows that the compound series Γ^m will be $=$ the mth power of $\overline{1+x}|^{\frac{1}{n}}$, or $= \overline{1+x}|^{\frac{m}{n}}$.

But (by the foregoing discourse, art 115, page 299) $\overline{1+x}|^{\frac{m}{n}}$ is equal the simple series Λ.

Therefore the compound series Γ^m is equal the simple series Λ.

Therefore, if we change the signs + and − of all the even terms, or terms involving the odd powers of x, in both these serieses Γ^m and Λ, into their contraries, the serieses thereby produced will be equal to each other.

But

But the series produced by this change in the signs of the terms of the compound series Γ^m will be the compound series Δ^m, by what has been shewn above in art. 11, 12, 13, &c — — — 21. And the series produced by this change in the signs of the terms of the simple series Λ is the simple series Π.

Therefore the simple series Π will be $=$ the compound series Δ^m.

But because the simple series Δ is $= \overline{1 - x}|^{\frac{1}{n}}$, it follows that the compound series Δ^m must be $=$ the mth power of $\overline{1 - x}|^{\frac{1}{n}}$, or $=$ the quantity $\overline{1 - x}|^{\frac{m}{n}}$.

Therefore the simple series Π will be equal the quantity $\overline{1 - x}|^{\frac{m}{n}}$. Q. E. D.

32. We have now demonstrated in a manner that, I hope, will be thought satisfactory, 1st, that the quantity $\overline{1 - x}|^{\frac{1}{n}}$, or the nth root of the residual quantity $1 - x$, is equal to the infinite series $1 - \frac{1}{n} A x - \frac{n-1}{2n} B x^2 - \frac{2n-1}{3n} C x^3 - \frac{3n-1}{4n} D x^4 - \frac{4n-1}{5n} E x^5 -$ &c *ad infinitum*, in which all the terms that come after the first term 1, are marked with the sign $-$, or subtracted from the said first term; and secondly that the quantity $\overline{1 - x}|^{\frac{m}{n}}$, or the mth power of the nth root of the said residual quantity, is equal to a simple series (which we have called Π) consisting of the very same terms as the series (which we have called Λ) that is equal to the quantity $\overline{1 + x}|^{\frac{m}{n}}$, or to the same mth power of the same nth root of the binomial quantity $1 + x$, and derived from the said series Λ by changing the signs of those terms of it which involve the odd powers of x, or the signs of its second, fourth, sixth, and other following even terms, into their contraries. Nothing therefore seems now to remain to be done with respect to this subject, but to illustrate these two serieses by applying them to a few particular cases, or examples, in the same manner as we illustrated the serieses which are equal to the quantities $\overline{1 + x}|^{\frac{1}{n}}$ and $\overline{1 + x}|^{\frac{m}{n}}$ in the beginning of the foregoing discourse concerning the binomial theorem in art. 8, 9, 10, 11, 12, &c. — — — 19 of the said discourse, pages 201, 202, 203, &c. — — — 209. These examples may be as follows.

Examples of the extraction of some particular roots of the residual quantity $1 - x$ by means of the series given above in art. 2.

33. In the first place we will extract the square root of the residual quantity $1 - x$ by means of the series given in art. 2, to wit, the series $1 - \frac{1}{n} A x - \frac{n-1}{2n}$

$\overline{\frac{n-1}{2n}}\,Bx^2 - \overline{\frac{2n-1}{3n}}\,Cx^3 - \overline{\frac{3n-1}{4n}}\,Dx^4 - \overline{\frac{4n-1}{5n}}\,Ex^5$ — &c *ad infinitum*.

Now in this case $\overline{1-x}|^{\frac{1}{n}}$ is $= \overline{1-x}|^{\frac{1}{2}}$, or n is $= 2$. Therefore $2n$ is $(= 2 \times 2) = 4$, and $3n$ is $(= 3 \times 2) = 6$, and $4n$ is $(= 4 \times 2) = 8$, and $5n$ is $(= 5 \times 2) = 10$, and consequently $n - 1$ is $(= 2 - 1) = 1$, and $2n - 1$ is $(= 4 - 1) = 3$, and $3n - 1$ is $(= 6 - 1) = 5$, and $4n - 1$ is $(= 8 - 1) = 7$. We shall therefore have $1 - \frac{1}{n}Ax - \overline{\frac{n-1}{2n}}\,Bx^2 - \overline{\frac{2n-1}{3n}}\,Cx^3 - \overline{\frac{3n-1}{4n}}\,Dx^4 - \overline{\frac{4n-1}{5n}}\,Ex^5$ — &c $= 1 - \frac{1}{2}Ax - \frac{1}{4}Bx^2 - \frac{3}{6}Cx^3 - \frac{5}{8}Dx^4 - \frac{7}{10}Ex^5$ — &c $= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}$ — &c.

Therefore, if the series given above in art. 2 is really equal to $\overline{1-x}|^{\frac{1}{n}}$, the quantity $\overline{1-x}|^{\frac{1}{2}}$, or the square-root of the residual quantity $1 - x$, will be equal to the series $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}$ — &c. Q. E. I.

34. Now "that this series is really equal to the square root of $1 - x$," will appear by multiplying it into itself. For we shall find that the product of the said multiplication will be $1 - x$. This multiplication may be performed in the manner following.

$$
\begin{array}{ccccccc}
1 & -\frac{x}{2} & -\frac{x^2}{8} & -\frac{x^3}{16} & -\frac{5x^4}{128} & -\frac{7x^5}{256} & -\ \&c \\
1 & -\frac{x}{2} & -\frac{x^2}{8} & -\frac{x^3}{16} & -\frac{5x^4}{128} & -\frac{7x^5}{256} & -\ \&c \\
\hline
1 & -\frac{x}{2} & -\frac{x^2}{8} & -\frac{x^3}{16} & -\frac{5x^4}{128} & -\frac{7x^5}{256} & -\ \&c \\
 & -\frac{x}{2} & +\frac{x^2}{4} & +\frac{x^3}{16} & +\frac{x^4}{32} & +\frac{5x^5}{256} & +\ \&c \\
 & & -\frac{x^2}{8} & +\frac{x^3}{16} & +\frac{x^4}{64} & +\frac{x^5}{128} & +\ \&c \\
 & & & -\frac{x^3}{16} & +\frac{x^4}{32} & +\frac{x^5}{128} & +\ \&c \\
 & & & & -\frac{5x^4}{128} & +\frac{5x^5}{256} & +\ \&c \\
 & & & & & -\frac{7x^5}{256} & +\ \&c \\
\hline
1 & -x & * & * & * & * & \&c.
\end{array}
$$

It appears therefore that the series $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}$ — &c, is really and truly, as far as relates to the six first terms of it, the square-root of the residual quantity $1 - x$; and consequently that the series given above in art.

art. 2 for the value of $\overline{1-x}|^{\frac{1}{n}}$, or $\sqrt[n]{1-x}$, is true in the case of square-roots.

35. In the next place we will investigate the value of $\overline{1-x}|^{\frac{1}{3}}$, or the cube-root of the residual quantity $1-x$, by means of the same series $1 - \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 - \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 - \frac{4n-1}{5n}Ex^5$ — &c *ad infinitum*.

Now in this case n is $= 3$; and consequently we shall have $2n = 6$, and $3n = 9$, and $4n = 12$, and $5n = 15$, and $n-1 = 2$, $2n-1 = 5$, $3n-1 = 8$, and $4n-1 = 11$. And therefore the series $1 - \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 - \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 - \frac{4n-1}{5n}Ex^5$ — &c, will in this case be $= 1 - \frac{1}{3}Ax - \frac{2}{6}Bx^2 - \frac{5}{9}Cx^3 - \frac{8}{12}Dx^4 - \frac{11}{15}Ex^5$ — &c $= 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729}$ — &c. Q. E. I.

36. Now "that this series is really equal to the cube root of $1-x$," will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be equal to $1-x$. These multiplications will be as follow.

$$
\begin{array}{cccccccl}
1 & -\frac{x}{3} & -\frac{x^2}{9} & -\frac{5x^3}{81} & -\frac{10x^4}{243} & -\frac{22x^5}{729} & - & \&c \\
1 & -\frac{x}{3} & -\frac{x^2}{9} & -\frac{5x^3}{81} & -\frac{10x^4}{243} & -\frac{22x^5}{729} & - & \&c \\
\hline
1 & -\frac{x}{3} & -\frac{x^2}{9} & -\frac{5x^3}{81} & -\frac{10x^4}{243} & -\frac{22x^5}{729} & - & \&c \\
 & -\frac{x}{3} & +\frac{x^2}{9} & +\frac{x^3}{27} & +\frac{5x^4}{243} & +\frac{10x^5}{729} & + & \&c \\
 & & -\frac{x^2}{9} & +\frac{x^3}{27} & +\frac{x^4}{81} & +\frac{5x^5}{729} & + & \&c \\
 & & & -\frac{5x^3}{81} & +\frac{5x^4}{243} & +\frac{5x^5}{729} & + & \&c \\
 & & & & -\frac{10x^4}{243} & +\frac{10x^5}{729} & + & \&c \\
 & & & & & -\frac{22x^5}{729} & + & \&c \\
\hline
1 & -\frac{2x}{3} & -\frac{x^2}{9} & -\frac{4x^3}{81} & -\frac{7x^4}{243} & -\frac{14x^5}{729} & & \&c.
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
1 & - \frac{2x}{3} & - & \frac{x^2}{9} & - & \frac{4x^3}{81} & - & \frac{7x^4}{243} & - & \frac{14x^5}{729} & & \&c \\
1 & - \frac{x}{3} & - & \frac{x^2}{9} & - & \frac{5x^3}{81} & - & \frac{10x^4}{243} & - & \frac{22x^5}{729} & - & \&c \\
\hline
1 & - \frac{2x}{3} & - & \frac{x^2}{9} & - & \frac{4x^3}{81} & - & \frac{7x^4}{243} & - & \frac{14x^5}{729} & & \&c \\
 & - \frac{x}{3} & + & \frac{2x^2}{9} & + & \frac{x^3}{27} & + & \frac{4x^4}{243} & + & \frac{7x^5}{729} & + & \&c \\
 & & - & \frac{x^2}{9} & + & \frac{2x^3}{27} & + & \frac{x^4}{81} & + & \frac{4x^5}{729} & + & \&c \\
 & & & & - & \frac{5x^3}{81} & + & \frac{10x^4}{243} & + & \frac{5x^5}{729} & + & \&c \\
 & & & & & & - & \frac{10x^4}{243} & + & \frac{20x^6}{729} & + & \&c \\
 & & & & & & & & - & \frac{22x^5}{729} & + & \&c \\
\hline
1 & - x & & * & & * & & * & & * & & \&c.
\end{array}
$$

It appears therefore that the series $1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} -$ &c, is really and truly, as far as relates to the first six terms of it, the cube-root of the residual quantity $1 - x$, and consequently that the series given above in art. 2 for the value of $\overline{1 - x}|^{\frac{1}{n}}$, or $\sqrt[n]{\overline{1 - x}}$, is true in the case of cube-roots as well as in that of square-roots.

Examples of the extraction of the roots of some particular powers of the residual quantity $1 - x$ *by means of the series given above in art.* 25, *and which in art.* 29, 30, *and* 31 *is denoted by the Greek capital letter* Π.

37. In the next place we will investigate the value of the quantity $\overline{1 - x}|^{\frac{2}{3}}$, or the cube-root of the square of the residual quantity $1 - x$, or (which comes to the same thing) the square of its cube-root, by means of the series given above in art. 25, and denoted by the Greek capital letter Π in art. 29, 30, and 31.

Now it is shewn in the foregoing discourse, art. 12, page 203, that $\overline{1 + x}|^{\frac{2}{3}}$, or the cube-root of the square of the binomial quantity $1 + x$, is equal to the series $1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} -$ &c, *ad infinitum*. Therefore the quantity $\overline{1 - x}|^{\frac{2}{3}}$ must, according to art. 25, be equal to a series derived from this

this series by changing the signs of its second, fourth, sixth, and other following even terms into their contraries, that is, to the series $1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} -$ &c, in which all the terms after the first term 1 are marked with the sign —, or are subtracted from the said first term. Therefore, if the proposition contained in art. 25, is true, and the series just now set down, and which, in the foregoing articles, is denoted by Π, is really equal to $\overline{1 - x}\big|^{\frac{m}{n}}$, the quantity $\overline{1 - x}\big|^{\frac{2}{3}}$, or the cube-root of the square of the residual quantity $1 - x$, will be equal to the series $1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} -$ &c.

38. Now "that this series is really equal to $\overline{1 - x}\big|^{\frac{2}{3}}$, or to the cube root of "the square of the residual quantity $1 - x$, or to the cube-root of the trinomial "quantity $1 - 2x + xx$," (for both the quantities $1 + 2x + xx$ and $1 - 2x + xx$ are equally called *trinomial* quantities, though in the latter quantity the middle term $2x$ is marked with the sign —, or subtracted from the sum of the two other terms) will appear by multiplying the said series twice into itself. These multiplications will be as follow.

$$
\begin{array}{cccccccccccccc}
1 & - & \frac{2x}{3} & - & \frac{x^2}{9} & - & \frac{4x^3}{81} & - & \frac{7x^4}{243} & - & \frac{14x^5}{729} & - & \&c \\
1 & - & \frac{2x}{3} & - & \frac{x^2}{9} & - & \frac{4x^3}{81} & - & \frac{7x^4}{243} & - & \frac{14x^5}{729} & - & \&c \\
\hline
1 & - & \frac{2x}{3} & - & \frac{x^2}{9} & - & \frac{4x^3}{81} & - & \frac{7x^4}{243} & - & \frac{14x^5}{729} & - & \&c \\
 & - & \frac{2x}{3} & + & \frac{4x^2}{9} & + & \frac{2x^3}{27} & + & \frac{8x^4}{243} & + & \frac{14x^5}{729} & + & \&c \\
 & & & - & \frac{x^2}{9} & + & \frac{2x^3}{27} & + & \frac{x^4}{81} & + & \frac{4x^5}{729} & + & \&c \\
 & & & & & - & \frac{4x^3}{81} & + & \frac{8x^4}{243} & + & \frac{4x^5}{729} & + & \&c \\
 & & & & & & & - & \frac{7x^4}{243} & + & \frac{14x^5}{729} & + & \&c \\
 & & & & & & & & & - & \frac{14x^5}{729} & + & \&c \\
\hline
1 & - & \frac{4x}{3} & + & \frac{2x^2}{9} & + & \frac{4x^3}{81} & + & \frac{5x^4}{243} & + & \frac{8x^5}{729} & & \&c.
\end{array}
$$

$1 - \frac{4x}{3}$

$$\begin{array}{lcccccl}
1 - \frac{4x}{3} & + \frac{2x^2}{9} & + \frac{4x^3}{81} & + \frac{5x^4}{243} & + \frac{8x^5}{729} & & \&c \\
1 - \frac{2x}{3} & - \frac{x^2}{9} & - \frac{4x^3}{81} & - \frac{7x^4}{243} & - \frac{14x^5}{729} & & - \&c \\
\hline
1 - \frac{4x}{3} & + \frac{2x^2}{9} & + \frac{4x^3}{81} & + \frac{5x^4}{243} & + \frac{8x^5}{729} & & \&c \\
\quad - \frac{2x}{3} & + \frac{8x^2}{9} & - \frac{4x^3}{27} & - \frac{8x^4}{243} & - \frac{10x^5}{729} & & - \&c \\
 & - \frac{x^2}{9} & + \frac{4x^3}{27} & - \frac{2x^4}{81} & - \frac{4x^5}{729} & & - \&c \\
 & & - \frac{4x^3}{81} & + \frac{16x^4}{243} & - \frac{8x^5}{729} & & - \&c \\
 & & & - \frac{7x^4}{243} & + \frac{28x^5}{729} & & - \&c \\
 & & & & - \frac{14x^5}{729} & & + \&c \\
\hline
1 - 2x + xx & * & * & * & & & \&c.
\end{array}$$

It appears therefore that the series $1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c$ is really and truly the cube-root of the trinomial quantity $1 - 2x + xx$, or of the square of the residual quantity $1 - x$, and consequently that the proposition contained in art. 25 is true, or that the series II is really equal to the quantity $\overline{1-x}\big|^{\frac{m}{n}}$, in the case of the cube-root of the square of a residual quantity, or when m, the numerator of the fraction $\frac{m}{n}$ (which is the index of the power of $1 - x$ in the quantity $\overline{1-x}\big|^{\frac{m}{n}}$) is $= 2$, and n, the denominator of the said fraction, is $= 3$.

39. As another example of the investigation of the value of $\overline{1-x}\big|^{\frac{m}{n}}$ by means of the series II, we will suppose m to be $= 3$, and n to be $= 5$, or $\overline{1-x}\big|^{\frac{m}{n}}$ to be equal to $\overline{1-x}\big|^{\frac{3}{5}}$, or to the fifth root of the cube of the residual quantity $1 - x$, or to the fifth root of the quadrinomial quantity $1 - 3x + 3x^2 - x^3$.

Now it appears by art. 14 of the foregoing discourse concerning the binomial theorem, page 205, that the quantity $\overline{1+x}\big|^{\frac{3}{5}}$ is $=$ the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c$. Therefore the series II will be $1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c$, and consequently, by art. 25, the quantity $\overline{1-x}\big|^{\frac{3}{5}}$ will be equal to the series $1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c$.

40. Now

40. Now "that this series is really equal to the quantity $\overline{1-x}|^{\frac{3}{5}}$, or to the "fifth root of the cube of the residual quantity $1-x$, or to the fifth root of the "quadrinomial quantity $1-3x+3x^2-x^3$," will appear by raising the said series to the fifth power by multiplying it, first, into itself, whereby we shall obtain its square, and afterwards multiplying the said square into itself, whereby we shall obtain its fourth power, and, lastly, multiplying the said fourth power of it into the said series itself, whereby we shall obtain its fifth power. For we shall find that the product of these three multiplications will be the said quadrinomial quantity $1-3x+3x^2-x^3$. These multiplications will be as follow.

$$
\begin{array}{lllllll}
1 & -\frac{3x}{5} & -\frac{3x^2}{25} & -\frac{7x^3}{125} & -\frac{21x^4}{625} & -\frac{357x^5}{15625} & -\ \&c \\
1 & -\frac{3x}{5} & -\frac{3x^2}{25} & -\frac{7x^3}{125} & -\frac{21x^4}{625} & -\frac{357x^5}{15625} & -\ \&c \\
\hline
1 & -\frac{3x}{5} & -\frac{3x^2}{25} & -\frac{7x^3}{125} & -\frac{21x^4}{625} & -\frac{357x^5}{15625} & -\ \&c \\
 & -\frac{3x}{5} & +\frac{9x^2}{25} & +\frac{9x^3}{125} & +\frac{21x^4}{625} & +\frac{63x^5}{3125} & +\ \&c \\
 & & -\frac{3x^2}{25} & +\frac{9x^3}{125} & +\frac{9x^4}{625} & +\frac{21x^5}{3125} & +\ \&c \\
 & & & -\frac{7x^3}{125} & +\frac{21x^4}{625} & +\frac{21x^5}{3125} & +\ \&c \\
 & & & & -\frac{21x^4}{625} & +\frac{63x^5}{3125} & +\ \&c \\
 & & & & & -\frac{357x^5}{15625} & +\ \&c \\
\hline
1 & -\frac{6x}{5} & +\frac{3x^2}{25} & +\frac{4x^3}{125} & +\frac{9x^4}{625} & +\frac{126x^5}{15625} & \&c.
\end{array}
$$

$$
\begin{array}{lllllll}
1 & -\frac{6x}{5} & +\frac{3x^2}{25} & +\frac{4x^3}{125} & +\frac{9x^4}{625} & +\frac{126x^5}{15625} & \&c \\
1 & -\frac{6x}{5} & +\frac{3x^2}{25} & +\frac{4x^3}{125} & +\frac{9x^4}{625} & +\frac{126x^5}{15625} & \&c \\
\hline
1 & -\frac{6x}{5} & +\frac{3x^2}{25} & +\frac{4x^3}{125} & +\frac{9x^4}{625} & +\frac{126x^5}{15625} & \&c \\
 & -\frac{6x}{5} & +\frac{36x^2}{25} & -\frac{18x^3}{125} & -\frac{24x^4}{625} & -\frac{54x^5}{3125} & -\ \&c \\
 & & +\frac{3x^2}{25} & -\frac{18x^3}{125} & +\frac{9x^4}{625} & +\frac{12x^5}{3125} & +\ \&c \\
 & & & +\frac{4x^3}{125} & -\frac{24x^4}{625} & +\frac{12x^5}{3125} & +\ \&c \\
 & & & & +\frac{9x^4}{625} & -\frac{54x^5}{3125} & +\ \&c \\
 & & & & & +\frac{126x^5}{15625} & -\ \&c \\
\hline
1 & -\frac{12x}{5} & +\frac{42x^2}{25} & -\frac{28x^3}{125} & -\frac{21x^4}{625} & -\frac{168x^5}{15625} & \&c.
\end{array}
$$

This

This is the fourth power of the series $1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625}$ — &c.

$$\begin{array}{cccccccc}
1 & - \frac{12x}{5} & + \frac{42x^2}{25} & - \frac{28x^3}{125} & - \frac{21x^4}{625} & - \frac{168x^5}{15625} & & \&c \\
1 & - \frac{3x}{5} & - \frac{3x^2}{25} & - \frac{7x^3}{125} & - \frac{21x^4}{625} & - \frac{357x^5}{15625} & - & \&c \\
\hline
1 & - \frac{12x}{5} & + \frac{42x^2}{25} & - \frac{28x^3}{125} & - \frac{21x^4}{625} & - \frac{168x^5}{15625} & - & \&c \\
 & - \frac{3x}{5} & + \frac{36x^2}{25} & - \frac{126x^3}{125} & + \frac{84x^4}{625} & + \frac{63x^5}{3125} & + & \&c \\
 & & - \frac{3x^2}{25} & + \frac{36x^3}{125} & - \frac{126x^4}{625} & + \frac{84x^5}{3125} & + & \&c \\
 & & & - \frac{7x^3}{125} & + \frac{84x^4}{625} & - \frac{294x^5}{3125} & + & \&c \\
 & & & & - \frac{21x^4}{625} & + \frac{252x^5}{3125} & - & \&c \\
 & & & & & - \frac{357x^5}{15625} & + & \&c \\
\hline
1 & - 3x & + 3x^2 & - x^3 & * & * & &
\end{array}$$

It appears therefore that the series $1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625}$ — &c, is really and truly the fifth root of the quadrinomial quantity $1 - 3x + 3x^2 - x^3$, or of the cube of the residual quantity $1 - x$, and consequently that the residual theorem laid down in art. 25, to wit, that the quantity $\overline{1 - x}|^{\frac{m}{n}}$ is equal to the series II, is true in the case of the fifth root of the cube of a residual quantity, or when m, the numerator of the fraction $\frac{m}{n}$ (which is the index of the power of $1 - x$ in the quantity $\overline{1 - x}|^{\frac{m}{n}}$) is $= 3$, and n, the denominator of the said fraction, is $= 5$.

41. In the foregoing examples the denominator n of the fraction $\frac{m}{n}$ (which is the index of the power of the residual quantity $1 - x$ in the quantity $\overline{1 - x}|^{\frac{m}{n}}$) has been greater than m, the numerator of the said fraction. We will now give an example, or two, of the expression of the value of the quantity $\overline{1 - x}|^{\frac{m}{n}}$ by means of the series II, when the denominator n of the fraction, or index, $\frac{m}{n}$, is less than its numerator m.

And, first, we will investigate the value of $\overline{1 - x}|^{\frac{3}{2}}$, or of the square-root of the cube of the residual quantity $1 - x$ by means of the said series II.

Now it appears by art. 16 of the foregoing discourse concerning the binomial

theorem, page 207, that the quantity $\overline{1+x}|^{\frac{3}{2}}$ is equal to the series $1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} +$ &c. Therefore the series II will be $1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} +$ &c, and consequently, if the proposition contained in art. 25 is true, $\overline{1-x}|^{\frac{3}{2}}$, or the square-root of the cube of the residual quantity $1-x$, or the square-root of the quadrinomial quantity $1 - 3x + 3x^2 - x^3$, will be equal to the series $1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} +$ &c.

42. Now "that this series is really equal to the square-root of the cube of "$1-x$, or to the square-root of the quadrinomial quantity $1 - 3x + 3x^2 -$ "x^3," will appear by multiplying the said series into itself. For we shall find that the product of the said multiplication will be the said quadrinomial quantity. This multiplication will be as follows.

$$
\begin{array}{rrrrrrr}
1 & -\frac{3x}{2} & +\frac{3x^2}{8} & +\frac{x^3}{16} & +\frac{3x^4}{128} & +\frac{3x^5}{256} & +\&c \\
1 & -\frac{3x}{2} & +\frac{3x^2}{8} & +\frac{x^3}{16} & +\frac{3x^4}{128} & +\frac{3x^5}{256} & +\&c \\
\hline
1 & -\frac{3x}{2} & +\frac{3x^2}{8} & +\frac{x^3}{16} & +\frac{3x^4}{128} & +\frac{3x^5}{256} & +\&c \\
 & -\frac{3x}{2} & +\frac{9x^2}{4} & -\frac{9x^3}{16} & -\frac{3x^4}{32} & -\frac{9x^5}{256} & -\&c \\
 & & +\frac{3x^2}{8} & -\frac{9x^3}{16} & +\frac{9x^4}{64} & +\frac{3x^5}{128} & +\&c \\
 & & & +\frac{x^3}{16} & -\frac{3x^4}{32} & +\frac{3x^5}{128} & +\&c \\
 & & & & +\frac{3x^4}{128} & -\frac{9x^5}{256} & +\&c \\
 & & & & & +\frac{3x^5}{256} & -\&c \\
\hline
1 & -3x & +3x^2 & -x^3 & * & * &
\end{array}
$$

It appears therefore that the series $1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} +$ &c is really and truly the square-root of the quadrinomial quantity $1 - 3x + 3x^2 - x^3$, or of the cube of the residual quantity $1-x$, and consequently that the residual theorem laid down in art. 25, to wit, that the quantity $\overline{1-x}|^{\frac{m}{n}}$ is equal to the series II, is true in the case of the square-root of the cube of the residual quantity $1-x$, or when m, the numerator of the fraction $\frac{m}{n}$ (which is the index of the power of $1-x$ in the quantity $\overline{1-x}|^{\frac{m}{n}}$) is $= 3$, and n, the denominator of the said fraction, is $= 2$.

43. I

43. I shall add one more example of the series II, in which the numerator m of the index $\frac{m}{n}$ shall be greater than its denominator n.

Let it be required to find, by means of the said series II, the value of $\overline{1-x}\rceil^{\frac{5}{3}}$, or of the cube-root of the fifth power of the residual quantity $1-x$, or of the cube-root of the sextinomial quantity $1-5x+10x^2-10x^3+5x^4-x^5$.

It appears by art. 18 of the foregoing discourse concerning the binomial theorem, page 208, that the quantity $\overline{1+x}\rceil^{\frac{5}{3}}$ is equal to the series $1+\frac{5x}{3}+\frac{5x^2}{9}-\frac{5x^3}{81}+\frac{5x^4}{243}-\frac{7x^5}{729}+$ &c.

Therefore the series II will be $1-\frac{5x}{3}+\frac{5x^2}{9}+\frac{5x^3}{81}+\frac{5x^4}{243}+\frac{7x^5}{729}+$ &c; and consequently $\overline{1-x}\rceil^{\frac{5}{3}}$, or the cube-root of the fifth power of the residual quantity $1-x$, or the cube-root of the sextinomial quantity $1-5x+10x^2-10x^3+5x^4-x^5$, will be equal to the series $1-\frac{5x}{3}+\frac{5x^2}{9}+\frac{5x^3}{81}+\frac{5x^4}{243}+\frac{7x^5}{729}+$ &c.

44. Now "that this series is really equal to the cube-root of the fifth power "of $1-x$, or to the cube-root of the sextinomial quantity $1-5x+10x^2-$ "$10x^3+5x^4-x^5$," will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be the said sextinomial quantity. These multiplications will be as follow.

$$
\begin{array}{rrrrrrl}
1 & -\frac{5x}{3} & +\frac{5x^2}{9} & +\frac{5x^3}{81} & +\frac{5x^4}{243} & +\frac{7x^5}{729} & +\ \&c \\
1 & -\frac{5x}{3} & +\frac{5x^2}{9} & +\frac{5x^3}{81} & +\frac{5x^4}{243} & +\frac{7x^5}{729} & +\ \&c \\
\hline
1 & -\frac{5x}{3} & +\frac{5x^2}{9} & +\frac{5x^3}{81} & +\frac{5x^4}{243} & +\frac{7x^5}{729} & +\ \&c \\
 & -\frac{5x}{3} & +\frac{25x^2}{9} & -\frac{25x^3}{27} & -\frac{25x^4}{243} & -\frac{25x^5}{729} & -\ \&c \\
 & & +\frac{5x^2}{9} & -\frac{25x^3}{27} & +\frac{25x^4}{81} & +\frac{25x^5}{729} & +\ \&c \\
 & & & +\frac{5x^3}{81} & -\frac{25x^4}{243} & +\frac{25x^5}{729} & +\ \&c \\
 & & & & +\frac{5x^4}{243} & -\frac{25x^5}{729} & +\ \&c \\
 & & & & & +\frac{7x^5}{729} & -\ \&c \\
\hline
1 & -\frac{10x}{3} & +\frac{35x^2}{9} & -\frac{140x^3}{81} & +\frac{35x^4}{243} & +\frac{14x^5}{729} & \&c.
\end{array}
$$

$1-\frac{10x}{3}$

$$\begin{array}{llllllll}
1 & -\frac{10x}{3} & +\frac{35x^2}{9} & -\frac{140x^3}{81} & +\frac{35x^4}{243} & +\frac{14x^5}{729} & & \&c \\
1 & -\frac{5x}{3} & +\frac{5x^2}{9} & +\frac{5x^3}{81} & +\frac{5x^4}{243} & +\frac{7x^5}{729} & + & \&c \\
\hline
1 & -\frac{10x}{3} & +\frac{35x^2}{9} & -\frac{140x^3}{81} & +\frac{35x^4}{243} & +\frac{14x^5}{729} & & \&c \\
 & -\frac{5x}{3} & +\frac{50x^2}{9} & -\frac{175x^3}{27} & +\frac{700x^4}{243} & -\frac{175x^5}{729} & - & \&c \\
 & & +\frac{5x^2}{9} & -\frac{50x^3}{27} & +\frac{175x^4}{81} & -\frac{700x^5}{729} & + & \&c \\
 & & & +\frac{5x^3}{81} & -\frac{50x^4}{243} & +\frac{175x^5}{729} & - & \&c \\
 & & & & +\frac{5x^4}{243} & -\frac{50x^5}{729} & + & \&c \\
 & & & & & +\frac{7x^5}{729} & - & \&c \\
\hline
1 & -5x & +10x^2 & -10x^3 & +5x^4 & -x^5 & & \&c.
\end{array}$$

It appears therefore that the series $1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c$ is really and truly the cube-root of the fifth power of the residual quantity $1 - x$, or the cube-root of the sextinomial quantity $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$, and consequently that the residual theorem laid down in art. 25, to wit, that the quantity $\overline{1-x}|^{\frac{m}{n}}$ is equal to the series II, is true in the case of the cube-root of the fifth power of the residual quantity $1 - x$, or when m, the numerator of the fraction $\frac{m}{n}$ (which is the index of the power of $1 - x$ in the quantity $\overline{1-x}|^{\frac{m}{n}}$) is $= 5$, and n, the denominator of the said fraction, is $= 3$.

45. These examples will, I apprehend, be sufficient to illustrate the two theorems delivered above in art. 2 and 25 of this discourse concerning the quantities $\overline{1-x}|^{\frac{1}{n}}$ and $\overline{1-x}|^{\frac{m}{n}}$, or the nth root of the residual quantity $1 - x$ and the mth power of the said nth root. And therefore, having already given demonstrations of these theorems in art. 11, 12, 13, &c. . . . and 26, 27, 30, 31, 32, &c, I shall here put an end to this discourse.

End of the discourse concerning the residual theorem in the case of fractional powers.

A
METHOD
OF EXTENDING
CARDAN's RULE

FOR RESOLVING THE CUBICK EQUATION

$$y^3 + qy = r, \text{ or } qy + y^3 = r,$$

TO THE RESOLUTION OF THE CUBICK EQUATION

$$qy - y^3 = r, \text{ when } \frac{rr}{4} \text{ is of any magnitude less than } \frac{q^3}{54},$$

$$\text{or } \frac{1}{2} \times \frac{q^3}{27}, \text{ or when } r \text{ is less than } \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}};$$

By the Help of Sir Isaac Newton's binomial and residual Theorems in the Case of Roots, which have been demonstrated in the two preceeding Discourses.

BY FRANCIS MASERES, ESQ. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

A
METHOD
OF EXTENDING
CARDAN's RULE, &c.

ART. 1. THE binomial and residual theorems, which have been demonstrated in the preceeding tracts both with respect to integral and to fractional powers of the quantities $1 + x$ and $1 - x$, are of very extensive use in many other branches of the mathematicks as well as in the construction of the logarithms, or measures, of ratios. And, amongst other subjects, they may be applied to the resolution of cubick equations in which the square of the unknown quantity is wanting, to wit, of the equations $y^3 + qy = r$, and $y^3 - qy = r$, and $qy - y^3 = r$, so as to enable us to find the values of y, in all the possible cases of these equations, or in all the different relative magnitudes of the co-efficient q, and the absolute term r, that can be supposed, not excepting that case of the second equation $y^3 - qy = r$ in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or rr is less than $\frac{4q^3}{27}$, or r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and which (from its not being capable of a direct and immediate resolution by the help of Cardan's rule for resolving the said equation $y^3 - qy = r$, in the first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, or rr is greater than $\frac{4q^3}{27}$, or r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) has obtained amongst algebräists the name of *the irreducible case*. Even in this case we may always, by the help of Sir Isaac Newton's binomial and residual theorems in the case of roots (which have been investigated in the two preceeding discourses) derive from one or other, of Cardan's rules an expression of the true value of y in the said equation $y^3 - qy = r$. The method of doing this, when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, has been explained at considerable length in a paper of

mine

mine publiſhed in the Philoſophical Tranſactions for the year 1778, which proceeds upon clear and intelligible principles, without any mention of impoſſible roots or impoſſible quantities of any kind, or even of negative quantities. But when $\frac{rr}{4}$ is leſs than $\frac{q^3}{2\times 27}$, or $\frac{q^3}{54}$, or rr is leſs than ($\frac{4q^3}{2\times 27}$, or) $\frac{2q^3}{27}$, or r is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, the method explained in that paper will not enable us to find the value of y in that equation; becauſe the ſeries obtained in that paper for the ſaid value will not in that caſe be a converging ſeries. Nor had I, at the time of publiſhing that paper, diſcovered any method of deriving from Cardan's rules an expreſſion of the value of y in the ſaid equation $y^3 - qy = r$ in this ſecond branch of the irreducible caſe of it, or when $\frac{rr}{4}$ was leſs than $\frac{q^3}{2\times 27}$, or $\frac{q^3}{54}$, or r was leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$. But I have ſince found out a way of doing it by the help of Cardan's firſt rule, which gives us the value of y in the equation $y^3 + qy = r$, or $qy + y^3 = r$. For from the expreſſion of this value of y in the ſaid equation $qy + y^3 = r$ we may, in the caſe ſuppoſed, or when $\frac{rr}{4}$ is leſs than $\frac{q^3}{2\times 27}$, or $\frac{q^3}{54}$, or r is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, derive an expreſſion of the value of the leſſer of the two roots of the equation $qy - y^3 = r$, by the help of the two ſerieſes for expreſſing the cube-roots of the binomial quantity $1 + x$ and the reſidual quantity $1 - x$ obtained by means of the binomial and reſidual theorems: and from the value of the ſaid leſſer root of the equation $qy - y^3 = r$, we may afterwards derive the value of y in the equation $y^3 - qy = r$ by the reſolution of a quadratick equation. And this extenſion of Cardan's firſt rule (which gives us the value of y in the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$), to the diſcovery of the leſſer root of the cubick equation $qy - y^3 = r$, when r is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, may be made in a clear and intelligible manner, without any mention of impoſſible roots, or other impoſſible quantities, or even of negative quantities, as well as the former extenſion of Cardan's ſecond rule (which gives us the value of y in the equation $y^3 - qy = r$ in the firſt caſe of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, or r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) to the diſcovery of the value of y in the firſt branch of the ſecond caſe of that equation, or when $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, but greater than $\frac{q^3}{2\times 27}$, or $\frac{q^3}{54}$. The manner of making this extenſion of Cardan's firſt rule to the diſcovery of the leſſer root of the equation $qy - y^3 = r$, I ſhall now endeavour to explain.

The

The values of the quantities $\overline{1 + x}|^{\frac{1}{3}}$ *and* $\overline{1 - x}|^{\frac{1}{3}}$, or $\sqrt[3]{1 + x}$ and $\sqrt[3]{1 - x}$, *expressed in infinite serieses by means of the binomial and residual theorems.*

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2. It has been shewn above in the discourse concerning the binomial theorem in the case of fractional powers, art. 51, page 242, that $\overline{1 + x}|^{\frac{1}{3}}$, or $\sqrt[3]{1 + x}$, or the cube-root of the binomial quantity $1 + x$, is equal to the infinite series $1 + \frac{1}{3}Ax - \frac{2}{6}Bx^2 + \frac{5}{9}Cx^3 - \frac{8}{12}Dx^4 + \frac{11}{15}Ex^5 - \frac{14}{18}Fx^6 + \frac{17}{21}Gx^7 - \frac{20}{24}Hx^8 + \frac{23}{27}Ix^9 - \frac{26}{30}Kx^{10} + \frac{29}{33}Lx^{11} - \frac{32}{36}Mx^{12} + \frac{35}{39}Nx^{13} - \frac{38}{42}Ox^{14} + \frac{41}{45}Px^{15} - \frac{44}{48}Qx^{16} + \frac{47}{51}Rx^{17} - \frac{50}{54}Sx^{18} +$ &c *ad infinitum*, or $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{154x^6}{6561} + \frac{374x^7}{19{,}683} - \frac{935x^8}{59{,}049} + \frac{21{,}505x^9}{1{,}594{,}323} - \frac{55{,}913x^{10}}{4{,}782{,}969} + \frac{147{,}407x^{11}}{14{,}348{,}907} - \frac{1{,}179{,}256x^{12}}{129{,}140{,}163} + \frac{3{,}174{,}920x^{13}}{387{,}420{,}489} - \frac{8{,}617{,}640x^{14}}{1{,}162{,}261{,}467} + \frac{70{,}664{,}648x^{15}}{10{,}460{,}353{,}203} - \frac{194{,}327{,}782x^{16}}{31{,}381{,}059{,}609} + \frac{537{,}259{,}162x^{17}}{94{,}143{,}178{,}827} - \frac{13{,}431{,}479{,}050x^{18}}{2{,}541{,}865{,}828{,}329} +$ &c *ad infinitum*. Therefore (by what is shewn in the foregoing discourse concerning the residual theorem in the case of fractional powers, art. 23, page 360) the quantity $\overline{1 - x}|^{\frac{1}{3}}$, or $\sqrt[3]{1 - x}$, or the cube-root of the residual quantity $1 - x$, will be equal to a series consisting of the very same terms as the foregoing series, but with the sign $-$ prefixed to every term after the first term 1, instead of every other term, that is, to the series $1 - \frac{1}{3}Ax - \frac{2}{6}Bx^2 - \frac{5}{9}Cx^3 - \frac{8}{12}Dx^4 - \frac{11}{15}Ex^5 - \frac{14}{18}Fx^6 - \frac{17}{21}Gx^7 - \frac{20}{24}Hx^8 - \frac{23}{27}Ix^9 - \frac{26}{30}Kx^{10} - \frac{29}{33}Lx^{11} - \frac{32}{36}Mx^{12} - \frac{35}{39}Nx^{13} - \frac{38}{42}Ox^{14} - \frac{41}{45}Px^{15} - \frac{44}{48}Qx^{16} - \frac{47}{51}Rx^{17} - \frac{50}{54}Sx^{18} -$ &c *ad infinitum*, or $1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \frac{154x^6}{6561} - \frac{374x^7}{19{,}683} - \frac{935x^8}{59{,}049} - \frac{21{,}505x^9}{1{,}594{,}323} - \frac{55{,}913x^{10}}{4{,}782{,}969} - \frac{147{,}407x^{11}}{14{,}348{,}907} - \frac{1{,}179{,}256x^{12}}{129{,}140{,}163} - \frac{3{,}174{,}920x^{13}}{387{,}420{,}489} - \frac{8{,}617{,}640x^{14}}{1{,}162{,}261{,}467} - \frac{70{,}664{,}648x^{15}}{10{,}460{,}353{,}203} - \frac{194{,}327{,}782x^{16}}{31{,}381{,}059{,}609} - \frac{537{,}259{,}162x^{17}}{94{,}143{,}178{,}827} - \frac{13{,}431{,}479{,}050x^{18}}{2{,}541{,}865{,}828{,}329} -$ &c.

The

The value of $\sqrt[3]{1+x} - \sqrt[3]{1-x}$ *expressed in an infinite series.*

3. Therefore the difference of the two quantities $\sqrt[3]{1+x}$ and $\sqrt[3]{1-x}$, or the excess of $\sqrt[3]{1+x}$, (or the cube-root of the binomial quantity $1+x$,) above $\sqrt[3]{1-x}$, (or the cube root of the residual quantity $1-x$, will be) equal to the excess of the former of these two serieses above the latter, that is (if, for the sake of brevity, we denote the co-efficients of the second, and third, and fourth, and other following terms of these two serieses by the single letters B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, &c), to the excess of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - Gx^6 + Hx^7 - Ix^8 + Kx^9 - Lx^{10} + Mx^{11} - Nx^{12} + Ox^{13} - Px^{14} + Qx^{15} - Rx^{16} + Sx^{17} - Tx^{18} +$ &c above the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - Gx^6 - Hx^7 - Ix^8 - Kx^9 - Lx^{10} - Mx^{11} - Nx^{12} - Ox^{13} - Px^{14} - Qx^{15} - Rx^{16} - Sx^{17} - Tx^{18} -$ &c, and consequently to the series $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + 2Ox^{13} + 2Qx^{15} + 2Sx^{17}$ + &c.

Of the root of the cubick equation $y^3 + qy = r$, *or* $qy + y^3 = r$, *according to Cardan's first rule.*

4. The first of the two rules for the resolution of certain cubick equations usually known by the name of *Cardan's rules* was not invented by Cardan himself, but, about 30 years before the publication of his book, by one *Scipio Ferreus*, of *Bononia*, or *Bologna*, in Italy; as Cardan himself informs us. It relates to the resolution of the cubick equation $y^3 + qy = r$, and is true in all the possible cases of that equation, or in all the different relative magnitudes of q (the co-efficient of the unknown quantity y), and the absolute term r, and consequently when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, which is the case that we are about to consider. And it gives us two different expressions of the value of y in the said equation, to wit, 1st, $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}}\ \frac{-q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}}}$, and, 2dly, $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}} - \sqrt[3]{-\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}}$, or (if, for the sake of brevity, we put $ee = \frac{rr}{4}$, and $ss = \frac{q^3}{27} + \frac{rr}{4}$) 1st, $\sqrt[3]{e+s}\ \frac{-q}{3\sqrt[3]{e+s}}$, and,

 2dly,

2dly, $\sqrt[3]{e+s} - \sqrt[3]{-e+s}$, or, rather, (becaufe ss is greater than $\frac{rr}{4}$, or ee, and confequently s is greater than $\frac{r}{2}$, or e, and therefore ought to be placed before it) 1ft, $\sqrt[3]{s+e}\ \frac{-q}{3\sqrt[3]{s+e}}$, and, 2dly, $\sqrt[3]{s+e} - \sqrt[3]{s-e}$. See my differtation on the ufe of the negative fign in Algebra, art. 208, 209, 210, 211, 212, pages 178, 179, 180.

It is the fecond of thefe two expreffions, to wit, the expreffion $\sqrt[3]{s+e} - \sqrt[3]{s-e}$, that we fhall have occafion to confider in the courfe of the following articles.

The value of the root of the faid cubick equation expreffed by an infinite feries by means of the binomial and refidual theorems.

5. Now $s+e$ is $= s \times \overline{1+\frac{e}{s}}$, and $s-e$ is $= s \times \overline{1-\frac{e}{s}}$. Therefore $\sqrt[3]{s+e}$ will be $= \sqrt[3]{s} \times \sqrt[3]{1+\frac{e}{s}}$, and $\sqrt[3]{s-e}$ will be $= \sqrt[3]{s} \times \sqrt[3]{1-\frac{e}{s}}$; and confequently $\sqrt[3]{s+e} - \sqrt[3]{s-e}$ will be $= \sqrt[3]{s} \times \sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{s} \times \sqrt[3]{1-\frac{e}{s}} = \sqrt[3]{s} \times \overline{\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}}}$. Therefore the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$, will be $= \sqrt[3]{s} \times \overline{\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}}}$.

But, fince $\sqrt[3]{1+x} - \sqrt[3]{1-x}$ is equal to the feries $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + 2Ox^{13} + 2Qx^{15} + 2Sx^{17} +$ &c *ad infinitum* (as is fhewn in art. 3) it follows (by fubftituting $\frac{e}{s}$ inftead of x in the terms of the faid equation) that $\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}}$ will be equal to the feries $\frac{2Be}{s} + \frac{2De^3}{s^3} + \frac{2Fe^5}{s^5} + \frac{2He^7}{s^7} + \frac{2Ke^9}{s^9} + \frac{2Me^{11}}{s^{11}} + \frac{2Oe^{13}}{s^{13}} + \frac{2Qe^{15}}{s^{15}} + \frac{2Se^{17}}{s^{17}} +$ &c *ad infinitum.* Therefore $\sqrt[3]{s} \times \overline{\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}}}$ will be equal to $\sqrt[3]{s} \times$ the feries $\frac{2Be}{s} + \frac{2De^3}{s^3} + \frac{2Fe^5}{s^5} + \frac{2He^7}{s^7} + \frac{2Ke^9}{s^9} + \frac{2Me^{11}}{s^{11}} + \frac{2Oe^{13}}{s^{13}} + \frac{2Qe^{15}}{s^{15}} + \frac{2Se^{17}}{s^{17}} +$ &c *ad infinitum*; and confequently the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$, will be equal to $\sqrt[3]{s} \times$ the faid feries $\frac{2Be}{s} + \frac{2De^3}{s^3} + \frac{2Fe^5}{s^5} + \frac{2He^7}{s^7} + \frac{2Ke^9}{s^9} + \frac{2Me^{11}}{s^{11}} + \frac{2Oe^{13}}{s^{13}} + \frac{2Qe^{15}}{s^{15}} + \frac{2Se^{17}}{s^{17}} +$ &c *ad*

ad infinitum, or to $2\sqrt[3]{s} \times$ the feries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \frac{Qe^{15}}{s^{15}} + \frac{Se^{17}}{s^{17}} +$ &c *ad infinitum*, or (reftoring the values of the co-efficients B, C, D, E, F, G, H, &c) $2\sqrt[3]{s} \times$ the feries $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19{,}683s^7} + \frac{21{,}505e^9}{1{,}594{,}323s^9} + \frac{147{,}407e^{11}}{14{,}348{,}907s^{11}} + \frac{3{,}174{,}920e^{13}}{387{,}420{,}489s^{13}} + \frac{70{,}664{,}648e^{15}}{10{,}460{,}353{,}203s^{15}} + \frac{537{,}259{,}162e^{17}}{94{,}143{,}178{,}827s^{17}} +$ &c *ad infinitum.*

We will now proceed to give an example of the refolution of a cubick equation of the faid form $y^3 + qy = r$, or $qy + y^3 = r$, by means of this expreffion.

An example of the refolution of a cubick equation of the foregoing form $y^3 + qy = r$, or $qy + y^3 = r$, by means of the expreffion $2\sqrt[3]{s} \times$ the feries $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19{,}683s^7} + \frac{21{,}505e^9}{1{,}594{,}323s^9} + \frac{147{,}407e^{11}}{14{,}348{,}907s^{11}} + \frac{3{,}174{,}920e^{13}}{387{,}420{,}489s^{13}} + \frac{70{,}664{,}648e^{15}}{10{,}460{,}353{,}203s^{15}} + \frac{537{,}259{,}162e^{17}}{94{,}143{,}178{,}827s^{17}} +$ &c ad infinitum.

6. Let the equation that is to be refolved by means of this expreffion, be $y^3 + 15y = 4$, or $15y + y^3 = 4$.

Here q is $= 15$, and r is $= 4$; and confequently $\frac{q}{3}$ is $(= \frac{15}{3}) = 5$, and $\frac{r}{2}$ is $(= \frac{4}{2}) = 2$, and $\frac{q^3}{27}$ is $(= \overline{\tfrac{q}{3}}|^3 = \overline{5}|^3) = 125$, and $\frac{rr}{4}$ is $(= \overline{\tfrac{r}{2}}|^2 = \overline{2}|^2) = 4$.

We fhall therefore have ss $(= \frac{q^3}{27} + \frac{rr}{4} = 125 + 4) = 129$, and s $(= \sqrt{129}) = 11.357{,}816{,}691$, and $\sqrt[3]{s}$ $(= \sqrt[3]{11.357{,}816{,}691}) = 2.247{,}835$, and confequently $2\sqrt[3]{s}$ $(= 2 \times 2.247{,}835) = 4.495{,}670$.

And we fhall alfo have e $(= \frac{r}{2}) = 2$, and confequently $\frac{e}{s}$ $(= \frac{2.000{,}000{,}000}{11.357{,}816{,}691}) = 0.176{,}090{,}181$, and $\frac{ee}{ss}$ $(= \overline{0.176{,}090{,}181}|^2) = 0.031{,}006{,}752$, and $\frac{e^3}{s^3}$ $(= \frac{e}{s} \times \frac{e^2}{s^2} = 0.176{,}090{,}181 \times 0.031{,}006{,}752) = 0.005{,}459{,}984$, and $\frac{e^5}{s^5}$ $(= \frac{e^3}{s^3} \times \frac{e^2}{s^2} = 0.005{,}459{,}984 \times 0.031{,}006{,}752) = 0.000{,}169{,}296$, and $\frac{e^7}{s^7}$ $(= \frac{e^5}{s^5} \times \frac{e^2}{s^2} = 0.000{,}169{,}296 \times 0.031{,}006{,}752) = 0.000{,}005{,}249$, and $\frac{e^9}{s^9}$ $(= \frac{e^7}{s^7} \times \frac{e^2}{s^2} = 0.000{,}005{,}249 \times 0.031{,}006{,}752) = 0.000{,}000{,}162$,

and $\frac{e^{11}}{s^{11}}$ ($= \frac{e^9}{s^9} \times \frac{e^2}{s^2} = 0.000,000,162 \times 0.031,006,752$) $= 0.000,000,005$, and $\frac{e^{13}}{s^{13}}$ ($= \frac{e^{11}}{s^{11}} \times \frac{e^2}{s^2} = 0.000,000,005 \times 0.031,006,752$) $= 0.000,000,000$, and $\frac{e}{3s}$ ($= \frac{1}{3} \times \frac{e}{s} = \frac{1}{3} \times 0.176,090,181$) $= 0.058,696,727$, and $\frac{5e^3}{81s^3}$ ($= \frac{5}{81} \times \frac{e^3}{s^3} = \frac{5}{81} \times 0.005,459,984 = \frac{5 \times 0.005,459,984}{81} = \frac{0.027,299,920}{81}$) $= 0.000,337,036$, and $\frac{22e^5}{729s^5}$ ($= \frac{22}{729} \times \frac{e^5}{s^5} = \frac{22}{729} \times 0.000,169,296 = \frac{22 \times 0.000,169,296}{729} = \frac{0.003,724,512}{729}$) $= 0.000,005,109$, and $\frac{374e^7}{19,683s^7}$ ($= \frac{374}{19,683} \times \frac{e^7}{s^7} = \frac{374}{19,683} \times 0.000,005,249 = \frac{374 \times 0.000,005,249}{19,683} = \frac{0.001,963,126}{19,683}$) $= 0.000,000,099$, and $\frac{21,505e^9}{1,594,323s^9}$ ($= \frac{21,505}{1,594,323} \times \frac{e^9}{s^9} = \frac{21,505}{1,594,323} \times 0.000,000,162 = \frac{21,505 \times 0.000,000,162}{1,594,323} = \frac{0.003,483,810}{1,594,323}$) $= 0.000,000,002$, and $\frac{147,407e^{11}}{14,348,907s^{11}}$ ($= \frac{147,407}{14,348,907} \times \frac{e^{11}}{s^{11}} = \frac{147,407}{14,348,907} \times 0.000,000,005 = \frac{147,407 \times 0,000,000,005}{14,348,907} = \frac{0.000,737,035}{14,348,907}$) $= 0.000,000,000$. Therefore the series $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19,683s^7} + \frac{21,505e^9}{1,594,323s^9} + \frac{147,407e^{11}}{14,438,907s^{11}} +$ &c will be ($= 0.058,696,727 + 0.000,337,036 + 0.000,005,109 + 0.000,000,099 + 0.000,000,002, + 0.000,000,000, +$ &c) $= 0.059,038,973$; and consequently $2\sqrt{^2}\, s$ multiplied into the said series will be $= 2\sqrt{^3}\, s$ multiplied into $0.059,038,973 = 4.495,670 \times 0.059,038,973 = 0.265,419,739$. Therefore y, or the root of the cubick equation $y^3 + 15y = 4$, or $15y + y^3 = 4$, will be $= 0.265,419,739$. Q. E. I.

7. This value of y approaches very nearly to the truth. For, if we suppose y to be equal to $0.265,419,739$, we shall have $yy = 0.070,447,637,850$, and $y^3 = 0.018,698,193$, and $15y$ ($= 15 \times 0.265,419,739$) $= 3.981,296,085$, and consequently $y^3 + 15y$ ($= 0.018,698,193 + 3.981,296,085$) $= 3.999,994,278$; which differs from the absolute term, 4, of the proposed equation $y^3 + 15y = 4$ by only $0,000,005,722$, or less than $0.000,006$, or $\frac{6}{1000,000}$, or 6 millionth parts of an unit, or 6 four-millionth parts, or 3 two-millionth parts, of the said absolute term 4 itself.

8. If we were to make the value of y already found, to wit, $0.265,419,739$, the basis of a further approximation to its true value according to Mr. Raphson's method of resolving equations, by supposing y to be equal to $0.265,419,739 + z$, and substituting this compound quantity instead of y in the proposed equation $y^3 + qy = 4$, and then resolving the equation resulting from such substitution in the same manner as a simple equation by omitting all the terms that involve either the square or cube of z, we should find that z was equal to $\frac{0.000,005,722,000,000,000}{15.211,342,913}$

$\frac{0.000,005,722,000,000,000}{15.211,342,913} = 0.000,000,376$, and consequently that y, or 0.265, 419,739 + z, would be (= 0.265,419,739 + 0.000,000,376) = 0.265,420, 115. We may therefore consider this last number, 0.265,420,115, as being the true value of y as far as the said true value can be expressed in nine places of figures.

End of the resolution of the equation $y^3 + 15y = 4$, or $15y + y^3 = 4$.

9. It has been shewn in the foregoing articles, that, if e be put $= \frac{r}{2}$, and ss be $= \frac{q^3}{27} + \frac{rr}{4}$, or $\frac{q^3}{27} + ee$, the root y of the cubick equation $y^3 + qy = r$ will be equal to $2\sqrt{^3}\, s \times$ the series $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19,683s^7} + \frac{21,505e^9}{1,594,323s^9} + \frac{147,407e^{11}}{14,348,907s^{11}}$ + &c *ad infinitum*, or $2\sqrt{^3}\, s \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \frac{Qe^{15}}{s^{15}} + \frac{Se^{17}}{s^{17}}$ + &c *ad infinitum*, the terms of which are the second, fourth, sixth, eighth, tenth, and other following terms of the series which is equal to $\overline{1 + \frac{e}{s}}|^{\frac{1}{3}}$, or $\sqrt{^3}\,\overline{1 + \frac{e}{s}}$, or the cube-root of the binomial quantity $1 + \frac{e}{s}$. Now from this expression, which is equal to the value of y in the equation $y^3 + qy = r$, or $qy + y^3 = r$, we may, by a peculiar train of reasoning, derive another expression, very much resembling the former, which shall be equal to the lesser of the two roots of the equation $qx - x^3 = r$ in which the letters q and r denote the same known quantities as in the foregoing equation $y^3 + qy = r$, or $qy + y^3 = r$. The method of doing this I shall now endeavour to explain.

Of the cubick equation $qx - x^3 = r$.

10. The equation $qx - x^3 = r$ is not always possible, whatever be the magnitudes of q and r, but only when r is equal to, or less than, the quantity $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or that value of the compound quantity $qx - x^3$ which results from the substitution of $\frac{\sqrt{q}}{\sqrt{3}}$, or $\sqrt{\frac{q}{3}}$, in its terms instead of x. If r is equal to this quantity, the equation $qx - x^3 = r$ will have only one root, to wit, $\frac{\sqrt{q}}{\sqrt{3}}$, or $\sqrt{\frac{q}{3}}$; and, when the absolute term r is less than the said quantity $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qx - x^3 = r$ will have two roots, of which the lesser will be less than $\frac{\sqrt{q}}{\sqrt{3}}$,

$\frac{\sqrt{q}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{q}}{\sqrt{3}}$, but lefs than $\sqrt{q}$. See my differtation on the ufe of the negative fign in Algebra, art. 114, page 92. It is the leffer of thefe two roots of the equation $qx - x^3 = r$ that I now propofe to find in thofe cafes of the faid equation in which the abfolute term is lefs not only than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ (which is its greateft poffible magnitude), but than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, by means of an expreffion to be derived from the foregoing expreffion $2\sqrt{^3}\, s \times$ the feries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \frac{Qe^{15}}{s^{15}} + \frac{Se^{17}}{s^{17}} +$ &c *ad infinitum*, which is equal to the root y of the equation $y^3 + qy = r$, or $qy + y^3 = r$, the letters q and r being fuppofed to ftand for the fame quantities in both equations.

The value of the leffer root of the faid equation expreffed in an infinite feries.

11. Now, when r is lefs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or rr is lefs than $\frac{2q^3}{27}$, or $\frac{rr}{4}$ is lefs than $(\frac{1}{4} \times \frac{2q^3}{27}$, or $\frac{q^3}{2 \times 27}$, or) $\frac{q^3}{54}$, let zz be taken $= \frac{q^3}{27} - \frac{rr}{4}$, and let z be fubftituted every where, inftead of s, in the foregoing expreffion $2\sqrt{^3}\, s \times$ the feries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{11}} +$ &c *ad infinitum*, whereby the faid expreffion will be changed into the expreffion $2\sqrt{^3}\, z \times$ the feries $\frac{Be}{z} + \frac{De^3}{z^3} + \frac{Fe^5}{z^5} + \frac{He^7}{z^7} + \frac{Ke^9}{z^9} + \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} +$ &c *ad infinitum*; and, laftly, let the fign — be prefixed to the fecond, fourth, fixth, eighth, tenth, and every following even term of this laft feries, inftead of the fign +. The new expreffion thereby obtained, to wit, the expreffion $2\sqrt{^3}\, z \times$ the feries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c will be equal to the leffer root of the equation $qx - x^3 = r$. This propofition we will now endeavour to demonftrate.

A proof that the infinite feries fet forth in the foregoing article is a converging feries.

12. In the firft place it will be proper to fhew that this feries will be a converging feries. Now this may be fhewn in the manner following.

Since $\frac{rr}{4}$, or ee, is lefs than $\frac{q^3}{54}$, it follows that $\frac{rr}{2}$, or $2ee$, will be lefs than $\frac{2q^3}{54}$, or $\frac{q^3}{27}$. Therefore (fubtracting ee, or $\frac{rr}{4}$, from both fides) $2ee - ee$ will be lefs than $\frac{q^3}{27} - \frac{rr}{4}$, that is ee will be lefs than zz. Confequently e will be lefs than

than z, and the fraction $\frac{e}{z}$ will be less than 1. Therefore the fractions $\frac{e}{z}$, $\frac{e^3}{z^3}$, $\frac{e^5}{z^5}$, $\frac{e^7}{z^7}$, $\frac{e^9}{z^9}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{13}}{z^{13}}$, &c will form a decreasing progression, and consequently *à fortiori*, the several terms of the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c (of which the numeral co-efficients B, D, F, H, K, M, O, &c, also form a decreasing progression) will also form a decreasing progression, or a converging series. Q. E. D.

Therefore the expression $2\sqrt{^3}\, z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c will converge to a certain finite magnitude, and consequently may be equal to the lesser root of the cubick equation $qx - x^3 = r$. It remains that we shew that it is so.

Preparations for demonstrating that the foregoing infinite series is equal to the lesser root of the equation $qx - x^3 = r$.

13. In order to demonstrate that the expression $2\sqrt{^3}\, z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c is equal to the lesser root of the equation $qx - x^3 = r$, it will be necessary to resume the consideration of the expression $2\sqrt{^3}\, s \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{M^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum*, from which it is derived.

Now, since the expression $2\sqrt{^3}\, s \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^6} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum* is equal to the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$, it follows that, if we were, first, to multiply the said expression $2\sqrt{^3}\, s \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}}$ + &c *ad infinitum* into the co-efficient q, and then to raise the said expression $2\sqrt{^3}\, s \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum* to its cube, or third power, and lastly, to add the said cube to the said product, the sum thence arising would be equal to r, to whatever number of terms the said series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c may be continued. For, if this sum were not equal to r, it would not be true that the expression $2\sqrt{^3}\, s \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum*, was equal to the root of the equation $qy + y^3 = r$.

These operations may be performed in the following manner.

14. Since

14. Since $\frac{q^3}{27} + \frac{rr}{4}$ is $= ss$, or (becaufe $\frac{rr}{4}$ is $= ee$), $\frac{q^3}{27} + ee$ is $= ss$, we fhall have $\frac{q^3}{27} = ss - ee$, and $q^3 = 27 \times \overline{ss - ee} = 27 \times ss \times \overline{1 - \frac{ee}{ss}}$, and confequently $q = 3 \times \overline{ss}|^{\frac{1}{3}} \times \overline{1 - \frac{ee}{ss}}|^{\frac{1}{3}} = 3 \times s^{\frac{2}{3}} \times \overline{1 - \frac{ee}{ss}}|^{\frac{1}{3}} = 3s^{\frac{2}{3}} \times \overline{1 - \frac{ee}{ss}}|^{\frac{1}{3}} = 3s^{\frac{2}{3}} \times$ the feries $1 - \frac{Bee}{ss} - \frac{Ce^4}{s^4} - \frac{De^6}{s^6} - \frac{Ee^8}{s^8} - \frac{Fe^{10}}{s^{10}} - \frac{Ge^{12}}{s^{12}} -$ &c *ad infinitum*. Therefore the product qy will be $= 3s^{\frac{2}{3}} \times$ the feries $1 - \frac{Bee}{ss} - \frac{Ce^4}{s^4} - \frac{De^6}{s^6} - \frac{Ee^8}{s^8} - \frac{Fe^{10}}{s^{10}} - \frac{Ge^{12}}{s^{12}} -$ &c *ad infinitum* $\times y = 3s^{\frac{2}{3}} \times$ the feries $1 - \frac{Bee}{ss} - \frac{Ce^4}{s^4} - \frac{De^6}{s^6} - \frac{Ee^8}{s^8} - \frac{Fe^{10}}{s^{10}} - \frac{Ge^{12}}{s^{12}} -$ &c *ad infinitum* $\times$ the expreffion $2s^{\frac{1}{3}} \times$ the feries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum* $= 6s \times$ the feries $1 - \frac{Bee}{ss} - \frac{Ce^4}{s^4} - \frac{De^6}{s^6} - \frac{Ee^8}{s^8} - \frac{Fe^{10}}{s^{10}} - \frac{Ge^{12}}{s^{12}} -$ &c *ad infinitum* $\times$ the feries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum* $= 6s$ into the following compound feries, to wit,

$$
\begin{array}{llllllll}
\frac{Be}{s} & - \frac{B^2e^3}{s^3} & - \frac{BCe^5}{s^5} & - \frac{BDe^7}{s^7} & - \frac{BEe^9}{s^9} & - \frac{BFe^{11}}{s^{11}} & - \frac{BGe^{13}}{s^{13}} & - \&c \\
 & + \frac{De^3}{s^3} & - \frac{BDe^5}{s^5} & - \frac{CDe^7}{s^7} & - \frac{D^2e^9}{s^9} & - \frac{DEe^{11}}{s^{11}} & - \frac{DFe^{13}}{s^{13}} & - \&c \\
 & & + \frac{Fe^5}{s^5} & - \frac{BFe^7}{s^7} & - \frac{CFe^9}{s^9} & - \frac{DFe^{11}}{s^{11}} & - \frac{EFe^{13}}{s^{13}} & - \&c \\
 & & & + \frac{He^7}{s^7} & - \frac{BHe^9}{s^9} & - \frac{CHe^{11}}{s^{11}} & - \frac{DHe^{13}}{s^{13}} & - \&c \\
 & & & & + \frac{Ke^9}{s^9} & - \frac{BKe^{11}}{s^{11}} & - \frac{CKe^{13}}{s^{13}} & - \&c \\
 & & & & & + \frac{Me^{11}}{s^{11}} & - \frac{BMe^{13}}{s^{13}} & - \&c \\
 & & & & & & + \frac{Oe^{13}}{s^{13}} & - \&c;
\end{array}
$$

or, if, for the fake of brevity, we denote this compound feries by the Greek capital letter Γ, we fhall have the product $qy = 6s \times$ the compound feries Γ.

15. In the foregoing compound feries Γ, which is the product of the multiplication of the feries $1 - \frac{Bee}{ss} - \frac{Ce^4}{s^4} - \frac{De^6}{s^6} - \frac{Ee^8}{s^8} - \frac{Fe^{10}}{s^{10}} - \frac{Ge^{12}}{s^{12}} -$ &c (of which all the terms after the firft term 1 are marked with the fign $-$, or fubtracted from the faid firft term) into the feries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c (of which all the terms after the firft term $\frac{Be}{s}$ are marked with

with the ſign +, or added to the ſaid firſt term), it is evident that the firſt term of every horizontal row of terms will be marked with the ſign +, and that every following term of the ſame row will be marked with the ſign —, to whatever number of terms the ſaid horizontal rows of terms may be continued, and likewiſe that the firſt term of every new horizontal row of terms is placed immediately under the ſecond term of the next preceeding horizontal row, becauſe they both involve the ſame power of the fraction $\frac{e}{s}$; whence it follows that all the terms in every vertical column of terms, except the loweſt term, will be marked with the ſign —, and that the ſaid loweſt term will be marked with the ſign +. This will appear moſt evidently upon performing the multiplication of the former of the ſaid ſerieſes into the latter.

16. If we multiply the foregoing compound ſeries called Γ into $6s$, the product will be the following compound ſeries, to wit,

$$\begin{array}{l}
6Be - \frac{6B^2e^3}{s^2} - \frac{6BCc^5}{s^4} - \frac{6BDe^7}{s^6} - \frac{6BEe^9}{s^8} - \frac{6BFe^{11}}{s^{10}} - \frac{6BGe^{13}}{s^{12}} - \&c \\
\qquad + \frac{6De^3}{s^2} - \frac{6BDe^5}{s^4} - \frac{6CDe^7}{s^6} - \frac{6D^2e^9}{s^8} - \frac{6DEe^{11}}{s^{10}} - \frac{6DFe^{13}}{s^{12}} - \&c \\
\qquad\qquad + \frac{6Fe^5}{s^4} - \frac{6BFe^7}{s^6} - \frac{6CFe^9}{s^8} - \frac{6DFe^{11}}{s^{10}} - \frac{6EFe^{13}}{s^{12}} - \&c \\
\qquad\qquad\qquad + \frac{6He^7}{s^6} - \frac{6BHe^9}{s^8} - \frac{6CHe^{11}}{s^{10}} - \frac{6DHe^{13}}{s^{12}} - \&c \\
\qquad\qquad\qquad\qquad + \frac{6Ke^9}{s^8} - \frac{6BKe^{11}}{s^{10}} - \frac{6CKe^{13}}{s^{12}} - \&c \\
\qquad\qquad\qquad\qquad\qquad + \frac{6Me^{11}}{s^{10}} - \frac{6BMe^{13}}{s^{12}} - \&c \\
\qquad\qquad\qquad\qquad\qquad\qquad + \frac{6Oe^{13}}{s^{12}} - \&c.
\end{array}$$

Let this compound ſeries, for the ſake of brevity, be denoted by the Greek capital letter Δ. Then, ſince the product qy is equal to $6s \times$ the compound ſeries Γ, it will alſo be equal to the compound ſeries Δ.

17. In the next place we muſt find the cube of the expreſſion $2\sqrt[3]{s} \times$ the ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum*, which is equal to y.

Now the cube of this expreſſion is $= 8s \times$ the cube of the ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c; which cube may be found by multiplying the ſaid ſeries twice into itſelf in the manner following.

The

The multiplication of the series $\frac{Be}{s}+\frac{De^3}{s^3}+\frac{Fe^5}{s^5}+\frac{He^7}{s^7}+\frac{Ke^9}{s^9}+\frac{Me^{11}}{s^{11}}+\frac{Oe^{13}}{s^{13}}+$ *&c twice into itself, in order to obtain its cube.*

$$\frac{Be}{s}+\frac{De^3}{s^3}+\frac{Fe^5}{s^5}+\frac{He^7}{s^7}+\frac{Ke^9}{s^9}+\frac{Me^{11}}{s^{11}}+\frac{Oe^{13}}{s^{13}}+\&c$$

$$\frac{Be}{s}+\frac{De^3}{s^3}+\frac{Fe^5}{s^5}+\frac{He^7}{s^7}+\frac{Ke^9}{s^9}+\frac{Me^{11}}{s^{11}}+\frac{Oe^{13}}{s^{13}}+\&c$$

$$\frac{B^2e^2}{s^2}+\frac{BDe^4}{s^4}+\frac{BFe^6}{s^6}+\frac{BHe^8}{s^8}+\frac{BKe^{10}}{s^{10}}+\frac{BMe^{12}}{s^{12}}+\&c$$

$$+\frac{BDe^4}{s^4}+\frac{D^2e^6}{s^6}+\frac{DFe^8}{s^8}+\frac{DHe^{10}}{s^{10}}+\frac{DKe^{12}}{s^{12}}+\&c$$

$$+\frac{BFe^6}{s^6}+\frac{DFe^8}{s^8}+\frac{F^2e^{10}}{s^{10}}+\frac{FHe^{12}}{s^{12}}+\&c$$

$$+\frac{BHe^8}{s^8}+\frac{DHe^{10}}{s^{10}}+\frac{FHe^{12}}{s^{12}}+\&c$$

$$+\frac{BKe^{10}}{s^{10}}+\frac{DKe^{12}}{s^{12}}+\&c$$

$$+\frac{BMe^{12}}{s^{12}}+\&c$$

$$\frac{B^2e^2}{s^2}+\frac{2BDe^4}{s^4}+\frac{2BFe^6}{s^6}+\frac{2BHe^8}{s^8}+\frac{2BKe^{10}}{s^{10}}+\frac{2BMe^{12}}{s^{12}}+\&c$$

$$+\frac{D^2e^6}{s^6}+\frac{2DFe^8}{s^8}+\frac{2DHe^{10}}{s^{10}}+\frac{2DKe^{12}}{s^{12}}+\&c$$

$$+\frac{F^2e^{10}}{s^{10}}+\frac{2FHe^{12}}{s^{12}}+\&c$$

$$\frac{Be}{s}+\frac{De^3}{s^3}+\frac{Fe^5}{s^5}+\frac{He^7}{s^7}+\frac{Ke^9}{s^9}+\frac{Me^{11}}{s^{11}}+\frac{Oe^{13}}{s^{13}}+\&c$$

$$\frac{B^3e^3}{s^3}+\frac{2B^2De^5}{s^5}+\frac{2B^2Fe^7}{s^7}+\frac{2B^2He^9}{s^9}+\frac{2B^2Ke^{11}}{s^{11}}+\frac{2B^2Me^{13}}{s^{13}}+\&c$$

$$+\frac{BD^2e^7}{s^7}+\frac{2BDFe^9}{s^9}+\frac{2BDHe^{11}}{s^{11}}+\frac{2BDKe^{13}}{s^{13}}+\&c$$

$$+\frac{BF^2e^{11}}{s^{11}}+\frac{2BFHe^{13}}{s^{13}}+\&c$$

$$+\frac{B^2De^5}{s^5}+\frac{2BD^2e^7}{s^7}+\frac{2BDFe^9}{s^9}+\frac{2BDHe^{11}}{s^{11}}+\frac{2BDKe^{13}}{s^{13}}+\&c$$

$$+\frac{D^3e^9}{s^9}+\frac{2D^2Fe^{11}}{s^{11}}+\frac{2D^2He^{13}}{s^{13}}+\&c$$

$$+\frac{DF^2e^{13}}{s^{13}}+\&c$$

$$+\frac{B^2Fe^7}{s^7}+\frac{2BDFe^9}{s^9}+\frac{2BF^2e^{11}}{s^{11}}+\frac{2BFHe^{13}}{s^{13}}+\&c$$

$$+\frac{D^2Fe^{11}}{s^{11}}+\frac{2DF^2e^{13}}{s^{13}}+\&c$$

$$\begin{array}{rrrrrrr}
 & & & + \frac{B^2He^9}{s^9} & + \frac{2BDHe^{11}}{s^{11}} & + \frac{2BFHe^{13}}{s^{13}} & + \&c \\
 & & & & & + \frac{D^2He^{13}}{s^{13}} & + \&c \\
 & & & & + \frac{B^2Ke^{11}}{s^{11}} & + \frac{2BDKe^{13}}{s^{13}} & + \&c \\
 & & & & & + \frac{B^2Me^{13}}{s^{13}} & + \&c \\
\hline
\frac{B^3e^3}{s^3} + \frac{3B^2De^5}{s^5} & + \frac{3B^2Fe^7}{s^7} & + \frac{3B^2He^9}{s^9} & + \frac{3B^2Ke^{11}}{s^{11}} & + \frac{3B^2Me^{13}}{s^{13}} & & + \&c \\
 & + \frac{3BD^2e^7}{s^7} & + \frac{6BDFe^9}{s^9} & + \frac{6BDHe^{11}}{s^{11}} & + \frac{6BDKe^{13}}{s^{13}} & & + \&c \\
 & & + \frac{D^3e^9}{s^9} & + \frac{3BF^2e^{11}}{s^{11}} & + \frac{6BFHe^{13}}{s^{13}} & & + \&c \\
 & & & + \frac{3D^2Fe^{11}}{s^{11}} & + \frac{3D^2He^{13}}{s^{13}} & & + \&c \\
 & & & & + \frac{3DF^2e^{13}}{s^{13}} & & + \&c. \\
\hline
\end{array}$$

This laſt compound ſeries is the cube of the ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c *ad infinitum*.

Therefore $8s$ into the cube of the ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c will be $= 8s \times$ the foregoing compound ſeries; or, if, for the ſake of brevity, we denote the ſaid compound ſeries by the Greek capital letter Λ, $8s \times$ the cube of the ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c will be $= 8s \times$ the compound ſeries Λ; that is, the cube of the expreſſion $2\sqrt{^3}\, s \times$ the ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c, or the cube of y, will be $= 8s \times$ the compound ſeries Λ.

18. If the foregoing compound ſeries Λ be multiplied into $8s$, the product will be the following compound ſeries, to wit,

$$\begin{array}{rrrrrr}
\frac{8B^3e^3}{s^2} + \frac{24B^2De^5}{s^4} & + \frac{24B^2Fe^7}{s^6} & + \frac{24B^2He^9}{s^8} & + \frac{24B^2Ke^{11}}{s^{10}} & + \frac{24B^2Me^{13}}{s^{12}} & + \&c \\
 & + \frac{24BD^2e^7}{s^6} & + \frac{48BDFe^9}{s^8} & + \frac{48BDHe^{11}}{s^{10}} & + \frac{48BDKe^{13}}{s^{12}} & + \&c \\
 & & + \frac{8D^3e^9}{s^8} & + \frac{24BF^2e^{11}}{s^{10}} & + \frac{48BFHe^{13}}{s^{12}} & + \&c \\
 & & & + \frac{24D^2Fe^{11}}{s^{10}} & + \frac{24D^2He^{13}}{s^{12}} & + \&c \\
 & & & & + \frac{24DF^2e^{13}}{s^{12}} & + \&c,
\end{array}$$

which

which we will denote by the Greek capital letter Π. Then will y^3 be equal the compound ſeries Π.

19. Since the compound ſeries Δ, obtained in art. 16, is equal to the product qy, and the compound ſeries Π, obtained in art. 18, is equal to y^3, it follows that the ſum of the two compound ſerieſes Δ and Π will be equal to $qy + y^3$, and conſequently to its equal, the abſolute term r; that is, the ſerieſes $\Delta + \Pi$ will be $= r$. Therefore the ſeries Π will be $= r -$ the ſeries Δ. But r is equal to the firſt term, $6\,Be$, of the ſeries Δ. For B is $\frac{1}{3}$, and e is $= \frac{r}{2}$, and conſequently $6\,Be$ is $(= 6 \times \frac{1}{3} \times \frac{r}{2} = \frac{6r}{6}) = r$. Therefore the ſeries Π is $= 6\,Be -$ the ſeries Δ; that is, the compound ſeries Π is equal to the exceſs of the firſt term, $6\,Be$, of the compound ſeries Δ above the whole of the ſaid ſeries. But $6\,Be -$ the ſeries Δ, or the exceſs of the firſt term, $6\,Be$, of the ſeries Δ above the whole of the ſaid ſeries, will be a compound ſeries conſiſting of all the terms of the compound ſeries Δ, except the firſt term $6\,Be$, with their ſigns + and − every where changed. Therefore the compound ſeries Π will be equal to a compound ſeries conſiſting of all the terms of the compound ſeries Δ, except the firſt term $6\,Be$, with their ſigns + and − every where changed; that is, the compound ſeries

$$\begin{array}{r}
\frac{8B^3e^3}{s^2} + \frac{24B^2De^5}{s^4} + \frac{24B^2Fe^7}{s^6} + \frac{24B^2He^9}{s^8} + \frac{24B^2Ke^{11}}{s^{10}} + \frac{24B^2Me^{13}}{s^{12}} + \&c \\
+ \frac{24BD^2e^7}{s^6} + \frac{48BDFe^9}{s^8} + \frac{48BDHe^{11}}{s^{10}} + \frac{48BDKe^{13}}{s^{12}} + \&c \\
+ \frac{8D^3e^9}{s^8} + \frac{24BF^2e^{11}}{s^{10}} + \frac{48BFHe^{13}}{s^{12}} + \&c \\
+ \frac{24D^2Fe^{11}}{s^{10}} + \frac{24D^2He^{13}}{s^{12}} + \&c \\
+ \frac{24DF^2e^{13}}{s^{12}} + \&c
\end{array}$$

will be equal to the compound ſeries

$$\begin{array}{r}
\frac{6B^2e^3}{s^2} + \frac{6BCe^5}{s^4} + \frac{6BDe^7}{s^6} + \frac{6BEe^9}{s^8} + \frac{6BFe^{11}}{s^{10}} + \frac{6BGe^{13}}{s^{12}} + \&c \\
- \frac{6De^3}{s^2} + \frac{6BDe^5}{s^4} + \frac{6CDe^7}{s^6} + \frac{6D^2e^9}{s^8} + \frac{6DEe^{11}}{s^{10}} + \frac{6DFe^{13}}{s^{12}} + \&c \\
- \frac{6Fe^5}{s^4} + \frac{6BFe^7}{s^6} + \frac{6CFe^9}{s^9} + \frac{6DFe^{11}}{s^{10}} + \frac{6EFe^{13}}{s^{12}} + \&c \\
- \frac{6He^7}{s^6} + \frac{6BHe^9}{s^8} + \frac{6CHe^{11}}{s^{10}} + \frac{6DHe^{13}}{s^{12}} + \&c \\
- \frac{6Ke^9}{s^8} + \frac{6BKe^{11}}{s^{10}} + \frac{6CKe^{13}}{s^{12}} + \&c \\
- \frac{6Me^{11}}{s^{10}} + \frac{6BMe^{13}}{s^{12}} + \&c \\
- \frac{6Oe^{13}}{s^{12}} + \&c;
\end{array}$$

which, for the ſake of brevity, we will denote by the Greek capital letter Σ. And then we ſhall have the compound ſeries Π = the compound ſeries Σ.

Of

Of the signs + and — that are to be prefixed to the several terms of the foregoing compound series Σ.

20. In this compound series Σ all the terms of the first horizontal row of terms are to be added together, and consequently all the terms after the first term, $\frac{6B^2e^3}{s^2}$, are to be marked with the sign +, to whatever number of terms the said horizontal row of terms may be continued; and the first terms of the second, third, fourth, fifth, and other following horizontal rows of terms (to whatever number of horizontal rows the said compound series may be continued) will be marked with the sign —, and all the following terms of the said horizontal rows after the first terms will be marked with the sign +; and consequently all the terms of every vertical column of terms in this compound series, except the lowest term, will be marked with the sign +, and the said lowest term of each vertical column will be marked with the sign —.

This follows necessarily from art. 15 and 16; because the signs + and — that are prefixed to the terms of the compound series Δ, set down in art. 16, are the same with the signs of the corresponding terms of the preceeding compound series Γ, set down in art. 14, from which the compound series Δ is derived by only multiplying its terms into $6s$; and the compound series Σ is derived from the compound series Δ by omitting its first term $6Be$, and changing the signs of all its following terms. Consequently the signs to be prefixed to the terms of the compound series Σ must be contrary to those which are to be prefixed to the corresponding terms of the compound series Γ, and therefore must be contrary to those which are described in art. 15, or must be such as they are described to be in the present article.

Of the equality between the co-efficients of the terms of the compound series Π and the co-efficients of the corresponding terms of the compound series Σ.

21. Since ee is $= \frac{rr}{4}$, and ss is $= \frac{q^3}{27} + \frac{rr}{4}$, we may, by lessening the value of $\frac{rr}{4}$, or of r, without altering that of q, lessen the value of the fraction $\frac{\frac{rr}{4}}{\frac{q^3}{27} + \frac{rr}{4}}$; or $\frac{ee}{ss}$, and consequently that of $\frac{e}{s}$, as far as we please. Yet in all these values of the fraction $\frac{e}{s}$ it will always be true that the compound series Π, which involves in its terms the fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, which will be equal to the compound series Σ, which involves in its terms the same fractions. It therefore follows from this constant equality between these two serieses in all possible

poſſible magnitudes (how ſmall ſoever) of the fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, that the term, or terms, that involve any given powers of e and s in one of the two ſerieſes muſt be equal to the terms that involve the ſame powers of them in the other ſeries. And conſequently the ſeveral co-efficients of the terms which involve any given powers of e and s in one of the two ſerieſes muſt be equal to the ſeveral co-efficients of the terms that involve the ſame powers of e and s in the other ſeries. Thus, for example, the co-efficient of the fraction $\frac{e^3}{s^2}$ in the ſeries Π, to wit, $8\,B^3$, muſt be equal to the two co-efficients of the ſame fraction $\frac{e^3}{s^2}$ in the ſeries Σ, to wit, $6\,B^2 - 6\,D$; and in like manner the co-efficient of the fraction $\frac{e^5}{s^4}$ in the ſeries Π, to wit, $24\,B^2D$, muſt be equal to the three co-efficients of the ſame fraction $\frac{e^5}{s^4}$ in the ſeries Σ, to wit, $6\,BC + 6\,BD - 6\,F$; and the two co-efficients of the fraction $\frac{e^7}{s^6}$ in the ſeries Π, to wit, $24\,B^2F + 24\,BD^2$, muſt be equal to the four co-efficients of the ſame fraction $\frac{e^7}{s^6}$ in the ſeries Σ, to wit, $6\,BD + 6\,CD + 6\,BF - 6\,H$; and the ſame thing muſt take place with reſpect to the co-efficients of the following fractions $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c *ad infinitum*, or to whatever number of terms the two ſerieſes may be continued.

Examples of the ſaid equality of the co-efficients of the terms of the ſaid two compound ſerieſes.

22. Of this equality between the co-efficients of the ſame fractions, conſiſting of the powers of e and s, in theſe two compound ſerieſes Π and Σ, it may not be amiſs to give a few inſtances by actually computing the values of the ſaid co-efficients; which may be done in the manner following.

The capital letters B, C, D, E, F, and H, are equal to $\frac{1}{3}$, $\frac{1}{9}$, $\frac{5}{81}$, $\frac{10}{243}$, $\frac{22}{729}$, and $\frac{374}{19683}$, reſpectively. Therefore B^3 will be $= \frac{1}{27}$, and $8\,B^3$ will be $= \frac{8}{27}$; and $6\,B^2$ will be $(= 6 \times \frac{1}{9}) = \frac{2}{3}$, and $6\,D$ will be $(= 6 \times \frac{5}{81} = 2 \times \frac{5}{27}) = \frac{10}{27}$, and conſequently $6\,B^2 - 6\,D$ will be $(= \frac{2}{3} - \frac{10}{27} = \frac{18}{27} - \frac{10}{27}) = \frac{8}{27}$; that is, $8\,B^3$, the co-efficient of the fraction $\frac{e^3}{s^2}$ in the compound ſeries Π, and $6\,B^2 - 6\,D$, the co-efficient of the ſame fraction $\frac{e^3}{s^2}$ in the compound ſeries Σ, will, each of them, be equal to the ſame quantity $\frac{8}{27}$.

In like manner 24 B^2D, or the co efficient of the fraction $\frac{e^5}{s^4}$ in the compound series Π, is $(= 24 \times \frac{1}{3 \times 3} \times \frac{5}{81} = \frac{8}{3} \times \frac{5}{81}) = \frac{40}{243}$; and 6 BC + 6 BD − 6 F, or the co-efficient of the same fraction $\frac{e^5}{s^4}$ in the compound series Σ, will be found to be equal to the same quantity. For 6 BC + 6 BD − 6 F is $= 6 \times \frac{1}{3} \times \frac{1}{9} + 6 \times \frac{1}{3} \times \frac{5}{81} - 6 \times \frac{22}{729} (= \frac{2}{9} + \frac{10}{81} - \frac{44}{243} = \frac{54}{243} + \frac{30}{243} - \frac{44}{243} = \frac{84}{243} - \frac{44}{243}) = \frac{40}{243}$.

Q. E. D.

And 24 B^2F + 24 BD^2, or the co-efficient of the fraction $\frac{e^7}{s^6}$ in the compound series Π, is $(= 24 \times \frac{1}{9} \times \frac{22}{729} + 24 \times \frac{1}{3} \times \frac{5 \times 5}{81 \times 81} = \frac{8}{3} \times \frac{22}{729} + 8 \times \frac{25}{81 \times 81} = \frac{176}{2187} + \frac{200}{6561} = \frac{528}{6561} + \frac{200}{6561}) = \frac{728}{6561}$; and 6 BD + 6 CD + 6 BF − 6 H, or the co-efficient of the same fraction $\frac{e^7}{s^6}$ in the other compound series Σ, is equal to the same quantity. For it is $= 6 \times \frac{1}{3} \times \frac{5}{81} + 6 \times \frac{1}{9} \times \frac{5}{81} + 6 \times \frac{1}{3} \times \frac{22}{729} - 6 \times \frac{374}{19{,}683} (= 2 \times \frac{5}{81} + \frac{2}{3} \times \frac{5}{81} + 2 \times \frac{22}{729} - \frac{2 \times 374}{6561} = \frac{10}{81} + \frac{10}{243} + \frac{44}{729} - \frac{748}{6561} = \frac{810}{6561} + \frac{270}{6561} + \frac{396}{6561} - \frac{748}{6561} = \frac{1476}{6561} - \frac{748}{6561}) = \frac{728}{6561}$.

Q. E. D.

And the same equality will be found to take place between the co-efficients of the following fractions $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, *ad infinitum*, in the compound series Π and the co-efficients of the same fractions in the compound series Σ, respectively, to whatever number of terms the said serieses may be continued.

The reduction of the compound series Π *to a simple series* $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}} +$ *&c ad infinitum.*

23. In the compound series Π, obtained in art. 18, all the terms after the first term, $\frac{8B^3e^3}{s^2}$, are marked with the sign + and added to the said first term. Therefore, if we reduce the said compound series to a simple series, with single letters for the co-efficients of its terms, by denoting the co-efficient, 8 B^3, of the fraction $\frac{e^3}{s^2}$ in the first term $\frac{8B^3e^3}{s^2}$, by the single letter P, and the co-efficient,

24 B^2D,

24 B^2D, of the fraction $\frac{e^5}{s^4}$ in the second term, $\frac{24 B^2 D e^5}{s^4}$, by the single letter Q, and the compound co-efficient, 24 B^2F + 24 BD^2, of the fraction $\frac{e^7}{s^6}$ in the third term, by the single letter R, and the compound co-efficients of the following fractions $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, in the fourth, fifth, and sixth, and other following terms of the same series, by the single letters S, T, V, &c, the simple series that will be equal to the said compound series Π will be $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}}$ + &c, in which all the terms after the first term, $\frac{Pe^3}{s^2}$, are marked with the sign +, or are added to the said first term.

The same simple series $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}}$ *+ &c ad infinitum will also be equal to the compound series Σ.*

24. But by art. 19, the compound series Π is equal to the compound series Σ. Therefore the simple series $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}}$ + &c will be equal to the compound series Σ, which is set down in the latter part of art. 19.

And the several co-efficients P, Q, R, S, T, V, *&c, of the terms of the said simple series will be respectively equal to the several compound co-efficients of the corresponding terms of the said compound series Σ.*

25. And, further, it is shewn in art. 21, that the co-efficients of the several fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, in the compound series Π are respectively equal to the co-efficients of the same fractions in the other compound series Σ. Therefore the co-efficients P, Q, R, S, T, V, &c, of the several fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{e^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, in the simple series $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}}$ + &c (which are respectively equal to the co-efficients of the same fractions in the compound series Π) will be also respectively equal to the co-efficients of the same fractions in the compound series Σ. And hence it follows that, if we were to reduce the said compound series Σ to a simple series by the addition and subtraction of the several members of each of its vertical columns

lumns according to the ſigns + and − which are prefixed to them, we ſhould find the ſum of the co-efficients of the terms that are marked with the ſign + in each of the vertical columns of the ſaid compound ſeries (and which are all the terms of the column, except the laſt, or loweſt) to be greater than the co-efficient of the laſt, or loweſt, term in the column, which is marked with the ſign −, and the reſulting differences, or exceſſes, of the ſums of the co-efficients of the terms marked with the ſign + in each column above the co-efficients of the loweſt terms in them, which are marked with the ſign —, to be equal to the co-efficients P, Q, R, S, T, V, &c, of the fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, in the ſimple ſeries $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}} +$ &c reſpectively; that is, we ſhould find the compound co-efficient of the fraction $\frac{e^3}{s^2}$ in the compound ſeries Σ, to wit, $6B^2 - 6D$, to be $= P$; and the compound co-efficient of the fraction $\frac{e^5}{s^4}$ in the ſame ſeries Σ, to wit, $6BC + 6BD - 6F$, to be $= Q$; and the compound co-efficient of the fraction $\frac{e^7}{s^6}$ in the ſame ſeries, to wit, $6BD + 6CD + 6BF - 6H$, to be $= R$; and the compound co-efficient of the fraction $\frac{e^9}{s^8}$, to wit, $6BE + 6D^2 + 6CF + 6BH - 6K$, to be $= S$; and the compound co-efficient of the fraction $\frac{e^{11}}{s^{10}}$, to wit, $6BF + 6DE + 6DF + 6CH + 6BK - 6M$, to be $= T$; and the compound co-efficient of the fraction $\frac{e^{13}}{s^{12}}$, to wit, $6BG + 6DF + 6EF + 6DH + 6CK + 6BM - 6O$, to be $= V$; and, in like manner, the compound co-efficients of the following fractions $\frac{e^{15}}{s^{14}}$, $\frac{e^{17}}{s^{16}}$, $\frac{e^{19}}{s^{18}}$, $\frac{e^{21}}{s^{20}}$, $\frac{e^{23}}{s^{22}}$, &c, *ad infinitum* in the ſame compound ſeries Σ to be reſpectively equal to the co-efficients of the ſame fractions in the following terms of the ſimple ſeries $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}} +$ &c *ad infinitum*.

Concluſions contained in the three foregoing articles.

26. It appears from the three foregoing articles, that the ſimple ſeries $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}} +$ &c will be equal to each of the two compound ſerieſes, Π and Σ, and likewiſe that each of its terms will be equal to the correſpondent term, or term that involves the ſame powers of *e* and *s*, in each of the ſaid two compound ſerieſes, and conſequently that the co-efficient of each of its terms will be equal to the co-efficient of the correſpondent term, or term involving the ſame powers of *e* and *s*, in each of the ſaid two compound ſerieſes.

Examination

Examination of the expression $2\sqrt{^3}\, z \times$ *the series* $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — *&c. ad infinitum.*

27. These things being thoroughly understood, we must now turn our attention to the expression $2\sqrt{^3}\, z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ + &c *ad infinitum*, which we have asserted to be equal to the lesser root of the cubick equation $qx - x^3 = r$. This we must now proceed to demonstrate. And first we will endeavour to prove that this expression is equal to one of the two roots of the said equation; and afterwards we will shew that it cannot be equal to the greater of those two roots, and consequently that it must be equal to the lesser of them.

28. Now it will be evident that this expression $2\sqrt{^3}\, z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c *ad infinitum* is equal to one of the roots of the cubick equation $qx - x^3 = r$, if we can shew that, being substituted instead of x in the compound quantity $qx - x^3$, it will make that quantity be equal to the absolute term r, or that the product of the multiplication of the said expression into the co-efficient q will be greater than the cube of the said expression, and that its excess above the said cube will be equal to the absolute term r. This therefore is what I shall now endeavour to demonstrate.

The product of the multiplication of q into the said expression set forth in an infinite series.

29. The value of zz in the expression $2\sqrt{^3}\, z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c is $\frac{q^3}{27} - \frac{rr}{4}$, by art. 11, in which this series is derived from the expression $2\sqrt{^3}\, s \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}}$ + &c, which is equal to the root of the equation $qy + y^3 = r$. But ee is $= \frac{rr}{4}$, and consequently $\frac{q^3}{27} - ee$ is $= \frac{q^3}{27} - \frac{rr}{4}$. Therefore zz will be $= \frac{q^3}{27} - ee$, and consequently $\frac{q^3}{27}$ will be $= zz + ee$, and q^3 will be $= 27$

$= 27 \times \overline{zz + ee} = 27 \times zz \times \overline{1 + \frac{ee}{zz}}$, and q will be $= 3 \times \overline{zz}|^{\frac{1}{3}} \times \overline{1 + \frac{ee}{zz}}|^{\frac{1}{3}} = 3 \times z^{\frac{2}{3}} \times \overline{1 + \frac{ee}{zz}}|^{\frac{1}{3}} =$ (by the binomial theorem in the case of roots) $3 \times \overline{z}|^{\frac{2}{3}} \times$ the series $1 + \frac{1}{3} A \frac{ee}{zz} - \frac{2}{6} B \times \frac{e^4}{z^4} + \frac{5}{9} C \times \frac{e^6}{z^6} - \frac{8}{12} D \times \frac{e^8}{z^8} + \frac{11}{15} E \times \frac{e^{10}}{z^{10}} - \frac{14}{18} F \times \frac{e^{12}}{z^{12}} + \frac{17}{21} G \times \frac{e^{14}}{z^{14}} - \frac{20}{24} H \times \frac{e^{16}}{z^{16}} + \&c = 3 \times \overline{z}|^{\frac{2}{3}} \times$ the series $1 + \frac{Bee}{zz} - \frac{Ce^4}{z^4} + \frac{De^6}{z^6} - \frac{Ee^8}{z^8} + \frac{Fe^{10}}{z^{10}} - \frac{Ge^{12}}{z^{12}} + \frac{He^{14}}{z^{14}} - \frac{Ie^{16}}{z^{16}} + \&c.$ Therefore the product of the multiplication of q into the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ will be $= 3 \times \overline{z}|^{\frac{2}{3}} \times$ the series $1 + \frac{Bee}{zz} - \frac{Ce^4}{z^4} + \frac{De^6}{z^6} - \frac{Ee^8}{z^8} + \frac{Fe^{10}}{z^{10}} - \frac{Ge^{12}}{z^{12}} + \frac{He^{14}}{z^{14}} - \frac{Ie^{16}}{z^{16}} + \&c \times 2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c = 6z \times$ the series $1 + \frac{Bee}{zz} - \frac{Ce^4}{z^4} + \frac{De^6}{z^6} - \frac{Ee^8}{z^8} + \frac{Fe^{10}}{z^{10}} - \frac{Ge^{12}}{z^{12}} + \frac{He^{14}}{z^{14}} - \frac{Ie^{16}}{z^{16}} + \&c \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c = 6z \times$ the compound series

$$
\begin{array}{llllllll}
\frac{Be}{z} & + \frac{B^2e^3}{z^3} & - \frac{BCe^5}{z^5} & + \frac{BDe^7}{z^7} & - \frac{BEe^9}{z^9} & + \frac{BFe^{11}}{z^{11}} & - \frac{BGe^{13}}{z^{13}} & + \&c \\
 & - \frac{De^3}{z^3} & - \frac{BDe^5}{z^5} & + \frac{CDe^7}{z^7} & - \frac{D^2e^9}{z^9} & + \frac{DEe^{11}}{z^{11}} & - \frac{DFe^{13}}{z^{13}} & + \&c \\
 & & + \frac{Fe^5}{z^5} & + \frac{BFe^7}{z^7} & - \frac{CFe^9}{z^9} & + \frac{DFe^{11}}{z^{11}} & - \frac{EFe^{13}}{z^{13}} & + \&c \\
 & & & - \frac{He^7}{z^7} & - \frac{BHe^9}{z^9} & + \frac{CHe^{11}}{z^{11}} & - \frac{DHe^{13}}{z^{13}} & + \&c \\
 & & & & + \frac{Ke^9}{z^9} & + \frac{BKe^{11}}{z^{11}} & - \frac{CKe^{13}}{z^{13}} & + \&c \\
 & & & & & - \frac{Me^{11}}{z^{11}} & - \frac{BMe^{13}}{z^{13}} & + \&c \\
 & & & & & & + \frac{Oe^{13}}{z^{13}} & + \&c.
\end{array}
$$

Let us, for the sake of brevity, denote this compound series by the small Greek letter γ. And we shall then have the product of the multiplication of q into the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum* equal to $6z \times$ the compound series γ.

A com-

A comparison between the foregoing compound series γ and the compound series Γ obtained above in art. 14.

30. This compound series γ will consist of terms that will involve the several fractions $\frac{e}{z}$, $\frac{e^3}{z^3}$, $\frac{e^5}{z^5}$, $\frac{e^7}{z^7}$, $\frac{e^9}{z^9}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{13}}{z^{13}}$, &c, or the several odd powers of the fraction $\frac{e}{z}$, in the same manner as the terms of the compound series Γ, obtained above in art. 14, involve the corresponding fractions $\frac{e}{s}$, $\frac{e^3}{s^3}$, $\frac{e^5}{s^5}$, $\frac{e^7}{s^7}$, $\frac{e^9}{s^9}$, $\frac{e^{11}}{s^{11}}$, $\frac{e^{13}}{s^{13}}$, &c, or the several odd powers of the fraction $\frac{e}{s}$: and the co-efficients of the several powers of $\frac{e}{z}$ in the present series γ will be the same with the co-efficients of the same powers of $\frac{e}{s}$ in the former series Γ, though they will not be every where marked with the same signs + and —. This equality, or, rather, identity, of the co-efficients of the corresponding terms in both these compound serieses arises from the equality, or identity, of the co-efficients of the terms of the two simple serieses, by the multiplication of which into each other the said compound series γ is produced, with the co-efficients of the corresponding terms of the two simple serieses, by the multiplication of which into each other the former compound series Γ is produced. For, since the co-efficients of the terms of the two serieses $1 + \frac{Bee}{zz} - \frac{Ce^4}{z^4} + \frac{De^6}{z^6} - \frac{Ee^8}{z^8} + \frac{Fe^{10}}{z^{10}} - \frac{Ge^{12}}{z^{12}} +$ &c and $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c (by the multiplication of which into each other the compound series γ is produced) are the same with the co-efficients of the corresponding terms of the two serieses $1 - \frac{Bee}{ss} - \frac{Ce^4}{s^4} - \frac{De^6}{s^6} - \frac{Ee^8}{s^8} - \frac{Fe^{10}}{s^{10}} - \frac{Ge^{12}}{s^{12}} -$ &c and $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}}$ + &c (by the multiplication of which into each other the compound series Γ was produced) it is evident that the co-efficients of the several corresponding members of the two compound serieses Γ and γ produced by these multiplications must be the same combinations of the co-efficients B, C, D, E, F, G, &c and B, D, F, H, K, M, O, &c of the terms of the two original serieses, or in other words, must be the same quantities in both serieses.

Of the signs + and — that are to be prefixed to the terms of the compound series γ.

31. But the signs + and — that are to be prefixed to the several terms of the compound series γ will not be every where the same as those that are to be prefixed to the corresponding terms of the other compound series Γ, but only in the

 terms

terms of the third, fifth, ſeventh, and other following odd vertical columns of terms, which involve the fractions $\frac{e^5}{s^5}$, $\frac{e^9}{s^9}$, $\frac{e^{13}}{s^{13}}$, &c and $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, $\frac{e^{13}}{z^{13}}$ &c. In the ſecond, fourth, ſixth, and other following even vertical columns of terms in the ſaid compound ſeries γ, which involve the fractions $\frac{e^3}{z^3}$, $\frac{e^7}{z^7}$, $\frac{e^{11}}{z^{11}}$, &c the ſigns + and — to be prefixed to the ſeveral terms in the ſaid vertical columns will be reſpectively contrary to thoſe which are to be prefixed to the correſponding terms of the ſecond, fourth, ſixth, and other following even vertical columns of the compound ſeries Γ, which involve the correſponding fractions $\frac{e^3}{s^3}$, $\frac{e^7}{s^7}$, $\frac{e^{11}}{s^{11}}$, &c. This will appear to be the caſe as far as the terms of the ſeventh vertical columns of the ſaid two compound ſerieſes, or the terms which involve the fractions $\frac{e^{13}}{s^{13}}$ and $\frac{e^{13}}{z^{13}}$, upon the inſpection of the ſaid two compound ſerieſes, as herein before ſet down in art. 14 and 29. For in the former of theſe ſerieſes the terms contained in the third, fifth, and ſeventh, vertical columns are $-\frac{BCe^5}{s^5} - \frac{BDe^5}{s^5} + \frac{Fe^5}{s^5}$, $-\frac{BEe^9}{s^9} - \frac{D^2e^9}{s^9} - \frac{CFe^9}{s^9} - \frac{BHe^9}{s^9} + \frac{Ke^9}{s^9}$, and $-\frac{BGe^{13}}{s^{13}} - \frac{DFe^{13}}{s^{13}} - \frac{EFe^{13}}{s^{13}} - \frac{DHe^{13}}{s^{13}} - \frac{CKe^{13}}{s^{13}} - \frac{BMe^{13}}{s^{13}} + \frac{Oe^{13}}{s^{13}}$; in each of which columns, or ſets, of terms, the ſign — is prefixed to every term, except the laſt, and the ſign + is prefixed to the laſt term; and in the latter of theſe ſerieſes the terms contained in the third, fifth, and ſeventh vertical columns are $-\frac{BCe^5}{z^5} - \frac{BDe^5}{z^5} + \frac{Fe^5}{z^5}$, $-\frac{BEe^9}{z^9} - \frac{D^2e^9}{z^9} - \frac{CFe^9}{z^9} - \frac{BHe^9}{z^9} + \frac{Ke^9}{z^9}$, and $-\frac{BGe^{13}}{z^{13}} - \frac{DFe^{13}}{z^{13}} - \frac{EFe^{13}}{z^{13}} - \frac{DHe^{13}}{z^{13}} - \frac{CKe^{13}}{z^{13}} - \frac{BMe^{13}}{z^{13}} + \frac{Oe^{13}}{z^{13}}$; in each of which columns, or ſets, of terms the ſign — is, in like manner, prefixed to all the terms, except the laſt, and the ſign + is prefixed to the laſt term. And in the former of theſe ſerieſes, the terms contained in the ſecond, fourth, and ſixth, vertical columns are $-\frac{B^2e^3}{s^3} + \frac{De^3}{s^3}$, $-\frac{BDe^7}{s^7} - \frac{CDe^7}{s^7} - \frac{BFe^7}{s^7} + \frac{He^7}{s^7}$, and $-\frac{BFe^{11}}{s^{11}} - \frac{DEe^{11}}{s^{11}} - \frac{DFe^{11}}{s^{11}} - \frac{CHe^{11}}{s^{11}} - \frac{BKe^{11}}{s^{11}} + \frac{Me^{11}}{s^{11}}$; in each of which columns, or ſets of terms, the ſign — is prefixed to all the terms, except the laſt, and the ſign + is prefixed to the laſt term: and in the latter of theſe ſerieſes the terms contained in the ſecond, fourth, and ſixth, vertical columns are $+\frac{B^2e^3}{z^3} - \frac{De^3}{z^3}$, $+\frac{BDe^7}{z^7} + \frac{CDe^7}{z^7} + \frac{BFe^7}{z^7} - \frac{He^7}{z^7}$, and $+\frac{BFe^{11}}{z^{11}} + \frac{DEe^{11}}{z^{11}} + \frac{DFe^{11}}{z^{11}} + \frac{CHe^{11}}{z^{11}} + \frac{BKe^{11}}{z^{11}} - \frac{Me^{11}}{z^{11}}$; in each of which columns, or ſets of terms the ſign + is prefixed to all the terms, except the laſt term, and the ſign — is prefixed to the laſt term.

And that the ſame thing will take place in the terms of all the following vertical columns of theſe two compound ſerieſes, to whatever number of terms they may be continued, may be ſhewn in the manner following.

32. It

32. It is ſhewn above in art. 15, that, in the compound ſeries Γ, all the terms in every vertical column of the ſeries, except the laſt, or loweſt, term, are marked with the ſign —, and the loweſt term is marked with the ſign +. And we have ſeen that the ſame thing takes place in the terms of the third vertical column and of the fifth vertical column, and of the ſeventh vertical column, of the compound ſeries γ, which involve in them the fractions $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, and $\frac{e^{13}}{z^{13}}$; to wit, that all the terms in each of theſe vertical columns, except the loweſt, have the ſign — prefixed to them, and that the loweſt term is marked with the ſign +; but that the contrary rule takes place in the terms of the ſecond vertical column, and of the fourth vertical column, and of the ſixth vertical column, of the ſaid compound ſeries γ, which involve in them the fractions $\frac{e^3}{z^3}$, $\frac{e^7}{z^7}$, and $\frac{e^{11}}{z^{11}}$; to wit, that all the terms in each of theſe vertical columns, except the loweſt, are marked with the ſign +, and the loweſt term is marked with the ſign —. We are now to prove that the ſame rules will hold as to the ſigns of the terms contained in the 9th, 11th, 13th, 15th, and other following odd vertical columns of terms, and the ſigns of the terms contained in the 8th, 10th, 12th, 14th, and other following even vertical columns of terms of the ſaid compound ſeries γ, to whatever number of terms, or columns of terms, the ſaid compound ſeries γ may be continued.

33. Now in the ſaid compound ſeries γ (which is ſet down in art. 29), it is evident, that in the 1ſt, and 3d, and 5th, and 7th, and other following odd horizontal rows of terms, the two firſt terms will be marked with the ſign +, and all the following terms will be marked with the ſign — and the ſign + alternately; and in the 2d, and 4th, and 6th, and other following even horizontal rows of terms, the two firſt terms will be marked with the ſign —, and all the following terms will be marked with the ſign + and the ſign — alternately. This is a neceſſary conſequence of the order in which the ſigns + and — follow each other, in the two ſimple ſerieſes, by the multiplication of which the compound ſeries γ is produced. For the ſimple ſeries which is the multiplicand of his multiplication, is the ſeries $1 + \frac{Bee}{zz} - \frac{Ce^4}{z^4} + \frac{De^6}{z^6} - \frac{Ee^8}{z^8} + \frac{Fe^{10}}{z^{10}} - \frac{Ge^{12}}{z^{12}} +$ &c; in which the ſecond term $\frac{Bee}{zz}$ is marked with the ſign +, and all the following terms are marked with the ſigns — and + alternately: and conſequently, whenever this ſeries is multiplied by a quantity to which the ſign + is prefixed, the product will be a ſeries of terms, in which the ſigns + and — will follow each other in the ſame order as in the multiplicand, that is, the ſign + will be prefixed to the two firſt terms of the ſaid product, and the ſign — and the ſign + will be prefixed to the third, fourth, fifth, ſixth, and other following terms of the ſaid product alternately; and, whenever this ſimple ſeries is multiplied by a quantity to which the ſign — is prefixed, the product will be a ſeries of terms in which the ſigns will be, reſpectively, contrary to the ſigns of the correſponding terms of the multiplicand, and therefore the ſign — will be prefixed to the two firſt

first terms of the said product, and the signs + and — will be prefixed to the third, fourth, fifth, sixth, and other following terms of the said product alternately. But the simple series into which the said simple series $1 + \frac{Bee}{zz} - \frac{Ce^4}{z^4} + \frac{De^6}{z^6} - \frac{Ee^8}{z^8} + \frac{Fe^{10}}{z^{10}} - \frac{Ge^{12}}{z^{12}} +$ &c is multiplied, in order to produce the compound series γ, is the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c, in which the terms are marked with the sign + and the sign — alternately. Therefore in the 1st, and 3d, and 5th, and 7th, and other following odd horizontal rows of terms in the compound series γ (which is produced by the said multiplication) the sign + will be prefixed to the two first terms of the said horizontal rows, and the sign — and the sign + will be prefixed to the third, and fourth, and fifth, and sixth, and other following terms of the said rows, alternately; and in the 2d, and 4th, and 6th, and 8th, and other following even horizontal rows of terms in the said compound series γ, the sign — will be prefixed to the two first terms of the said horizontal rows, and the sign + and the sign — will be prefixed to the third, and fourth, and fifth, and sixth, and other following terms of the said horizontal rows, alternately.

34. Now from this order of succession of the signs + and — to each other in the horizontal rows of terms of the compound series γ, we may deduce the order of their succession in the terms contained in the several vertical columns of the said compound series. This may be done in the manner following.

Since the two first terms of the several odd horizontal rows of terms in this compound series γ are marked with the sign +, and the two first terms of the several even horizontal rows are marked with the sign —, and the first terms of the second, and third, and fourth, and other following horizontal rows of terms are the last, or lowest, terms of the second, and third, and fourth, and other following vertical columns in the said series, it follows that the last, or lowest terms of the second, and third, and fourth, and other vertical columns will be marked with the sign — and the sign + alternately; and it follows likewise that the sign prefixed to the second term of every horizontal row of terms (being the same as the sign prefixed to the first term of the same horizontal row) will be contrary to the sign of the term placed immediately under it, or of the first term of the next horizontal row, and consequently that the sign prefixed to the last term but one of every vertical column (which is always the second term of one of the horizontal rows of terms) will be contrary to the last, or lowest, term of the same vertical column, which is the first term of the next lower horizontal row of terms.

Therefore, since the first terms of the second, and fourth, and sixth, and other following even horizontal rows of terms, or the last or lowest terms of the second, fourth, sixth, and other following even vertical columns, are marked with the sign —, the last terms but one of the same columns, to wit, the second, fourth, sixth, and other following even columns, will be marked with the sign +; and, since the first terms of the third, and fifth, and seventh, and other following odd horizontal rows of terms, or the last, or lowest terms of the third, fifth,

fifth, feventh, and other following odd vertical columns, are marked with the fign +, the laft terms but one of the fame columns, to wit, the third, and fifth, and feventh, and other following odd columns, will be marked with the fign —.

35. In the foregoing article 34, we have fhewn that the loweft terms of the fecond, fourth, fixth, and other following even columns of terms in the compound feries γ will be marked with the fign —, and the next higher terms, or the loweft terms but one, of the fame columns will be marked with the fign +; and that the loweft terms of the third, fifth, feventh, and other following odd columns of terms in the fame feries will be marked with the fign +, and the next higher terms, or the loweft terms but one, of the fame columns will be marked with the fign —. We muft now examine the figns that are to be prefixed to the other terms of the feveral columns that are higher than the two loweft terms.

Now the figns to be prefixed to all the terms of every column above the two loweft terms are the fame with each other and with the fign of the loweft term but one of the fame column; as may be fhewn in the manner following.

The fecond term of every horizontal row of terms in the compound feries γ is placed directly under the third term of the next horizontal row above it, and under the fourth term of the fecond horizontal row above it, and under the fifth term of the third horizontal row above it, and fo on *ad infinitum*, the number of terms in every new horizontal row above the faid fecond term that preceed the term that is placed immediately above fuch fecond term, increafing continually by an unit. But, becaufe the figns + and — are prefixed to the fecond, and third, and fourth, and fifth, and other following terms of the firft, the third, the fifth, the feventh, and other following odd horizontal rows of terms, alternately, and the figns — and + are prefixed to the fecond, and third, and fourth, and fifth, and other following terms of the fecond, the fourth, the fixth, the eighth, and other following even horizontal rows of terms, alternately, it is evident that the fign prefixed to the fecond term of every horizontal row of terms muft be the fame with the fign prefixed to the third term of the next horizontal row above it, and with the fign prefixed to the fourth term of the fecond horizontal row above it, and with the fign prefixed to the fifth term of the third horizontal row above it, and with the figns prefixed to the fixth, feventh, eighth, &c, terms of the fourth, fifth, fixth, &c, horizontal rows above it, *ad infinitum*. And therefore the fign + or —, that is to be prefixed to the fecond term of every horizontal row of terms in the faid compound feries γ will be the fame with the figns of all the terms placed immediately above the faid fecond term, or of all the terms of the fame vertical column that preceed it: or, the figns to be prefixed to all the terms of every vertical column of terms in the faid compound feries γ above the two loweft terms of the faid column will be the fame with each other, and with the fign of the loweft term but one of the faid column.

Q. E. D.

A reca-

A recapitulation of what has been shewn in the six foregoing articles concerning the analogy between the compound series Γ and the compound series γ.

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36. It follows from the six foregoing articles, 30, 31, 32, 33, 34, and 35, that, if we compare the compound series γ, obtained in art. 29, with the compound series Γ, obtained above in art. 14, the analogy and the differences between them will be as follows, to wit: In the 1st place, in the compound series γ the terms will involve the fractions $\frac{e}{z}$, $\frac{e^3}{z^3}$, $\frac{e^5}{z^5}$, $\frac{e^7}{z^7}$, $\frac{e^9}{z^9}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{13}}{z^{13}}$, &c, or the odd powers of the fraction $\frac{e}{z}$, instead of the fractions $\frac{e}{s}$, $\frac{e^3}{s^3}$, $\frac{e^5}{s^5}$, $\frac{e^7}{s^7}$, $\frac{e^9}{s^9}$, $\frac{e^{11}}{s^{11}}$, $\frac{e^{13}}{s^{13}}$, &c, or the odd powers of the fraction $\frac{e}{s}$, which are involved in the terms of the compound series Γ.

In the next place, the co-efficients of the several terms in the series γ that involve in them the several odd powers of the fraction $\frac{e}{z}$ in the compound series γ will be the very same with the co-efficients of the several corresponding terms, or terms involving the same odd powers of the fraction $\frac{e}{s}$, in the series Γ respectively.

And, lastly, the signs $-$ and $+$ to be prefixed to the several terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the series γ (which involve the fractions $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, $\frac{e^{13}}{z^{13}}$, &c) will be the same as are to be prefixed to the corresponding terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the series Γ, which involve the fractions $\frac{e^5}{s^5}$, $\frac{e^9}{s^9}$, $\frac{e^{13}}{s^{13}}$, &c; but the signs $+$ and $-$, which are to be prefixed to the several terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the series γ (which involve the fractions $\frac{e^3}{z^3}$, $\frac{e^7}{z^7}$, $\frac{e^{11}}{z^{11}}$, &c) will be respectively contrary to those which are to be prefixed to the corresponding terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the series Γ, which involve the fractions $\frac{e^3}{s^3}$, $\frac{e^7}{s^7}$, $\frac{e^{11}}{s^{11}}$, &c.

37. If we multiply the foregoing compound series called γ, (which was obtained in art. 29) by the quantity $6z$, the product will be the following compound series; to wit,

$$6Be + \frac{6B^2e^3}{z^2} - \frac{6BCe^5}{z^4} + \frac{6BDe^7}{z^6} - \frac{6BEe^9}{z^8} + \frac{6BFe^{11}}{z^{10}} - \frac{6BGe^{13}}{z^{12}} + \&c$$
$$- \frac{6De^3}{z^2} - \frac{6BDe^5}{z^4} + \frac{6CDe^7}{z^6} - \frac{6D^2e^9}{z^8} + \frac{6DEe^{11}}{z^{10}} - \frac{6DFe^{13}}{z^{12}} + \&c$$
$$+ \frac{6Fe^5}{z^4} + \frac{6BFe^7}{z^6} - \frac{6CFe^9}{z^8} + \frac{6DFe^{11}}{z^{10}} - \frac{6EFe^{13}}{z^{12}} + \&c$$
$$- \frac{6He^7}{z^6} - \frac{6BHe^9}{z^8} + \frac{6CHe^{11}}{z^{10}} - \frac{6DHe^{13}}{z^{12}} + \&c$$
$$+ \frac{6Ke^9}{z^8} + \frac{6BKe^{11}}{z^{10}} - \frac{6CKe^{13}}{z^{12}} + \&c$$
$$- \frac{Me^{11}}{z^{10}} - \frac{BMe^{13}}{z^{12}} + \&c$$
$$+ \frac{Oe^{13}}{z^{12}} + \&c.$$

Let this compound ſeries, for the ſake of brevity, be denoted by the ſmall Greek letter δ. Then, ſince the product of the multiplication of q into the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum* is equal to $6z \times$ the compound ſeries γ, it will alſo be equal to the compound ſeries δ.

Of the analogy between the foregoing compound ſeries δ and the compound ſeries Δ obtained above in art. 16.

38. Now ſince the compound ſeries Δ, obtained above in art. 16, is equal to $6s \times$ the compound ſeries Γ, and the compound ſeries δ is equal to $6z \times$ the compound ſeries γ, it follows that there will be the ſame analogy and the ſame differences between the two compound ſerieſes Δ and δ as between the two compound ſerieſes Γ and γ. And conſequently the analogy and the differences which are ſhewn in art. 36 to take place between the two compound ſerieſes Γ and γ, will take place alſo between the two compound ſerieſes Δ and δ, and therefore,

In the firſt place, the ſecond and other following terms of the compound ſeries δ will involve the fractions $\frac{e^3}{z^2}, \frac{e^5}{z^4}, \frac{e^7}{z^6}, \frac{e^9}{z^8}, \frac{e^{11}}{z^{10}}, \frac{e^{13}}{z^{12}}$, &c, inſtead of the correſponding fractions $\frac{e^3}{s^2}, \frac{e^5}{s^4}, \frac{e^7}{s^6}, \frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}}$, &c, which are involved in the ſecond and other following terms of the compound ſeries Δ.

And 2dly, the co-efficients of the ſeveral fractions $\frac{e^3}{z^2}, \frac{e^5}{z^4}, \frac{e^7}{z^6}, \frac{e^9}{z^8}, \frac{e^{11}}{z^{10}}, \frac{e^{13}}{z^{12}}$, &c, in the terms of the ſecond and other following vertical columns of the compound ſeries δ will be the ſame with the co-efficients of the correſponding fractions $\frac{e^3}{s^2}, \frac{e^5}{s^4}, \frac{e^7}{s^6}, \frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}}$, &c, reſpectively, in the terms of the ſecond and other following vertical columns of the compound ſeries Δ.

 And,

And, 3dly, the ſigns — and +, that are to be prefixed to the ſeveral terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the compound ſeries δ (which involve in them the fractions $\frac{e^5}{z^4}$, $\frac{e^9}{z^8}$, $\frac{e^{13}}{z^{12}}$, &c) will be the ſame as are to be prefixed to the correſponding terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the compound ſeries Δ which involve in them the correſponding fractions $\frac{e^5}{s^4}$, $\frac{e^9}{s^8}$, $\frac{e^{13}}{s^{12}}$, &c; and the ſigns + and — which are to be prefixed to the ſeveral terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the compound ſeries δ (which involve in them the fractions $\frac{e^3}{z^2}$, $\frac{e^7}{z^6}$, $\frac{e^{11}}{z^{10}}$, $\frac{e^{15}}{z^{14}}$, &c) will be, reſpectively, contrary to thoſe which are to be prefixed to the correſponding terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the compound ſeries Δ, which involve in them the fractions $\frac{e^3}{s^2}$, $\frac{e^7}{s^6}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{15}}{s^{14}}$, &c.

The product of the multiplication of the co-efficient q into the expreſſion $2\sqrt{^3}\,\overline{|z} \times$ *the ſeries* $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — *&c is equal to the ſimple ſeries* $6Be + \frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}}$ + &c.

39. The compound ſeries mentioned above in art. 19, and denoted by the Greek capital letter Σ, is equal to $6Be$ — the compound ſeries Δ. And it is ſhewn in art. 25 and 26 that the compound ſeries Σ, or $6Be - \Delta$, is equal to the ſimple ſeries $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}}$ + &c. Therefore (adding Δ to both ſides) we ſhall have $6Be = \Delta + \frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}}$ + &c, and Δ = the ſimple ſeries $6Be - \frac{Pe^3}{s^2} - \frac{Qe^5}{s^4} - \frac{Re^7}{s^6} - \frac{Se^9}{s^8} - \frac{Te^{11}}{s^{10}} - \frac{Ve^{13}}{s^{12}}$ — &c, in which all the terms after the firſt term $6Be$ are marked with the ſign —, or ſubtracted from the ſaid firſt term. But it has been ſhewn in the foregoing art. 38, that the co-efficients of the fractions $\frac{e^3}{z^2}$, $\frac{e^5}{z^4}$, $\frac{e^7}{z^6}$, $\frac{e^9}{z^8}$, $\frac{e^{11}}{z^{10}}$, $\frac{e^{13}}{z^{12}}$, &c, in the compound ſeries δ are equal to, or the ſame with, the co-efficients of the correſponding fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$ &c in the compound ſeries Δ, reſpectively; and that the ſigns — and +, that are to be prefixed to the ſeveral terms of the 3d, 5th, 7th, and other following odd vertical columns of the compound ſeries δ, (which involve in them the fractions

$\frac{e^5}{z^4}$,

$\frac{e^5}{z^4}$, $\frac{e^9}{z^8}$, $\frac{e^{13}}{z^{12}}$, &c) are the same with those which are to be prefixed to the corresponding terms of the 3d, 5th, 7th, and other following odd vertical columns of the compound series Δ, which involve in them the corresponding fractions $\frac{e^5}{s^4}$, $\frac{e^9}{s^8}$, $\frac{e^{13}}{s^{12}}$, &c; and that the signs which are to be prefixed to the several terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of the compound series δ (which involve in them the fractions $\frac{e^3}{z^2}$, $\frac{e^7}{z^6}$, $\frac{e^{11}}{z^{10}}$, $\frac{e^{15}}{z^{14}}$, &c) are, respectively, contrary to those which are to be prefixed to the corresponding terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of the compound series Δ, which involve in them the fractions $\frac{e^3}{s^2}$, $\frac{e^7}{s^6}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{15}}{s^{14}}$, &c. And in both the said compound serieses Δ and δ the first term is $6\,Be$. It follows therefore, that, since the compound series Δ is equal to the simple series $6\,Be - \frac{Pe^3}{s^2} - \frac{Qe^5}{s^4} - \frac{Re^7}{s^6} - \frac{Se^9}{s^8} - \frac{Te^{11}}{s^{10}} - \frac{Ve^{13}}{s^{12}} -$ &c *ad infinitum*, the compound series δ must be equal to the simple series $6\,Be + \frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} +$ &c *ad infinitum*. But, by art. 37, the product of the multiplication of the co-efficient q into the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c is equal to the compound series δ. Therefore the product of the multiplication of the co-efficient q into the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c will also be equal to the simple series $6\,Be + \frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} +$ &c *ad infinitum*.

Of the relation of the product of the multiplication of the co-efficient q into the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c to the cube of the said expression.

---

40. We are now to shew that the product of the multiplication of the co-efficient q into the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c is greater than the cube of the said expression, and that the difference is equal to r, or to $6\,Be$, which is $(= 6 \times \frac{1}{3} \times \frac{r}{2} = \frac{6}{6} \times r) = r$.

 Now

Now this will be evident, if we can shew that the simple series $6Be + \frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} +$ &c *ad infinitum* (which we have shewn to be equal to the said product) is greater than the said cube, and that the difference is equal to r, or $6Be$; or that the said series, when its first term $6Be$ is taken from it, will be equal to the said cube; or that the simple series $\frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} +$ &c *ad infinitum*, will be equal to the cube of the said expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c *ad infinitum*. This we will therefore now endeavour to prove in the manner following.

41. The cube of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c is $= 8z \times$ the cube of the said series. We must therefore raise the said series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c to its cube, or third power, by multiplying it twice into itself. This may be done as follows.

The multiplication of the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c twice into itself, in order to obtain its cube.

$$
\begin{array}{cccccccc}
\frac{Be}{z} & -\frac{De^3}{z^3} & +\frac{Fe^5}{z^5} & -\frac{He^7}{z^7} & +\frac{Ke^9}{z^9} & -\frac{Me^{11}}{z^{11}} & +\frac{Oe^{13}}{z^{13}} & -\&c \\
\frac{Be}{z} & -\frac{De^3}{z^3} & +\frac{Fe^5}{z^5} & -\frac{He^7}{z^7} & +\frac{Ke^9}{z^9} & -\frac{Me^{11}}{z^{11}} & +\frac{Oe^{13}}{z^{13}} & -\&c \\
\hline
\frac{B^2e^2}{z^2} & -\frac{BDe^4}{z^4} & +\frac{BFe^6}{z^6} & -\frac{BHe^8}{z^8} & +\frac{BKe^{10}}{z^{10}} & -\frac{BMe^{12}}{z^{12}} & +\&c & \\
 & -\frac{BDe^4}{z^4} & +\frac{D^2e^6}{z^6} & -\frac{DFe^8}{z^8} & +\frac{DHe^{10}}{z^{10}} & -\frac{DKe^{12}}{z^{12}} & +\&c & \\
 & & +\frac{BFe^6}{z^6} & -\frac{DFe^8}{z^8} & +\frac{F^2e^{10}}{z^{10}} & -\frac{FHe^{12}}{z^{12}} & +\&c & \\
 & & & -\frac{BHe^8}{z^8} & +\frac{DHe^{10}}{z^{10}} & -\frac{FHe^{12}}{z^{12}} & +\&c & \\
 & & & & +\frac{BKe^{10}}{z^{10}} & -\frac{DKe^{12}}{z^{12}} & +\&c & \\
 & & & & & -\frac{BMe^{12}}{z^{12}} & +\&c & \\
\end{array}
$$

$\frac{B^2e^2}{z^2}$

$$\frac{B^2e^2}{z^2} - \frac{2BDe^4}{z^4} + \frac{2BFe^6}{z^6} - \frac{2BHe^8}{z^8} + \frac{2BKe^{10}}{z^{10}} - \frac{2BMe^{12}}{z^{12}} + \&c$$
$$+ \frac{D^2e^6}{z^6} - \frac{2DFe^8}{z^8} + \frac{2DHe^{10}}{z^{10}} - \frac{2DKe^{12}}{z^{12}} + \&c$$
$$+ \frac{F^2e^{10}}{z^{10}} - \frac{2FHe^{12}}{z^{12}} + \&c$$

$$\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$$

$$\frac{B^3e^3}{z^3} - \frac{2B^2De^5}{z^5} + \frac{2B^2Fe^7}{z^7} - \frac{2B^2He^9}{z^9} + \frac{2B^2Ke^{11}}{z^{11}} - \frac{2B^2Me^{13}}{z^{13}} + \&c$$
$$+ \frac{BD^2e^7}{z^7} - \frac{2BDFe^9}{z^9} + \frac{2BDHe^{11}}{z^{11}} - \frac{2BDKe^{13}}{z^{13}} + \&c$$
$$+ \frac{BF^2e^{11}}{z^{11}} - \frac{2BFHe^{13}}{z^{13}} + \&c$$
$$- \frac{B^2De^5}{z^5} + \frac{2BD^2e^7}{z^7} - \frac{2BDFe^9}{z^9} + \frac{2BDHe^{11}}{z^{11}} - \frac{2BDKe^{13}}{z^{13}} + \&c$$
$$- \frac{D^3e^9}{z^9} + \frac{2D^2Fe^{11}}{z^{11}} - \frac{2D^2He^{13}}{z^{13}} + \&c$$
$$- \frac{DF^2e^{13}}{z^{13}} + \&c$$

$$+ \frac{B^2Fe^7}{z^7} - \frac{2BDFe^9}{z^9} + \frac{2BF^2e^{11}}{z^{11}} - \frac{2BFHe^{13}}{z^{13}} + \&c$$
$$+ \frac{D^2Fe^{11}}{z^{11}} - \frac{2DF^2e^{13}}{z^{13}} + \&c$$
$$- \frac{B^2He^9}{z^9} + \frac{2BDHe^{11}}{z^{11}} - \frac{2BFHe^{13}}{z^{13}} + \&c$$
$$- \frac{D^2He^{13}}{z^{13}} + \&c$$
$$+ \frac{B^2Ke^{11}}{z^{11}} - \frac{2BDKe^{13}}{z^{13}} + \&c$$
$$- \frac{B^2Me^{13}}{z^{13}} + \&c$$

$$\frac{B^3e^3}{z^3} - \frac{3B^2De^5}{z^5} + \frac{3B^2Fe^7}{z^7} - \frac{3B^2He^9}{z^9} + \frac{3B^2Ke^{11}}{z^{11}} - \frac{3B^2Me^{13}}{z^{13}} + \&c$$
$$+ \frac{3BD^2e^7}{z^7} - \frac{6BDFe^9}{z^9} + \frac{6BDHe^{11}}{z^{11}} - \frac{6BDKe^{13}}{z^{13}} + \&c$$
$$- \frac{D^3e^9}{z^9} + \frac{3BF^2e^{11}}{z^{11}} - \frac{6BFHe^{13}}{z^{13}} + \&c$$
$$+ \frac{3D^2Fe^{11}}{z^{11}} - \frac{3D^2He^{13}}{z^{13}} + \&c$$
$$- \frac{3DF^2e^{13}}{z^{13}} + \&c.$$

This laſt compound ſeries is the cube of the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum.*

Therefore

Therefore 8 z $\times$ the cube of the ſaid ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c will be $=$ 8 z $\times$ the foregoing compound ſeries; or, if, for the ſake of brevity, we denote the ſaid compound ſeries by the ſmall Greek letter λ, we ſhall have 8 z $\times$ the cube of the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c equal to 8 z $\times$ the compound ſeries λ; and conſequently the cube of the expreſſion $2z^{\frac{1}{3}}$ $\times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c *ad infinitum* will be $=$ 8 z $\times$ the compound ſeries λ.

A compariſon between the compound ſeries λ, obtained in the foregoing article 41, *and the compound ſeries Λ, obtained above in art.* 17.

42. We muſt now compare the compound ſeries λ, which is equal to the cube of the ſimple ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c, with the compound ſeries Λ, obtained above in art. 17, which is equal to the cube of the ſimple ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}}$ — &c, in order to diſcover the analogies and the differences that ſubſiſt between them.

Now, upon comparing together theſe two compound ſerieſes Λ and λ, we ſhall find that,

In the firſt place, wherever there is any power of the fraction $\frac{e}{s}$ in the compound ſeries Λ, there will be the ſame power of the fraction $\frac{e}{z}$ in the correſponding term of the compound ſeries λ.

And, in the ſecond place, the co-efficients of the terms of the compound ſeries λ are equal to, or the ſame with, the co-efficients of the correſponding terms of the compound ſeries Λ.

And theſe two analogies between theſe two compound ſerieſes muſt take place not only in the few firſt vertical columns of terms of the ſaid ſerieſes which are ſet down above in art. 17 and art. 41, but throughout all the terms of the ſaid ſerieſes, to whatever number of terms the ſaid ſerieſes may be continued: becauſe the original ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c (by the multiplication of which twice into itſelf the compound ſeries λ is produced) involves in its terms the ſame powers of the fraction $\frac{e}{z}$, and the ſame co-efficients B, D, F, H, K, M, O, &c, combined with thoſe powers reſpectively,

tively, as the other original ſeries $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}}$ + &c (by the multiplication of which twice into itſelf the compound ſeries A was produced) involves of the fraction $\frac{e}{s}$.

And, in the 3d place, all the terms of the compound ſeries A, after the firſt term $\frac{B^3e^3}{s^3}$, are marked with the ſign +, or added to the ſaid firſt term: but in the compound ſeries λ only the terms of the 3d, and 5th, and other following odd vertical columns of terms (which involve the fractions $\frac{e^7}{z^7}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{15}}{z^{15}}$, $\frac{e^{19}}{z^{19}}$, &c) are marked with the ſign +, or added to the firſt term $\frac{B^3e^3}{z^3}$; and the ſecond term $\frac{3B^2De^5}{z^5}$, and the terms in the 4th, 6th, 8th, and other following even vertical columns of terms in the ſaid ſeries (which involve the fractions $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, $\frac{e^{13}}{z^{13}}$, $\frac{e^{17}}{z^{17}}$, &c) are marked with the ſign −, or ſubtracted from the ſaid firſt term.

43. That in the compound ſeries λ the terms in the ſecond, third, and other following vertical columns of terms are marked with the ſigns − and + alternately, is evident upon inſpection as far as the multiplication is carried above in art. 41, that is, as far as the ſix firſt vertical columns of terms. And that the ſame alternate ſucceſſion of thoſe ſigns will take place in all the following columns of terms in the ſaid ſeries, to whatever number of terms the ſaid ſeries may be continued, will be evident from the following conſiderations.

Since in the multiplication of the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ − &c into itſelf in art. 41, in order to obtain its ſquare, the ſigns − and + follow each other alternately both in the multiplicand and the multiplicator (which are both the ſame ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7}$ + &c), it is evident that in the ſeveral products of the multiplicand by the firſt term $\frac{Be}{z}$ of the multiplicator, and by its third term $\frac{Fe^5}{z^5}$, and its fifth term $\frac{Ke^9}{z^9}$, and its ſeventh term $\frac{Oe^{13}}{z^{13}}$, and all its following odd terms (to which odd terms the ſign + is prefixed) the order of the ſigns + and − in the terms of the ſaid products muſt be the ſame as in the multiplicand itſelf; that is, the terms of the ſaid ſeveral products which conſtitute the 1ſt, 3d, 5th, 7th, and other following odd horizontal lines of the ſaid general product (before the ſimilar terms are collected together by addition at the bottom) will be marked with the ſigns + and − alternately. And it is likewiſe evident, that in the ſeveral products of the ſaid multiplicand by the ſecond term $\frac{De^3}{z^3}$ of the multiplicator (which is marked with the

the ſign —), and by the fourth term $\frac{He^7}{z^7}$, and the ſixth term $\frac{Me^{11}}{z^{11}}$, and the following even terms of the multiplicator (which are all marked with the ſign —), the order of the ſigns + and — in the terms of the ſaid products muſt be contrary to the order of them in the terms of the multiplicand itſelf; that is, the terms of the ſaid ſeveral products which conſtitute the 2d, 4th, 6th, 8th, and other following even horizontal rows of terms in the ſaid general product (before the ſimilar terms are collected together by addition at the bottom) will be marked with the ſigns — and + alternately. Therefore the firſt term of every new horizontal row of terms will be marked with a contrary ſign to that of the firſt term of the horizontal row next above it, and conſequently with the ſame ſign as the ſecond term of the ſaid horizontal row next above it, or as the term under which it is placed; the firſt term of every new horizontal row being placed immediately under the ſecond term of the horizontal row next above it. And thus it appears that the laſt term of every vertical row and the laſt term but one will be marked with the ſame ſign. And hence it follows (from the alternate ſucceſſion of the ſigns + and — to each other in the terms of all the horizontal rows of terms in the ſaid general product, before the terms are collected together at the bottom) that all the terms in every vertical column of terms will be marked with the ſame ſign as the loweſt term, and conſequently that the terms in the ſecond, third, fourth, fifth, ſixth, and other following vertical columns of terms will be marked with the ſign — and the ſign + alternately. Q. E. D.

44. And, ſince all the terms in each of the vertical columns of the general product in art. 41 (before the ſimilar terms are collected together at the bottom by addition) are marked with the ſame ſign + or —, and the terms in the 2d, 3d, 4th, 5th, 6th, and other following vertical columns of terms in the ſaid general product are marked with the ſign — and the ſign + alternately, it follows that in the compound ſeries ſet down under the ſaid general product (and which is equal to it and derived from it by adding together the ſimilar terms in each vertical column ſo as to convert them into ſingle terms) the ſeveral terms of the ſecond, and third, and fourth, and fifth, and ſixth, and other following vertical columns of terms will alſo be marked with the ſign — and the ſign + alternately. Q. E. D.

45. And, ſince in the compound ſeries laſt mentioned, which is equal to the ſquare of the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c, the ſecond and other following vertical columns of terms are marked with the ſign — and the ſign + alternately, and the compound ſeries λ (which is the cube of the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c) is produced by multiplying the ſaid compound ſeries into the original ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c, in which the ſecond and other following terms

2 are

are alſo marked with the ſign — and the ſign + alternately, it will follow, by repeating the reaſonings uſed in the two laſt articles 43 and 44, that in the product of this laſt multiplication, that is, in the compound ſeries λ, the terms of the ſecond, and third, and fourth, and fifth, and ſixth, and other following vertical columns will alſo be marked with the ſign — and the ſign + alternately, agreeably to what we have ſeen by inſpection of the ſaid compound ſeries λ (as ſet down in art. 41) to take place in the firſt ſix vertical columns of it, or as far as the terms that involve the fraction $\frac{e^{13}}{z^{13}}$, and, agreeably likewiſe, to what was aſſerted, in art. 42, concerning all the following vertical columns of the ſaid compound ſeries. Q. E. D.

Of the compound ſeries π, which is equal to $8z \times$ the compound ſeries λ, and conſequently to the cube of the expreſſion $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c ad infinitum.

46. If the foregoing compound ſeries λ, obtained in art. 41, be multiplied into $8z$, the product will be the following compound ſeries, to wit,

$$\begin{array}{rrrrrrl}
\frac{8B^3e^3}{z^3} - \frac{24B^2De^5}{z^4} & + \frac{24B^2Fe^7}{z^6} & - \frac{24B^2He^9}{z^8} & + \frac{24B^2Ke^{11}}{z^{10}} & - \frac{24B^2Me^{13}}{z^{12}} & + \&c \\
 & + \frac{24BD^2e^7}{z^6} & - \frac{48BDFe^9}{z^8} & + \frac{48BDHe^{11}}{z^{10}} & - \frac{48BDKe^{13}}{z^{12}} & + \&c \\
 & & - \frac{8D^3e^9}{z^8} & + \frac{24BF^2e^{11}}{z^{10}} & - \frac{48BFHe^{13}}{z^{12}} & + \&c \\
 & & & + \frac{24D^2Fe^{11}}{z^{10}} & - \frac{24D^2He^{13}}{z^{12}} & + \&c \\
 & & & & + \frac{24DF^2e^{13}}{z^{12}} & + \&c,
\end{array}$$

which, for the ſake of brevity, we will denote by the ſmall Greek letter π.

Then will the cube of the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c (which is equal to $8z \times$ the compound ſeries λ) be = the compound ſeries π.

47. Since the compound ſeries π is $= 8z \times$ the compound ſeries λ; and the terms of the ſecond, and third, and fourth, and other following vertical columns of the compound ſeries λ are marked with the ſign — and the ſign + alternately; it is evident that the terms of the ſecond, and third, and fourth, and other following vertical columns of the compound ſeries π will alſo be marked with the ſign — and the ſign + alternately: becauſe the multiplication into $8z$ can make no change in the ſigns of the quantities that are multiplied.

A comparifon between the compound feries π juft now obtained in art. 46, *and the compound feries* Π, *obtained above in art.* 18.

48. It has been fhewn above in art. 42, that the co-efficients of the feveral terms of the compound feries λ (fet down in art. 41) are refpectively equal to, or the fame with, the co-efficients of the correfponding terms of the compound feries Λ, fet down in art. 17. And therefore the co-efficients of the feveral terms of the compound feries π, or $8z \times$ the compound feries λ, which are equal to 8 times the co-efficients of the correfponding terms of the compound feries λ, muft be equal to the co-efficients of the correfponding terms in the compound feries Π, or $8s \times$ the compound feries Λ (fet down above in art. 18), which are equal to 8 times the co-efficients of the correfponding terms of the compound feries Λ.

A proof that the compound feries π is equal to the fimple feries $\frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} +$ &c *ad infinitum.*

49. It appears from the laft article 48 that the co-efficients of the feveral terms of the compound feries π, which is fet down in art. 46, are refpectively equal to, or the fame with, the co-efficients of the correfponding terms of the compound feries Π, which is fet down in art. 18; and it appears from art. 47 that all the terms in the fecond and other following vertical columns of the compound feries π will be marked with the fign $-$ and the fign $+$ alternately, to wit, all the terms in the 2d, 4th, 6th, 8th, and other following even vertical columns of it with the fign $-$, and all the terms of the 3d, 5th, 7th, 9th, and other following odd vertical columns of it with the fign $+$; whereas in the compound feries Π all the terms in the 2d, 3d, 4th, 5th, and all the following vertical columns, both odd and even, are marked with the fign $+$. It follows therefore that, if the compound feries Π be converted into a fimple feries by adding together into one fum all the terms of each feparate vertical column, and the compound feries π be likewife converted into a fimple feries by adding together into one fum all the terms of each feparate column, the co-efficients of the terms of the fecond fimple feries, which is equal to the compound feries π, will be refpectively equal to the co-efficients of the correfponding terms of the former fimple feries, which is equal to the compound feries Π: but in the fimple feries that is equal to the compound feries π, the fecond, and fourth, and fixth, and other following even terms will be marked with the fign $-$, and the third, and fifth, and feventh, and other following odd terms will be marked with the fign $+$; whereas in the fimple feries that is equal to the compound feries Π all the terms after the firft term will be marked with the fign. Therefore,

fore, if we ſuppoſe (as was done above in art. 23) the co-efficients of the terms of the ſimple ſeries that is equal to the compound ſeries Π to be P, Q, R, S, T, V, &c, and conſequently the ſaid ſimple ſeries to be $\frac{Pe^3}{s^2} + \frac{Qe^5}{s^4} + \frac{Re^7}{s^6} + \frac{Se^9}{s^8} + \frac{Te^{11}}{s^{10}} + \frac{Ve^{13}}{s^{12}}$ + &c *ad infinitum*, the ſimple ſeries that is equal to the compound ſeries π will be $\frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}}$ + &c *ad infinitum.*

A proof derived from the foregoing articles, that the expreſſion $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c is equal to one of the roots of the cubick equation $qx - x^3 = r$.

50. It is ſhewn in art. 46 that the compound ſeries π is equal to the cube of the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c. Therefore the ſimple ſeries $\frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}}$ + &c *ad infinitum* (which is equal to the compound ſeries π) will alſo be equal to the cube of the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c *ad infinitum.*

51. It was ſhewn above in art. 39 that the product of the multiplication of the co-efficient q into the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c *ad infinitum* is equal to the ſeries $6Be + \frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}}$ + &c *ad infinitum.* And now we have ſeen, in the laſt art. 50, that the cube of the ſaid expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c is equal to the ſeries $\frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}}$ + &c *ad infinitum*, which conſiſts of the very ſame terms as the ſeries $6Be + \frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}}$ + &c *ad infinitum*, except the firſt term $6Be$, and therefore is leſs than the ſaid laſt-mentioned

tioned ſeries by the difference 6 Be. It follows therefore, that the cube of the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c *ad infinitum*, is leſs than the product of the multiplication of the co-efficient q into the ſaid expreſſion, and that their difference is 6 Be, or r. And conſequently the ſaid expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c *ad infinitum* muſt be equal to one of the roots of the cubick equation $qx - x^3 = r$. Q. E. D.

52. Having now ſhewn that the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c *ad infinitum* is equal to one of the roots of the cubick equation $qx - x^3 = r$, it remains that we ſhew that it cannot be equal to the greater of the two roots of that equation, being always leſs than the leaſt poſſible magnitude of the ſaid greater root; whence it will follow that it muſt be equal to the leſſer root of the ſaid equation. This may be ſhewn in the manner following.

Of the leaſt poſſible magnitude, or the lower limit of the magnitude, of the greater root of the cubick equation $qx - x^3 = r$ upon the ſuppoſition that r is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$.

53. The greateſt poſſible magnitude of the abſolute term r of the cubick equation $qx - x^3 = r$ is $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or that value of the compound quantity $qx - x^3$ which reſults from a ſuppoſition that x is $= \frac{\sqrt{q}}{\sqrt{3}}$. If r is greater than this quantity $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qx - x^3 = r$ is impoſſible; and if r is exactly equal to $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qx - x^3 = r$ will have but one root, which will be $= \frac{\sqrt{q}}{\sqrt{3}}$; and, if r is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qx - x^3 = r$ will have two roots, of which the leſſer will be leſs than $\frac{\sqrt{q}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{q}}{\sqrt{3}}$, but leſs than $\sqrt{q}$. But in the foregoing articles it is ſuppoſed that the abſolute term r is leſs, not only than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ (which is its greateſt poſſible magnitude), but likewiſe than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, which is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ in the proportion of ($\sqrt{2}$ to 2, or of) 1 to $\sqrt{2}$. Therefore the greater root of the equation $qx - x^3 = r$ muſt, on the

the present supposition, be greater, not only than $\frac{\sqrt{q}}{\sqrt{3}}$, but than the greater root of it when r is $= \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{q}}$. Now, when r is $= \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, the greater root of the equation $qx - x^3 = r$, will be $\sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}}$. For, if we suppose x to be $= \sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}}$, we shall have $x^3 = 2\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, and qx $(= q \times \sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}} = \sqrt{2} \times \frac{q\sqrt{q}}{\sqrt{3}}) = 3\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, and consequently $qx - x^3$ $(= 3\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}} - 2\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}) = 1 \times \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$. Therefore, on the present supposition, to wit, that r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, the greater root of the equation $qx - x^3 = r$ must be greater than $\sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}}$, or than $\frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{q}$, or than $\sqrt{\frac{2}{3}} \times \sqrt{q}$, or than $\sqrt{0.666,666}$, &c $\times \sqrt{q}$, or than $0.8165 \times \sqrt{q}$; or, in other words, $\sqrt{\frac{2}{3}} \times \sqrt{q}$, or $0.8165 \times \sqrt{q}$, will be the least possible magnitude, or the lower limit of the magnitude, of the greater root of the equation $qx - x^3 = r$ upon the present supposition.

A proof that the said lower limit of the magnitude of the greater root of the cubick equation $qx - x^3 = r$ is greater than the greatest possible magnitude of the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c.

Now this quantity $\sqrt{\frac{2}{3}} \times \sqrt{q}$, or $0.8165 \times \sqrt{q}$, is greater than the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c *ad infinitum*; as may be shewn in the manner following.

54. This expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} -$ &c is equal to the series $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{8}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}} -$ &c.

Now, since zz is $= \frac{q^3}{27} - \frac{rr}{4}$, or $\frac{q^3}{27} - ee$, it is evident that, if ee, or $\frac{rr}{4}$, be supposed

ſuppoſed to increaſe, zz, or the exceſs of $\frac{q^3}{27}$ above ee or $\frac{rr}{4}$, will at the ſame time decreaſe. Therefore, if ee, or $\frac{rr}{4}$, is ſuppoſed to increaſe from o to its greateſt poſſible magnitude on the preſent ſuppoſition, to wit, to $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, the numerators of the terms of the ſeries $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{8}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}}$ — &c (which involve in them the powers of e) will continually increaſe, and their denominators (which are powers of the decreaſing quantity z) will at the ſame time continually decreaſe: and conſequently the terms of the ſaid ſeries will, on account both of the increaſe of their numerators and the decreaſe of their denominators, continually increaſe at the ſame time. Therefore the ſaid ſeries $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{8}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}}$ — &c will have attained its greateſt magnitude when ee, or $\frac{rr}{4}$, has attained its greateſt magnitude, or is $= \frac{q^3}{54}$; at which time zz, or $\frac{q^3}{27} - ee$, or $\frac{q^3}{27} - \frac{rr}{4}$, will be $= \frac{q^3}{27} - \frac{q^3}{54}$, or $\frac{2q^3}{54} - \frac{q^3}{54}$, or $\frac{q^3}{54}$, and conſequently will be equal to ee. Therefore the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c (which is equal to the ſeries $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{8}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}}$ — &c) will alſo have attained its greateſt magnitude when ee or $\frac{rr}{4}$, is equal to $\frac{q^3}{54}$, and zz is conſequently equal to ee. But, when zz is equal to ee, and ee is equal to $\frac{q^3}{54}$, the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c becomes $= B - D + F - H + K - M + O$ &c, and z is $(= e = \frac{r}{2}) = \sqrt{\frac{q^3}{54}}$, and conſequently $z^{\frac{1}{3}} = \sqrt[6]{\frac{q^3}{54}}$, and $2z^{\frac{1}{3}} = 2\sqrt[6]{\frac{q^3}{54}} = \frac{2\sqrt[6]{q^3}}{\sqrt[6]{54}} = \frac{2\sqrt[2]{q}}{\sqrt[6]{54}} = \frac{\sqrt[6]{64} \times \sqrt[2]{q}}{\sqrt[6]{54}} = \frac{\sqrt[6]{64}}{\sqrt[6]{54}} \times \sqrt{q} = \sqrt[6]{\frac{64}{54}} \times \sqrt{q} = \sqrt[6]{\frac{32}{27}} \times \sqrt{q} = \sqrt[6]{1.185,185,185} \times \sqrt{q} = 1.028,7 \times \sqrt{q}$; and conſequently the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c becomes in this caſe $= 1.0287 \times \sqrt{q} \times$ the ſeries $B - D + F - H + K - M + O$ — &c; or the greateſt poſſible magnitude of the expreſſion $2z^{\frac{1}{3}} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c is $= 1.0287 \times \sqrt{q} \times$ the ſeries $B - D + F - H + K - M + O$ — &c *ad infinitum*.

55. Now

55. Now because the terms B, D, F, H, K, M, O, &c, are a decreasing progression, every term being less than that which immediately preceeds it, the first term B alone of the said series B — D + F — H + K — M + O &c must be greater than the whole series. Therefore $1.0287 \times \sqrt{q} \times B$ will be greater than $1.0287 \times \sqrt{q} \times$ the whole series B — D + F — H + K — M + O — &c, and consequently than the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c *ad infinitum*. But B is $= \frac{1}{3}$. Therefore $1.0287 \times \sqrt{q} \times \frac{1}{3}$, or $\frac{1.0287}{3} \times \sqrt{q}$, or $0.3429 \times \sqrt{q}$, will be greater than the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c. But $0.3429 \times \sqrt{q}$ is much less than $0.8165 \times \sqrt{q}$, which is the least possible magnitude of the greater root of the equation $qx - x^3 = r$ upon the present supposition that $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, or that r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$. Therefore, *à fortiori*, the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c is less than $0.8165 \times \sqrt{q}$, or the least possible magnitude of the greater root of the equation $qx - x^3 = r$. Therefore the said expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c *ad infinitum* cannot be equal to the greater root of the said equation. But it has been shewn above in art. 51 that the said expression is equal to one of the roots of the said equation $qx - x^3 = r$ in this case of the said equation, or when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$. Therefore it must be equal to the lesser root of the said equation. Q. E. D.

End of the demonstration that the expression $2\sqrt[3]{z} \times$ *the series* $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ *— &c ad infinitum is equal to the lesser root of the cubick equation* $qx - x^3 = r$.

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56. I have now compleated the demonstration of the proposition laid down above in art. 11, to wit, that, if the absolute term r of the cubick equation $qx - x^3 = r$ be less, not only than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ (which is its greatest possible magnitude) but than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$; or if $\frac{rr}{4}$ be less, not only than $\frac{q^3}{27}$, but than $\frac{q^3}{2 \times 27}$, or

or $\frac{q^3}{54}$; and e be taken $= \frac{r}{2}$, and zz be $= \frac{q^3}{27} - \frac{rr}{4}$, or $\frac{q^3}{27} - ee$; the lesser root of the equation $qx - x^3 = r$ will be equal to the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c *ad infinitum*. It is indeed a very long and complicated demonstration. But I know not how to make it shorter without taking from the perspicuity of the reasonings used in it, which are both various and abstruse. And "that they should be so" will appear the less surprising, if we consider that they supply the place of those very obscure and intricate operations by which many writers of Algebra endeavour to find the roots of impossible quantities, such as $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$. See upon this subject *Monsieur Clairaut's Elémens d'Algébre*, Part V, art. ix, pages 286, 287, 288, 289, and a paper of *Monsieur Nicole* in the Memoirs of the Academy of Sciences at Paris for the year 1738, pages 99 and 100, from which Monsieur Clairaut has extracted what he has delivered upon this subject in the pages of his Algebra just now cited. And see also Dr. Wallis's Algebra, chapter 48, pages 179, 180, of the folio edition at London in 1685, and Professor Saunderson's Algebra, pages 744, 745, 746, 747, and Mac Laurin's Algebra, Part I. the supplement to the 14th chapter, pages 127, 128, 129, 130, and the Philosophical Transactions, No 451.

57. We will now proceed to give an example of the resolution of a cubick equation of the foregoing form $qx - x^3 = r$, when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, by means of the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c, which we have shewn in the foregoing articles to be equal to its lesser root.

An Example of the resolution of a cubick equation of the foregoing form, $qx - x^3 = r$, by means of the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{s^9} - \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}}$ — &c.

Let the equation that is to be resolved by means of this expression be $15x - x^3 = 4$.

Here q is $= 15$, and r is $= 4$; and consequently $\frac{q}{3}$ is $(= \frac{15}{3}) = 5$, and $\frac{r}{2}$ is $(= \frac{4}{2}) = 2$, and $\frac{q^3}{27}$ is $(= \left(\frac{q}{3}\right)^3 = 5^3) = 125$, and $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, is $(= \frac{125}{2}) = 62.5$, and $\frac{rr}{4}$ is $(= \left(\frac{r}{2}\right)^2 = 2^2) = 4$, which is less than 62.5, or $\frac{q^3}{54}$. Therefore

5 this

this equation may be resolved by means of the foregoing expression $2\sqrt{^3}z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c.

Now, since $\frac{q^3}{27}$ is $= 125$, and $\frac{rr}{4}$ is $= 4$, we shall have zz $(= \frac{q^3}{27} - \frac{rr}{4} = 125 - 4) = 121$, and consequently z $(= \sqrt{}\ 121) = 11$, and $\sqrt{^3}z$ $(= \sqrt{^3}\ 11) = 2.223,980,090,569,361$, &c, and $2\sqrt{^3}z$ $(= 2 \times 2.223,980,090,569,361$, &c) $= 4.447,960,181,138,722$, &c.

And we shall have ee $(= \frac{rr}{4}) = 4$, and consequently $e = 2$, and $\frac{e}{z} = \frac{2}{11} = 0.181,818,181,818$, and $\frac{e^2}{z^2} = \frac{4}{121}$.

Therefore $\frac{e^3}{z^3}$ will be $(= \frac{e}{z} \times \frac{ee}{zz} = 0\ 181,818,181,818 \times \frac{4}{121} = \frac{0.727,272,727,272}{121}) = 0.006,010,518,407$;

And $\frac{e^5}{z^5}$ will be $(= \frac{e^3}{z^3} \times \frac{e^2}{z^2} = 0.006,010,518,407 \times \frac{4}{121} = \frac{0.024,042,073,672}{121}) = 0.000,198,694,823$;

And $\frac{e^7}{z^7}$ will be $(= \frac{e^5}{z^5} \times \frac{e^2}{z^2} = 0.000,198,694,823 \times \frac{4}{121} = \frac{0.000,794,779,292}{121}) = 0.000,006,568,423$;

And $\frac{e^9}{z^9}$ will be $(= \frac{e^7}{z^7} \times \frac{e^2}{z^2} = 0.000,006,568,423 \times \frac{4}{121} = \frac{0.000,026,273,692}{121}) = 0.000,000,217,137$;

And $\frac{e^{11}}{z^{11}}$ will be $(= \frac{e^9}{z^9} \times \frac{e^2}{z^2} = 0.000,000,217,137 \times \frac{4}{121} = \frac{0.000,000,868,548}{121}) = 0.000,000,007,178$;

And $\frac{e^{13}}{z^{13}}$ will be $(= \frac{e^{11}}{z^{11}} \times \frac{e^2}{z^2} = 0.000,000,007,178 \times \frac{4}{121} = \frac{0.000,000,028,712}{121}) = 0.000,000,000,237$.

Therefore $\frac{Be}{z}$ will be $(= B \times 0.181,818,181,818 = \frac{1}{3} \times 0.181,818,181,818 = \frac{0.181,818,181,818}{3}) = 0.060,606,060,606$;

And $\frac{De^3}{z^3}$ will be $(= D \times 0.006,010,518,407 = \frac{5}{81} \times 0.006,010,518,407 = \frac{0.030,052,592,035}{81}) = 0.000,371,019,654$;

And $\frac{Fe^5}{z^5}$ will be $(= F \times 0.000,198,694,823 = \frac{22}{729} \times 0.000,198,694,823 = \frac{0.004,371,286,106}{729}) = 0.000,005,996,277$;

 And

And $\frac{He^7}{z^7}$ will be ($=$ H $\times$ 0.000,006,568,423 $= \frac{374}{19683} \times$ 0.000,006,568, 423 $= \frac{0.002,456,590,202}{19683}$) $=$ 0.000,000,124,807;

And $\frac{Ke^9}{z^9}$ will be ($=$ K $\times$ 0.000,000,217,137 $= \frac{21505}{1,594,323} \times$ 0.000,000, 217,137 $= \frac{0.004,669,531,185}{1,594,323}$) $=$ 0.000,000,002,928;

And $\frac{Me^{11}}{z^{11}}$ will be ($=$ M $\times$ 0.000,000,007,178 $= \frac{147,407}{14,348,907} \times$ 0.000,000, 007,178 $= \frac{0.001,058,087,446}{14,348,907}$) $=$ 0.000,000,000,073;

And $\frac{Oe^{13}}{z^{13}}$ will be ($=$ O $\times$ 0.000,000,000,237 $= \frac{3,174,920}{387,420,489} \times$ 0.000,000, 000,237 $= \frac{0.000,752,456,040}{387,420,489}$) $=$ 0.000,000,000,001.

Therefore the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c will be $=$ 0.060,606,060,606 — 0.000,371,019,654 + 0.000,005,996,277, — 0.000,000,124,807 + 0.000,000,002,928 — 0.000,000,000,073 + 0.000, 000,000,001 — &c $=$ 0.060,612,059,812 — 0.000,371,144,534 $=$ 0.060, 240,915,278; and consequently the expression $2\sqrt{^3}\, z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c will be $=$ 4.447,960,181,138, &c $\times$ 0.060,240,915,278 $=$ 0.267,949,192,431, &c. Therefore this number 0.267,949,192,431, &c is the lesser root of the proposed equation $15x - x^3 = 4$. Q. E. I.

The foregoing number 0.267,949,192,431, agrees with the true value of the lesser root of the equation $15x - x^3 = 4$ in all its twelve figures, the said root being equal to 0.267,949,192,431,122, &c, or 2 — 1.732,050,807,568,877, &c, or $2 - \sqrt{3}$. For, if we suppose x to be $= 2 - \sqrt{3}$, we shall have $15x$ ($= 15 \times \overline{2 - \sqrt{3}}) = 30 - 15\sqrt{3}$, and x^3 ($= 8 - 3 \times 4 \times \sqrt{3} + 3 \times 2 \times 3 - 3\sqrt{3} = 8 - 12\sqrt{3} + 18 - 3\sqrt{3}) = 26 - 15\sqrt{3}$, and $15x - x^3$ ($= 30 - 15\sqrt{3} - \overline{26 - 15\sqrt{3}} = 30 - 15\sqrt{3} - 26 + 15\sqrt{3} = 30 - 26) = 4$.

Another Example of the resolution of a cubick equation by means of the same expression.

58. As a second example of the foregoing method of resolving cubick equations, let it be proposed to resolve the equation $90x - x^3 = 98$.

Here q, the co-efficient of x, is $= 90$, and the absolute term r is $= 98$. Therefore $\frac{q}{3}$ will be ($= \frac{90}{3}$) $= 30$, and $\frac{q^3}{27}$ will be ($= \overline{\frac{q}{3}}|^3 = \overline{30}|^3$) $= 27,000$, and

and $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, will be $(= \frac{27,000}{2}) = 13,500$; and $\frac{r}{2}$ will be $(= \frac{98}{2}) = 49$, and $\frac{rr}{4}$ will be $(= \overline{49}|^2) = 2401$, which is less than 13,500, or $\frac{q^3}{54}$. Therefore the equation $90x - x^3 = 98$ may be resolved by means of the foregoing expression $2\sqrt{}^3 z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}}$ — &c.

Now, since $\frac{q^3}{27}$ is $= 27,000$, and $\frac{rr}{4}$ is $= 2401$, we shall have zz $(= \frac{q^3}{27} - \frac{rr}{4} = 27,000 - 2401) = 24599$, and z $(= \sqrt{}\ 24599) = 156.840,683$, and $\sqrt{}^3 z$ $(= \sqrt{}^3\ 156.840,683) = 5.392,865,326,078$, and $2\sqrt{}^3 z$ $(= 2 \times 5.392,865,326,078) = 10.785,730,652,156$.

And we shall have ee $(= \frac{rr}{4}) = 2401$, and consequently $e = 49$, and $\frac{e}{z} = \frac{49}{156.840,683} = 0.312,418,940,435$, and $\frac{e^2}{z^2} = \frac{2401}{24599}$.

Therefore $\frac{e^3}{z^3}$ will be $(= \frac{e}{z} \times \frac{e^2}{z^2} = 0.312,418,940,435 \times \frac{2401}{24599} = \frac{750.117,875,984,435}{24599}) = 0.030,493,836,171$;

And $\frac{e^5}{z^5}$ will be $(= \frac{e^3}{z^3} \times \frac{e^2}{z^2} = 0.030,493,836,171 \times \frac{2401}{24599} = \frac{73.115,600,646,571}{24599}) = 0.002,972,299,713$;

And $\frac{e^7}{z^7}$ will be $(= \frac{e^5}{z^5} \times \frac{e^2}{z^2} = 0.002,972,299,713 \times \frac{2401}{24599} = \frac{7.136,491,610,913}{24599}) = 0.000,290,113,078$;

And $\frac{e^9}{z^9}$ will be $(= \frac{e^7}{z^7} \times \frac{e^2}{z^2} = 0.000,290,113,078 \times \frac{2401}{24599} = \frac{0.696,561,500,278}{24599}) = 0.000,028,316,659$;

And $\frac{e^{11}}{z^{11}}$ will be $(= \frac{e^9}{z^9} \times \frac{e^2}{z^2} = 0.000,028,316,659 \times \frac{2401}{24599} = \frac{0.067,988,298,259}{24599}) = 0.000,002,763,864$;

And $\frac{e^{13}}{z^{13}}$ will be $(= \frac{e^{11}}{z^{11}} \times \frac{e^2}{z^2} = 0.000,002,763,864 \times \frac{2401}{24599} = \frac{0.006,636,037,464}{24599}) = 0.000,000,269,768$;

And $\frac{e^{15}}{z^{15}}$ $(= \frac{e^{13}}{z^{13}} \times \frac{e^2}{z^2} = 0.000,000,269,768 \times \frac{2401}{24599} = \frac{0.000,647,712,968}{24599}) = 0.000,000,026,330$.

Therefore $\frac{Be}{z}$ will be $(= B \times 0.312,418,940,435 = \frac{1}{3} \times 0.312,418,940,435 = \frac{0.312,418,940,435}{3}) = 0.104,139,646,811$;

And

And $\frac{De^3}{z^3}$ will be ($= D \times 0.030,493,836,171 = \frac{5}{81} \times 0.030,493,836,171 = \frac{0.152,469,180,855}{81}$) $= 0.001,882,335,566$;

And $\frac{Fe^5}{z^5}$ will be ($= F \times 0.002,972,299,713 = \frac{22}{729} \times 0.002,972,299,713 = \frac{0.065,390,593,686}{729}$) $= 0.000,089,699,031$;

And $\frac{He^7}{z^7}$ will be ($= H \times 0.000,290,113,078 = \frac{374}{19683} \times 0.000,290,113,078 = \frac{0.108,502,291,172}{19683}$) $= 0.000,005,512,487$;

And $\frac{Ke^9}{z^9}$ will be ($= K \times 0.000,028,316,659 = \frac{21505}{1,594,323} \times 0.000,028,316,659 = \frac{0.608,949,751,795}{1,594,323}$) $= 0.000,000,381,948$;

And $\frac{Me^{11}}{z^{11}}$ will be ($= M \times 0.000,002,763,864 = \frac{147,407}{14,348,907} \times 0.000,002,763,864 = \frac{0.407,412,900,648}{14,348,907}$) $= 0.000,000,028,393$;

And $\frac{Oe^{13}}{z^{13}}$ will be ($= O \times 0.000,000,269,768 = \frac{3,174,920}{387,420,489} \times 0.000,000,269,768 = \frac{0.856,491,818,560}{387,420,489}$) $= 0.000,000,002,210$;

And $\frac{Qe^{15}}{z^{15}}$ will be ($= Q \times 0.000,000,026,330 = \frac{70,664,648}{10,460,353,203} \times 0.000,000,026,330 = \frac{1.860,600,181,840}{10,460,353,203}$) $= 0.000,000,000,177$.

Therefore the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}}$ &c will be $= 0.104,139,646,811 - 0.001,882,335,566 + 0.000,089,699,031 - 0.000,005,512,487 + 0.000,000,381,948 - 0.000,000,028,393 + 0.000,000,002,210 - 0.000,000,000,177 +$ &c $= 0.104,229,730,000 - 0.001,887,876,623 = 0.102,341,853,377$; and consequently the expression $2\sqrt{}^2 z \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c will be $= 10.785,730,652,156 \times 0.102,341,853,377 = 1.103,831,664,966$, &c. Therefore this last number 1.103,831,664,966, &c, is the lesser root of the proposed equation $90x - x^3 = 98$. Q. E. I.

59. That this number 1.103,831,664,966 is nearly equal to the lesser root of the equation $90x - x^3 = 98$, will appear by substituting the first seven figures of it, to wit, 1.103,831, instead of x in the compound quantity $90x - x^3$. For we shall then have $90x$ ($= 90 \times 1.103,831$) $= 99.344,790$, and x^3 ($= \overline{1.103,831}|^3$) $= 1.344,955,018,877,185,191$, and consequently $90x - x^3$ (=

$$\begin{array}{r} 99.344,790,000,000,000,000 \\ -\ 1.344,955,018,877,185,191 \end{array}$$

$= 97.999,834,981,122,814,809$; which is very nearly equal to the absolute term

term 98. Therefore 1.103,831 muft be very nearly equal to one of the roots of the equation $90x - x^3 = 98$.

If we were to make the number 1.103,831 the firft ftep of a further approximation to the true value of x in this equation, according to Mr. Raphfon's method of approximation, we fhould find the firft eleven figures of the faid true value to be 1.103,832,911,1.

60. Thefe two examples are, as I apprehend, fufficient to illuftrate and confirm the foregoing method of refolving cubick equations of the foregoing form, $qx - x^3 = r$, when the abfolute term r is lefs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is lefs than $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, by means of the expreffion $2\sqrt{^3}z \times$ the feries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c *ad infinitum:* and therefore I fhall not add to the length of this difcourfe (which is already longer than I could have wifhed) by applying the faid expreffion to the refolution of any more examples.

Another expreffion of the value of the leffer root of the cubick equation $qx - x^3 = r$, derived from the foregoing expreffion of it.

61. But there is another expreffion for the value of the leffer root of the cubick equation $qx - x^3 = r$ in the cafe here fuppofed, which, as it may be eafily derived from the foregoing expreffion for it, to wit, $2\sqrt{^3}z \times$ the feries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c, ought not, I think, to be omitted. This expreffion does not confift entirely of an infinite feries (as the foregoing expreffion does), but partly of a finite algebraick expreffion, and partly of an infinite feries; and fewer terms of the infinite feries are neceffary to be computed and added together, in order to obtain the value of the expreffion to any propofed degree of exactnefs, than of the infinite feries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c contained in the foregoing expreffion. It is as follows, to wit, $\sqrt{^3}\overline{z+e} - \sqrt{^3}\overline{z-e} - 4\sqrt{^3}z \times$ the infinite feries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}} + \frac{Ve^{19}}{z^{19}} +$ &c; the terms of which feries are taken from the feries that is equal to $\overline{1 + \frac{e}{z}}|^{\frac{1}{3}}$, or the cube-root of the binomial quantity $1 + \frac{e}{z}$, to wit the feries $1 + \frac{Be}{z} - \frac{Ce^2}{z^2} + \frac{De^3}{z^3} - \frac{Ee^4}{z^4} + \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} + \frac{He^7}{z^7} - \frac{Ie^8}{z^8} + \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} + \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} + \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} + \frac{Qe^{15}}{z^{15}} - \frac{Re^{16}}{z^{16}} + \frac{Se^{17}}{z^{17}}$

$\frac{Se^{17}}{z^{17}} - \frac{Te^{18}}{z^{18}} + \frac{Ve^{19}}{z^{19}} -$ &c, by beginning with the fourth term $\frac{De^3}{z^3}$, and taking every fourth term from it. This expreſſion may be derived from the foregoing expreſſion $2\sqrt{^3}z \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c in the manner following.

The derivation of the expreſſion of the value of the leſſer root of the equation $qx - x^3 = r$ given in the preceeding article 61, from the former expreſſion of it.

62. By the binomial theorem in the caſe of roots we have $\sqrt{^3}\overline{1 + \frac{e}{z}} =$ the ſeries $1 + \frac{Be}{z} - \frac{Ce^2}{z^2} + \frac{De^3}{z^3} - \frac{Ee^4}{z^4} + \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} + \frac{He^7}{z^7} - \frac{Ie^8}{z^8} + \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} + \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} + \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} + \frac{Qe^{15}}{z^{15}} - \frac{Re^{16}}{z^{16}} + \frac{Se^{17}}{z^{17}} - \frac{Te^{18}}{z^{18}} + \frac{Ve^{19}}{z^{19}} -$ &c *ad infinitum*; and by the reſidual theorem in the caſe of roots we have $\sqrt{^3}\overline{1 - \frac{e}{z}}$ = the ſeries $1 - \frac{Be}{z} - \frac{Ce^2}{z^2} - \frac{De^3}{z^3} - \frac{Ee^4}{z^4} - \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} - \frac{He^7}{z^7} - \frac{Ie^8}{z^8} - \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} - \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} - \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} - \frac{Qe^{15}}{z^{15}} - \frac{Re^{16}}{z^{16}} - \frac{Se^{17}}{z^{17}} - \frac{Te^{18}}{z^{18}} - \frac{Ve^{19}}{z^{19}} -$ &c *ad infinitum*. Therefore, if we ſubtract $\sqrt{^3}\overline{1 - \frac{e}{z}}$ from $\sqrt{^3}\overline{1 + \frac{e}{z}}$, and this latter ſeries (which is equal to $\sqrt{^3}\overline{1 - \frac{e}{z}}$) from the former ſeries (which is equal to $\sqrt{^3}\overline{1 + \frac{e}{z}}$) the remainders will be equal to each other; that is, $\sqrt{^3}\overline{1 + \frac{e}{z}} - \sqrt{^3}\overline{1 - \frac{e}{z}}$ will be = the ſeries $\frac{2Be}{z} + \frac{2De^3}{z^3} + \frac{2Fe^5}{z^5} + \frac{2He^7}{z^7} + \frac{2Ke^9}{z^9} + \frac{2Me^{11}}{z^{11}} + \frac{2Oe^{13}}{z^{13}} + \frac{2Qe^{15}}{z^{15}} + \frac{2Se^{17}}{z^{17}} + \frac{2Ve^{19}}{z^{19}} +$ &c *ad infinitum*. Therefore $\frac{1}{2} \times \sqrt{^3}\overline{1 + \frac{e}{z}} - \frac{1}{2} \times \sqrt{^3}\overline{1 - \frac{e}{z}}$ will be = the ſeries $\frac{Be}{z} + \frac{De^3}{z^3} + \frac{Fe^5}{z^5} + \frac{He^7}{z^7} + \frac{Ke^9}{z^9} + \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} + \frac{Qe^{15}}{z^{15}} + \frac{Se^{17}}{z^{17}} + \frac{Ve^{19}}{z^{19}} +$ &c *ad infinitum*. Let the ſeries $\frac{2De^3}{z^3} + \frac{2He^7}{z^7} + \frac{2Me^{11}}{z^{11}} + \frac{2Qe^{15}}{z^{15}} + \frac{2Ve^{19}}{z^{19}} +$ &c *ad infinitum* be ſubtracted from both ſides. And we ſhall then have $\frac{1}{2}\sqrt{^3}\overline{1 + \frac{e}{z}} - \frac{1}{2}\sqrt{^3}\overline{1 - \frac{e}{z}}$ — the ſeries $\frac{2De^3}{z^3} + \frac{2He^7}{z^7} + \frac{2Me^{11}}{z^{11}} + \frac{2Qe^{15}}{z^{15}} + \frac{2Ve^{19}}{z^{19}} +$ &c *ad infinitum* = the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \frac{Se^{17}}{z^{17}} - \frac{Ve^{19}}{z^{19}} +$ &c *ad infinitum*. Therefore (multiplying both ſides into $2\sqrt{^3}z$) we ſhall

ſhall have $\sqrt[3]{z} \times \sqrt[3]{1+\frac{e}{z}} - \sqrt[3]{z} \times \sqrt[3]{1-\frac{e}{z}} - 2\sqrt[3]{z} \times$ the ſeries $\frac{2De^3}{z^3} + \frac{2He^7}{z^7} + \frac{2Me^{11}}{z^{11}} + \frac{2Qe^{15}}{z^{15}} + \frac{2Ve^{19}}{z^{19}} +$ &c *ad infinitum* $= 2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \frac{Se^{17}}{z^{17}} - \frac{Ve^{19}}{z^{19}} +$ &c *ad infinitum*, or $\sqrt[3]{z} \times \sqrt[3]{1+\frac{e}{z}} - \sqrt[3]{z} \times \sqrt[3]{1-\frac{e}{z}} - 4\sqrt[3]{z} \times$ the ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}} + \frac{Ve^{19}}{z^{19}} +$ &c *ad infinitum* $= 2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \frac{Se^{17}}{z^{17}} - \frac{Ve^{19}}{z^{19}} +$ &c *ad infinitum*. But, becauſe $z \times \overline{1+\frac{e}{z}}$ is $= z + e$, we ſhall have $\sqrt[3]{z} \times \sqrt[3]{1+\frac{e}{z}} = \sqrt[3]{z+e}$; and becauſe $z \times \overline{1-\frac{e}{z}}$ is $= z - e$, we ſhall have $\sqrt[3]{z} \times \sqrt[3]{1-\frac{e}{z}} = \sqrt[3]{z-e}$. Therefore $\sqrt[3]{z+e} - \sqrt[3]{z-e} - 4\sqrt[3]{z} \times$ the ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}} + \frac{Ve^{19}}{z^{19}} +$ &c *ad infinitum* will be $=$ the expreſſion $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \frac{Se^{17}}{z^{17}} - \frac{Ve^{19}}{z^{19}} +$ &c *ad infinitum*, and conſequently will alſo be equal to the leſſer root of the cubick equation $qx - x^3 = r$ in the caſe here ſuppoſed, or when the abſolute term r is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$.

Q. E. D.

An application of the laſt expreſſion of the value of the leſſer root of the equation $qx - x^3 = r$ to the reſolution of the above-mentioned numeral equations $15x - x^3 = 4$ and $90x - x^3 = 98$.

63. This new expreſſion $\sqrt[3]{z+e} - \sqrt[3]{z-e} - 4\sqrt[3]{z} \times$ the ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}} + \frac{Ve^{19}}{z^{19}} +$ &c may be applied to the reſolution of the two foregoing equations $15x - x^3 = 4$ and $90x - x^3 = 98$ in the manner following.

In the equation $15x - x^3 = 4$, we have ſeen above that e is $= 2$, and z is $= 11$, and $\frac{De^3}{z^3}$ is $= 0.000,371,019,654$, and $\frac{He^7}{z^7}$ is $= 0.000,000,124,807$, and $\frac{Me^{11}}{z^{11}}$ is $= 0.000,000,000,073$, and $\sqrt[3]{z}$ is $= \sqrt[3]{11} = 2.223,980,090,569,361$, &c. Therefore the ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} +$ &c is ($= 0.000,371,019,654 + 0.000,000,124,807 + 0.000,000,000,073 +$ &c) $= 0.000,371,144$.

144,534, &c, and $4\sqrt[3]{z} \times$ the ſaid ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} +$ &c is $(= 4 \times 2.223,980,090,569,361$, &c $\times 0.000,371,144,534 = 8.895,920,362,277,444, \times 0.000,371,144,534) = 0.003,301,672,217,358$, &c. And $z + e$ is $(= 11 + 2) = 13$, and $z - e$ is $(= 11 - 2) = 9$; and conſequently $\sqrt[3]{z+e}$ is $(= \sqrt[3]{13}) = 2.351,334,687,721$, and $\sqrt[3]{z-e}$ is $(= \sqrt[3]{9}) = 2.080,083,823,052$. Therefore $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ will be $(= 2.351,334,687,721 - 2.080,083,823,052) = 0.271,250,864,669$, and $\sqrt[3]{z+e} - \sqrt[3]{z-e} - 4\sqrt[3]{z} \times$ the ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} +$ &c will be $(= 0.271,250,864,669 - 0.003,301,672,217) = 0.267,949,192,452$. And conſequently this laſt number 0.267,949,192,452 will be equal to the leſſer of the two roots of the equation $15x - x^3 = 4$. Q. E. I.

This number 0.267,949,192,452 is exact in the firſt ten figures 0.267,949,192,4, the more exact value of the leſſer root of the ſaid equation being 0.267,949,192,431,122, &c, or $2 - 1.732,050,807,568,877$, &c, or $2 - \sqrt{3}$, as was ſhewn above in art. 57.

In the other equation $90x - x^3 = 98$ we have ſeen above that e is $= 49$, and z is $(= \sqrt{24599}) = 156.840,683$, &c, and $\frac{e}{z}$ is $(= \frac{49}{156.840,683}) = 0.312,418,940,435$, and $\frac{De^3}{z^3}$ is $= 0.001,882,335,566$, and $\frac{He^7}{z^7}$ is $= 0.000,005,512,487$, and $\frac{Me^{11}}{z^{11}}$ is $= 0.000,000,028,393$, and $\frac{Qe^{15}}{z^{15}}$ is $= 0.000,000,000,177$, and $\sqrt[3]{z}$ is $(= \sqrt[3]{156.840,683}) = 5.392,865,326,078$. Therefore the ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}}$ will be $(= 0.001,882,335,566 + 0.000,005,512,487, + 0.000,000,028,393 + 0.000,000,000,177) = 0.001,887,876,623$, and $4\sqrt[3]{z} \times$ the ſaid ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}}$ will be $(= 4 \times 5.392,865,326,078 \times 0.001,887,876,623 = 21.571,461,304,312 \times 0.001,887,876,623) = 0.040,724,257,520$.

And $z + e$ will be $(= 156.840,683 + 49) = 205.840,683$; and $z - e$ will be $(= 156.840,683 - 49) = 107.840,683$; and conſequently $\sqrt[3]{z+e}$ will be $(= \sqrt[3]{205.840,683}) = 5.904,417,671,968$, and $\sqrt[3]{z-e}$ will be $(= \sqrt[3]{107.840,683}) = 4.759,860,337,980$. Therefore $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ will be $(= 5.904,417,671,968 - 4.759,860,337,980) = 1.144,557,333,988$; and $\sqrt[3]{z+e} - \sqrt[3]{z-e} - 4\sqrt[3]{z} \times$ the ſeries $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}}$ will be $(= 1.144,557,333,988 - 0.040,724,257,520) = 1.103,833,076,468$. Therefore this laſt number 1.103,833,076,468 will be equal to the leſſer of the two roots of the equation $90x - x^3 = 98$. Q. E. I.

This value of the leſſer root of this equation is exact in the ſix firſt figures 1.103,83, and exceeds the true value of the ſaid root (which is 1.103,832,911,1,)

911,1,) by only the ſmall quantity 0.000,000,165,3, which is ſomewhat leſs than the difference whereby the former value found for this leſſer root in art. 58 by means of the expreſſion $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}}$ &c, to wit, 1.103,831,664,9, falls ſhort of the ſaid true value, that difference being 0.000,001,246,2.

A third expreſſion for the value of the leſſer root of the equation $qx - x^3 = r$, derived from the expreſſion obtained for it above in art. 55.

64. We may alſo derive another expreſſion for the value of the leſſer root of the equation $qx - x^3 = r$ in the caſe here ſuppoſed, from the foregoing expreſſion $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c, in the manner following.

The quantity $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ is $= \sqrt[3]{z} \times \sqrt[3]{1 + \frac{e}{z}} - \sqrt[3]{z} \times \sqrt[3]{1 - \frac{e}{z}} = \sqrt[3]{z} \times$ the ſeries $1 + \frac{Be}{z} - \frac{Ce^2}{z^2} + \frac{De^3}{z^3} - \frac{Ee^4}{z^4} + \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} + \frac{He^7}{z^7} - \frac{Ie^8}{z^8} + \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} + \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} + \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} + \frac{Qe^{15}}{z^{15}} -$ &c $- \sqrt[3]{z} \times$ the ſeries $1 - \frac{Be}{z} - \frac{Ce^2}{z^2} - \frac{De^3}{z^3} - \frac{Ee^4}{z^4} - \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} - \frac{He^7}{z^7} - \frac{Ie^8}{z^8} - \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} - \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} - \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} - \frac{Qe^{15}}{z^{15}} -$ &c $= \sqrt[3]{z} \times$ the ſeries $\frac{2Be}{z} + \frac{2De^3}{z^3} + \frac{2Fe^5}{z^5} + \frac{2He^7}{z^7} + \frac{2Ke^9}{z^9} + \frac{2Me^{11}}{z^{11}} + \frac{2Oe^{13}}{z^{13}} + \frac{2Qe^{15}}{z^{15}} +$ &c $= 2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} + \frac{De^3}{z^3} + \frac{Fe^5}{z^5} + \frac{He^7}{z^7} + \frac{Ke^9}{z^9} + \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} + \frac{Qe^{15}}{z^{15}} +$ &c. Therefore, if we add $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c to both ſides, we ſhall have $\sqrt[3]{z+e} - \sqrt[3]{z-e} + 2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c $= 2\sqrt[3]{z} \times$ the ſeries $\frac{2Be}{z} + \frac{2Fe^5}{z^5} + \frac{2Ke^9}{z^9} + \frac{2Oe^{13}}{z^{13}} +$ &c $= 4\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c. Therefore, if we ſubtract $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ from both ſides, we ſhall have $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^8} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c $= 4\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c $+ \sqrt[3]{z-e} - \sqrt[3]{z+e}$. But $2\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c has been ſhewn above to be

 equal

equal to the leſſer root of the equation $qx - x^3 = r$ in the caſe here ſuppoſed, or when $\frac{rr}{4}$ is leſs than $\frac{q^3}{54}$. Therefore the expreſſion $4\sqrt{}^3 z \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c *ad infinitum* $+ \sqrt{}^3\overline{z-e} - \sqrt{}^3\overline{z+e}$, or the exceſs of the quantity $4\sqrt{}^3 z \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c *ad infinitum* above the quantity $\sqrt{}^3\overline{z+e} - \sqrt{}^3\overline{z-e}$, will alſo be equal to the leſſer root of the equation $qx - x^3 = r$ in the ſame caſe of it. Q. E. D.

An application of the laſt, or third, expreſſion of the leſſer root of the equation $qx - x^3 = r$ to the reſolution of the above-mentioned numeral equations $15x - x^3 = 4$ and $90x - x^3 = 98$.

65. This laſt expreſſion, $4\sqrt{}^3 z \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c *ad infinitum*, may be applied to the reſolution of the two foregoing equations $15x - x^3 = 4$ and $90x - x^3 = 98$ in the manner following.

In the equation $15x - x^3 = r$ we have ſeen above that e is $= 2$, and z is $= 11$, and $\frac{Be}{z}$ is $= 0.060,606,060,606$, and $\frac{Fe^5}{z^5}$ is $= 0.000,005,996,277$, and $\frac{Ke^9}{z^9}$ is $= 0.000,000,002,928$, and $\frac{Oe^{13}}{z^{13}}$ is $= 0.000,000,000,001$, and $\sqrt{}^3 z$ is $= \sqrt{}^3 11 = 2.223,980,090,569,361$, &c. Therefore the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c will be $(= 0.060,606,060,606 + 0.000,005,996,277 + 0.000,000,002,928 + 0.000,000,000,001 +$ &c$) = 0.060,612,059,812$, &c, and $4\sqrt{}^3 z \times$ the ſaid ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c will be $(= 4 \times 2.223,980,090,569,361$, &c $\times 0.060,612,059,812$, &c $= 8.895,920,362,277,444 \times 0.060,612,059,812$, &c$) = 0.539,200,057,081$, &c.

And $z + e$ is $(= 11 + 2) = 13$, and $z - e$ is $(= 11 - 2) = 9$; and conſequently $\sqrt{}^3\overline{z+e}$ is $(= \sqrt{}^3 13) = 2.351,334,687,721$, and $\sqrt{}^3\overline{z-e}$ is $(= \sqrt{}^3 9) = 2.080,083,823,052$. Therefore $\sqrt{}^3\overline{z+e} - \sqrt{}^3\overline{z-e}$ is $(= 2.351,334,687,721 - 2.080,083,823,052) = 0.271,250,864,669$; and the expreſſion $4\sqrt{}^3 z \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c $- \sqrt{}^3\overline{z-e} + \sqrt{}^3\overline{z+e}$, or the exceſs of the quantity $4\sqrt{}^3 z \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c above the quantity $\sqrt{}^3\overline{z+e} - \sqrt{}^3\overline{z-e}$, will be $(= 0.539,200,057,081$, &c $- 0.271,250,864,669$, &c$) = 0.267,949,192,412$. Therefore this laſt number, $0.267,949,192,412$, will be equal to the leſſer root of the equation $15x - x^3 = 4$. Q. E. I.

This

This number 0.267,949,192,412 is exact in the firſt ten figures 0.267,949 192,4, the more exact value of the leſſer root of the ſaid equation being 0.267, 949,192,431,122, &c, or 2 — 1.732,050,807,568,877, &c, or $2 - \sqrt{3}$, as was ſhewn above in art. 57.

In the other equation $90x - x^3 = 98$ we have ſeen above that e is $= 49$, and z is $(= \sqrt{24599}) = 156.840,683$, &c, and $\frac{Be}{z}$ is $= 0.104,139,646,811$, and $\frac{Fe^5}{z^5}$ is $= 0.000,089,699,031$, and $\frac{Ke^9}{z^9}$ is $= 0.000,000,381,948$, and $\frac{Oe^{13}}{z^{13}}$ is $= 0.000,000,002,210$, and $\sqrt[3]{z}$ is $(= \sqrt[3]{156.840,683}) = 5.392,865, 326,078$. Therefore the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c will be $(= 0. 104,139,646,811 + 0.000,089,699,031 + 0.000,000,381,948 + 0.000,000, 002,210 +$ &c$) = 0.104,229,729,000$, and $4\sqrt[3]{z} \times$ the ſaid ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c will be $(= 4 \times 5.392,865,326,078 \times 0.104,229,729, 000 = 21.571,461,304,312 \times 0.104,229,729,000) = 2.248,387,565,882$, &c.

And $z + e$ is $(= 156.840,683 + 49) = 205.840,683$, and $z - e$ is $(= 156.840,683 - 49) = 107.840,683$; and conſequently $\sqrt[3]{z+e}$ is $(= \sqrt[3]{205.840,683}) = 5.904,417,671,968$, and $\sqrt[3]{z-e}$ is $(= \sqrt[3]{107.840,683}) = 4.759,860,337,980$, and $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ is $(= 5.904,417,671,968 - 4.759,860,337,980) = 1.144,557,333,988$. Therefore the expreſſion $4\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c $- \sqrt[3]{z-e} + \sqrt[3]{z+e}$, or the exceſs of the quantity $4\sqrt[3]{z} \times$ the ſeries $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c above the quantity $\sqrt[3]{z+e} - \sqrt[3]{z-e}$, will be $(= 2.248,387,565,882, - 1.144,557,333,988) = 1.103,830,231,894$. Therefore this laſt number 1.103,830,231,894 will be equal to the leſſer of the two roots of the equation $90x - x^3 = 98$. Q. E. I.

This number, 1.103,830,231,894, is exact in the firſt ſix figures 1.103,83, the more exact value of the leſſer root of the ſaid equation being 1.103,832, 911,1.

A ſhort view of the three expreſſions that have been here obtained for the value of the leſſer root of the cubick equation $qx - x^3 = r$.

66. It appears, therefore, that the leſſer root of the cubick equation $qx - x^3 = r$, in the caſe of it that has been here ſuppoſed, or when the abſolute term r is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, will be equal to either of the three following expreſſions; to wit,

 Firſt,

First, to $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c *ad infinitum*, which is an expression wholly transcendental;

And, 2dly, to the expression $\sqrt[3]{z+e} - \sqrt[3]{z-e} - 4\sqrt[3]{z} \times$ the series $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}} +$ &c *ad infinitum*; which is an expression partly algebräick and partly transcendental, and of which the greater part (to wit, $\sqrt[3]{z+e} - \sqrt[3]{z-e}$,) is algebräick;

And, 3dly, to the expression $4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} +$ &c *ad infinitum* $+ \sqrt[3]{z-e} - \sqrt[3]{z+e}$; which is also an expression partly transcendental, and partly algebräick, but of which the greater part is transcendental.

And we may observe that the algebräick part of the two latter expressions, to wit, the quantity $\sqrt[3]{z+e} - \sqrt[3]{z-e}$, is similar to the algebräick expression, $\sqrt[3]{s+e} - \sqrt[3]{s-e}$, given by Cardan's first rule, for the root of the cubick equation $qy + y^3 = r$.

67. The first of the three foregoing expressions of the value of the lesser root of the equation $qx - x^3 = r$, to wit, the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} +$ &c *ad infinitum*, is given by Monsieur Clairaut in his Elements of Algebra, Sect. X, page 288: and I have been informed that all the three expressions were published in the year 1738, in the Memoirs of the French Academy of Sciences, by Monsieur Nicole. But they are obtained by both those learned gentlemen by the intervention of ***negative quantities*** (or quantities less than nothing, or quantities resulting from the supposed subtraction of a greater quantity from a lesser), and of the square-roots of negative quantities, or what are called in books of algebra ***impossible quantities***; both which sorts of quantities I have throughout this discourse taken care to avoid mentioning.

Of the trisection of a circular arc by means of either of the three foregoing expressions for the value of the lesser root of the cubick equation $qx - x^3 = r$.

68. The foregoing expressions of the value of the lesser root of the cubick equation $qx - x^3 = r$ may be applied to the trisection of a circular arc, or to the finding of the chord of the third part of an arc of which the chord is given, in a circle of which the diameter is given, provided the given chord be less than the chord of the fourth part of the circumference of the circle, or of an arch of 90 degrees.

3 For,

For, if a be the radius of a circle, and consequently $2a$ its diameter, and k be the chord of any arc in it that is not greater than the semicircumference, and x be the chord of the third part of the arc of which k is the chord, the relation between the chords k and x will be expressed by this equation, $3aax - x^3 = aak$; as is demonstrated in many books of mathematicks, and amongst the rest, in my Dissertation on the Use of the Negative Sign in Algebra, published in the year 1758, pages 183, 184, art. 220, 221. Now let q be put $= 3aa$, and $r = aak$; and the equation $3aax - x^3 = aak$ will be converted into the equation $qx - x^3 = r$, in which $\frac{\sqrt{q}}{\sqrt{3}}$ represents a, or the radius of the circle, and $\frac{r}{\frac{q}{3}}$, or $r \times \frac{3}{q}$, or $\frac{3r}{q}$, represents $\frac{aak}{aa}$, or k, the given chord of the greater arch, which is to be trisected. Therefore, by computing either of the three foregoing expressions of the value of the lesser root of the equation $qx - x^3 = r$, we shall (if r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or aak is less than $\sqrt{2} \times a^3$, or k is less than $\sqrt{2} \times a$, or k is less than the chord of a quadrantal arc, or an arc of 90 degrees) obtain the value of the chord of the third part of the arch of which $\frac{3r}{q}$, or k, is the chord, in the said circle, of which $\frac{\sqrt{q}}{\sqrt{3}}$, or a, is the radius.

Q. E. I.

Of the analogy, or harmony, between circular arcs and logarithms, or between the measures of angles and the measures of ratios.

69. This application of the series obtained in the foregoing articles to the trisection of a circular arc is an instance of the analogy, or harmony, as Mr. Cotes calls it, that subsists between logarithms and circular arcs, or between the measures of ratios and the measures of angles. For from the expression $\sqrt[3]{s+e} - \sqrt[3]{s-e}$ given by Cardan for the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$ (the value of which expression is to be obtained by extracting the cube-roots of the given quantities $s + e$ and $s - e$, that is, by trisecting the ratios of $s + e$ to 1, and of $s - e$ to 1), we have derived the three expressions set down in art. 66, by either of which we may trisect the circular arc of which the given quantity $\frac{3r}{q}$, or k, is the chord, in a circle of which the given quantity $\frac{\sqrt{q}}{\sqrt{3}}$, or a, is the radius (provided the said arc is less than a quadrant) or may find the chord of the third part of the said arc by finding the lesser root of the equation $qx - x^3 = r$. And consequently problems that require the trisection of a circular arc, or of an angle, may, by means of the method here explained, be solved by the trisection of a ratio, or by the help of a table of logarithms.

A SCHO-

A SCHOLIUM.

70. That very learned and ingenious mathematician and philoſopher of the latter part of the laſt century, Mr. Leibnitz, of Hanover, ſeems to have been the firſt perſon who took notice of this connection between circular arcs and logarithms, or the meaſures of angles and the meaſures of ratios, which was afterwards more fully inſiſted upon and illuſtrated by Mr. Cotes in his *Harmonia Menſurarum*. For in the *Commercium Epiſtolicum de Analyſi promotâ* (firſt printed by order of the Royal Society of London, in the year 1712, and afterwards publiſhed by Tonſon and Watts in the year 1722) there is a very learned letter of Mr. Leibnitz to Mr. Henry Oldenburgh (at that time ſecretary to the Royal Society), dated on the 27th day of Auguſt in the year 1676, and directed to be communicated to Mr. Newton, at that time profeſſor of mathematicks in the univerſity of Cambridge (afterwards better known by the name of Sir Iſaac Newton); in which there is the following curious paſſage: "*Imaginariarum quantitatum in Realium Radicum expreſſiones ingredientium ſublationem, fruſtrà putem ſperari; imò quæri. Neque enim illæ ullo modo vel calculis vel conſtructionibus obſunt: et veræ realéſque ſunt quantitates, ſi inter ſe conjunguntur, ob deſtructiones virtuales. Quod multis elegantibus exemplìs et argumentis deprehendi.*

Exempli gratiâ, $\sqrt{1+\sqrt{-3}} + \sqrt{1-\sqrt{-3}}$ *eſt* $= \sqrt{6}$. *Tametſi enim neque ex binomio* $1+\sqrt{-3}$, *neque ex binomio* $1-\sqrt{-3}$, *radix extrahetur; nec proinde ſic deſtruetur [quantitas] imaginaria* $\sqrt{-3}$; *ſupponenda tamen eſt deſtructa eſſe virtualitèr; quod actu appareret, ſi fieri poſſet extractio. Aliâ tamen viâ hæc ſumma reperitur eſſe* $\sqrt{6}$. *Unde in cubicis binomiis, ubi realitas ejuſmodi formularum (tunc cum extractio ex ſingulis binomiis fieri nequit), ad oculum oſtendi non poteſt, mente tamen intelligitur. Quarè fruſtrà Carteſius aliique expreſſiones Cardanicas pro particularibus habuere. Si quis poſſet invenire quadraturam circuli et ejus partium ex datâ hyperbolæ et ejus partium quadraturâ, is poſſet eas tollere; modò in ipſam quadraturam imaginariæ illæ rurſùs ingrediantur.* See the *Commercium Epiſtolicum*, &c, pages 137, 138.

In this paſſage Mr. Leibnitz ſays that $\sqrt{1+\sqrt{-3}} + \sqrt{1-\sqrt{-3}}$ is $= \sqrt{6}$. Now, if we for a moment ſuppoſe it poſſible for ſuch quantities as $\sqrt{1+\sqrt{-3}}$ and $\sqrt{1-\sqrt{-3}}$ to exiſt, and (having made this ſuppoſition) we treat them as other algebräick quantities, it may be ſhewn in the following manner that their ſum is $= \sqrt{6}$.

Since $\overline{a+b}|^2$ is $= aa + 2ab + bb$, it follows (by ſubſtituting $\sqrt{1+\sqrt{-3}}$ for a, and $\sqrt{1-\sqrt{-3}}$ for b) that the ſquare of the quantity $\sqrt{1+\sqrt{-3}} + \sqrt{1-\sqrt{-3}}$ will be $= 1 + \sqrt{-3} + 2 \times \sqrt{1+\sqrt{-3}} \times \sqrt{1-\sqrt{-3}} + 1 - \sqrt{-3} = 2 + 2 \times \sqrt{1+\sqrt{-3}} \times \sqrt{1-\sqrt{-3}} = 2 + 2 \times \sqrt{\overline{1+\sqrt{-3}} \times \overline{1-\sqrt{-3}}} = 2 + 2 \times \sqrt{1 - -3} = 2 + 2 \times \sqrt{1+3} = 2 + 2 \times \sqrt{4} = 2 + 2 \times 2 = 2 + 4 = 6$. Q. E. D.

And in the ſame collection of letters called *Commercium Epiſtolicum*, &c, there is another letter of Mr. Leibnitz to Mr. Oldenburgh, dated on the 21ſt day of June

June in the following year 1677, in which there is the following paſſage relating to the ſame ſubject, which is ſtill more curious and remarkable than the former.

Antequam finiam, adjiciam uſum pulcherrimum ſerierum, qui imprimis Collinio noſtro non erit ingratus. Scis magnam eſſe difficultatem circà extrahendas radices ex binomiis cubicis, quando eas ingreditur quantitas imaginaria, orta ex radice quadraticâ negativæ quantitatis; ut $\sqrt[3]{a + \sqrt{-bb}} = M$, *et* $\sqrt[3]{a - \sqrt{-bb}} = N$; *ubi utraque quantitas M et N eſt ſingulatim impoſſibilis, ſumma autem, ut alibi oſtendi, eſt quantitas poſſibilis et realis, æqualis cuidem quæſitæ z. Ut verò ea eximatur, et ut extrabatur radix, nempe, ut inveniatur* $\frac{z}{2} + e \times \sqrt{-bb} = \sqrt[3]{a + \sqrt{-bb}}$, *et* $\frac{z}{2} - e \times \sqrt{-bb} = \sqrt[3]{a - \sqrt{-bb}}$ *(unde fit* $\sqrt[3]{a + \sqrt{-bb}} + \sqrt[3]{a - \sqrt{-bb}} = z$*) non poteſt adhiberi methodus Schotenii, Geometriæ Carteſianæ ſubjecta, quia opus eſt ad eam ut valor ipſius* $a + \sqrt{-bb}$ *exhibeatur ſaltèm approximando; quod notis methodis impoſſibile eſt. Quis enim valorem ipſius* $-bb$ *propè verum dabit? Neceſſe eſt enim invenire* $b \times \sqrt{-1}$; *quis autem exprimat* $\sqrt{-1}$ *appropinquando? Scripſi olim Collinio me remedium inveniſſe, quod etiàm ad omnes gradus ſuperiores valeat. Id ecce bic uno verbo!*

Ex binomiis $a + \sqrt{-bb}$ *extrabo radicem per ſeriem infinitam, ſive per theorema Newtonianum, ſive etiàm more meo priore, inſtituendo calculum ſecundùm naturam cujuſque gradûs, cum, ſcilicet, nondum theorema generale abſtraxiſſem: Quæ radix ponatur eſſe* $l + m \times \sqrt{-bb} + n + p \times \sqrt{-bb}$ *&c. Extrabatur jam et radix ex binomio altero,* $\sqrt[3]{a - \sqrt{bb}}$; *fiet illa* $l - m \times \sqrt{-bb} + n - p \times \sqrt{-bb}$ *&c; ut facilè demonſtrari poteſt ex calculo. Ergo, addendo bæc duo extracta, deſtruentur imaginariæ quantitates, et fiet* $z = 2l + 2n +$ *&c, quæ ſunt eæ ſeriei portiones in quibus nulla reperitur imaginaria. Invento ergò valore ipſius z, quantum ſatìs eſt, propinquo, quemadmodum Schotenius poſtulat, reliqua methodo Schotenianâ, perinde ac in illis binomiorum extrabendorum generibus, tranſigentur.* See the *Commercium Epiſtolicum*, &c, pages 202, and 203.

In the former of theſe paſſages Mr. Leibnitz ſeems to have entertained an opinion of the poſſibility of ſo extracting the roots of two impoſſible binomial quantities, that the impoſſible part of one of the roots ſhould be equal to the impoſſible part of the other root, and ſhould be marked with a contrary ſign + or —, and conſequently that the ſum of the ſaid two impoſſible roots ſhould be a poſſible quantity; but he does not ſeem to have then found out a method of doing this: but in the ſecond paſſage he informs Mr. Oldenburgh that he had found out a method of doing it, and he deſcribes the method to him: which conſiſts in extracting, in an infinite ſeries, by means of the binomial theorem, or otherwiſe, the cube root of the binomial quantity $a + \sqrt{-bb}$, and in extracting, in an infinite ſeries, by means of the reſidual theorem, or otherwiſe, the cube-root of the reſidual quantity $a - \sqrt{-bb}$, and in adding together the two ſerieſes obtained by theſe extractions, in which ſerieſes the impoſſible parts will mutually deſtroy each other, ſo as to leave a ſeries conſiſting of only real terms, to wit, the ſeries $2l + 2n +$ &c, for the value of the ſum of the ſaid two roots, or of the quantity $\sqrt[3]{a + \sqrt{-bb}} + \sqrt[3]{a - \sqrt{bb}}$. And

And *Monfieur Nicole* and *Monfieur Clairaut* have done nothing more than expatiate upon this method here given by Mr. Leibnitz, of obtaining thefe cube-roots by means of the binomial and refidual theorems. But both Mr. Leibnitz and thofe other authors, in explaining this method of proceeding, all equally fuppofe, for a while, the exiftence of impoffible quantities, fuch as $\sqrt{-bb}$, though they contrive to get rid of them in the conclufion: and this, I think, renders their methods of treating this fubject obfcure and unfatisfactory. But now in the prefent difcourfe, I have fhewn how Cardan's rule for refolving the cubick equations $y^3 + qy = r$, or $qy + y^3 = r$ (which is true in all the cafes of that equation, or in all the relative magnitudes of the co-efficient q and the abfolute term r) may be extended (by means of Sir Ifaac Newton's binomial and refidual theorems, in the cafe of roots) to the refolution of the cubick equation $qx - x^3 = r$, when r is lefs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is lefs than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, by finding its leffer root, from which the root of the oppofite equation $x^3 - qx$ may be derived by means of a quadratick equation, as is fhewn in my Differtation on the Ufe of the Negative Sign in Algebra, art. 218, pages 182, 183; and in the paper publifhed in the Philofophical Tranfactions for the year 1778, I have fhewn how Cardan's other rule for refolving the cubick equation $y^3 - qy = r$, or $x^3 - qx = r$ (which is true only when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) may be extended (by means of the fame binomial and refidual theorems in the cafe of roots) to the refolution of the equation $y^3 - qy = r$, or $x^3 - qx = r$, when r is lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$; and confequently, by both thefe difcourfes together, I have enabled the reader to extend one or other of Cardan's two rules to the refolution of the equation $x^3 - qx = r$, when r is of any magnitude lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, whether greater or lefs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is of any magnitude lefs than $\frac{q^3}{27}$, whether greater or lefs than $\frac{q^3}{54}$; and this ***without any mention of either impoffible or negative quantities:*** which, I hope, will be confidered as a valuable improvement on Mr. Leibnitz's and Monfieur Nicole's method of effecting the fame purpofe, by all fuch cultivators of Algebra as are fond of feeing it treated in a fcientifick manner, or with perfpicuity in the ideas and accuracy in the reafonings throughout all the various proceffes of it, inftead of being ufed as a fort of manual exercife, or mechanic art, by which conclufions are obtained we know not how, like corn that is put into a mill, and ground to meal, without the owner's comprehenfion of the manner in which the operation is performed.

A METHOD

A

METHOD

OF EXTENDING

CARDAN's RULE

FOR RESOLVING THE CUBICK EQUATION

$$y^3 - qy = r,$$

IN THE FIRST CASE OF IT,

or when r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, *or* $\frac{rr}{4}$ *is equal to, or greater than,* $\frac{q^3}{27}$, *to the other case of the same equation, in which r is less than* $\frac{2q\sqrt{q}}{3\sqrt{3}}$, *or* $\frac{rr}{4}$ *is less than* $\frac{q^3}{27}$, *and which the said rule is not naturally fitted to resolve; provided that the absolute term r (though less than* $\frac{2q\sqrt{q}}{3\sqrt{3}}$*) be greater than* $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, *or that* $\frac{rr}{4}$ *(though less than* $\frac{q^3}{27}$*) be greater than* $\frac{1}{2} \times \frac{q^3}{27}$, *or than* $\frac{q^3}{54}$.

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

N. B. *This Discourse was first published in the Philosophical Transactions for the Year* 1778.

A

METHOD

OF EXTENDING

CARDAN's RULE, &c.

ART. I. IT is well known to all perſons converſant with Algebra, that Cardan's rule for reſolving the cubick equation $y^3 - qy = r$ is only fitted to reſolve it when $\frac{rr}{4}$ is equal to, or greater than $\frac{q^3}{27}$, or when the abſolute term r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and that it is of no uſe in the reſolution of the other caſe of this equation, in which $\frac{rr}{4}$ is of any magnitude leſs than $\frac{q^3}{27}$, or r is of any magnitude leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$. For in this caſe the reſidual quantity $\frac{rr}{4} - \frac{q^3}{27}$, or the exceſs of $\frac{rr}{4}$ above $\frac{q^3}{27}$ becomes (according to the uſual language of algebräiſts) a negative quantity, and conſequently its ſquare-root (which enters into Cardan's expreſſions of the root of the equation $y^3 - qy = r$) becomes impoſſible, and thoſe expreſſions (which are $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, and $\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, and $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} - \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$) become impoſſible, or, according to what appears to me a more correct way of ſpeaking (as I never could form any idea of a negative quantity, nor could ever underſtand by the ſign — or *minus*, any thing more than the ſubtraction of a leſſer quantity from a greater) the quantity $\frac{rr}{4} - \frac{q^3}{27}$ itſelf becomes impoſſible, or the ſuppoſition that $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$ (which is one of the foundations of Cardan's rule above-mentioned) is no longer true, and conſequently the rule itſelf, which is built upon it, can no longer take place.

2. Neverthelefs it is poffible by the help of Sir Ifaac Newton's binomial and refidual theorems in the cafe of roots (which have been demonftrated in the foregoing difcourfes of this volume) to extend this rule of Cardan to the latter cafe of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, or r is lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and which it is not, of itfelf, fitted to refolve; or, to fpeak with more accuracy, it is poffible, by means of the faid two theorems of Sir Ifaac Newton, to derive from one of the expreffions of the value of y given by the faid rule of Cardan for the refolution of the equation $y^3 - qy = r$ in the firft cafe of the faid equation, or when $\frac{rr}{4}$ is equal to, or greater than $\frac{q^3}{27}$, another expreffion fomewhat different from the former, but bearing a great refemblance to it, that fhall exhibit the true value of y in the fecond cafe of that equation, or when $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, provided it be not alfo lefs than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$. And this may be done by a train of juft and clear reafonings, and without any mention of impoffible, or even of negative quantities. To fhow how this may be effected is the defign of the following pages.

3. That the whole of this matter may be feen at one view, it will be convenient to fet forth the foundation and inveftigation of Cardan's rule for refolving the equation $y^3 - qy = r$ in the firft cafe of it, or when the abfolute term r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is equal to, or greater than, $\frac{q^3}{27}$; which may be done in the manner following.

Obfervations preparatory to the inveftigation of Cardan's rule for refolving the cubick equation $y^3 - qy = r$, when r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is equal to, or greater than, $\frac{q^3}{27}$.

4. Previoufly to the inveftigation of this celebrated rule, it will be proper to make the following obfervations.

OBSERVATION 1. In the cubick equation $y^3 - qy = r$ (which is a propofition affirming that y^3, or the cube of the unknown quantity y, is greater than qy, or the product of the multiplication of y by the known co-efficient q, and that the excefs, or difference, is equal to the known quantity r) it is evident that, fince y^3 is greater than qy, $\frac{y^3}{y}$, or yy, muft be greater than $\frac{qy}{y}$ or q, and confequently that y muft be greater than $\sqrt{q}$.

OBS. 2. While y increafes from $\sqrt{q}$ *ad infinitum*, y^3 will increafe continually from $q\sqrt{q}$ *ad infinitum*, and qy will alfo increafe continually from the fame quantity $q\sqrt{q}$ *ad infinitum*.

OBS. 3.

Obs. 3. And, while y increases from $\sqrt{q}$ *ad infinitum*, the excess of y^3 above qy will increase continually from nothing *ad infinitum*, without ever decreasing.

For, if we put $\dot{y}$ (or the letter y with a point placed over it) to denote the increment which y receives in any given portion of time, either small or great, during its increase, $q\dot{y}$ will be the increment which qy will receive in the same time, and $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ will be the increment which y^3 will receive in the same time; because, when y is increased to $y + \dot{y}$, qy will be increased to $q \times \overline{y + \dot{y}}$, or to $qy + q\dot{y}$, and y^3 will be increased from y^3 to $\overline{y + \dot{y}}|^3$, or $y^3 + 3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$. Now, since y^2 is always greater than q during the whole increase of y from being equal to $\sqrt{q}$ *ad infinitum*, it follows that $y^2 \times \dot{y}$, or $y^2\dot{y}$, will be greater than $q \times \dot{y}$, or $q\dot{y}$, during that whole increase. But $3y^2\dot{y}$ is greater than $y^2\dot{y}$, and $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ is still greater than $3y^2\dot{y}$. Therefore, *à fortiori*, $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ must be greater than $q\dot{y}$ during the whole increase of y; that is, the increment of y^3 will be greater than the contemporary increment of qy during the whole increase of y. Therefore the residual quantity $y^3 - qy$, or the excess of y^3 above qy, will continually increase from 0, without ever decreasing, while y increases from $\sqrt{q}$ to any greater magnitude.

Further, since $3y^2\dot{y} + 3y\dot{y} + \dot{y}^3$ is the increment of y^3, and $q\dot{y}$ is the increment of qy, and $y^2\dot{y}$ is greater than $q\dot{y}$, it follows that the excess of the increment of y^3 above the increment of qy will be greater than the excess of the increment of y^3 above $y^2\dot{y}$, or than the excess of $3y^2\dot{y} + 3y\dot{y} + \dot{y}^3$ above $y^2\dot{y}$, or than the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$. But the excess of the increment of y^3 above the increment of qy is the increment of the residual quantity $y^3 - qy$. Therefore the increment of the residual quantity $y^3 - qy$ is greater than the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$. But it is evident that the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ will increase continually *ad infinitum*, while y increases *ad infinitum*; so that no quantity can be assigned, how great soever, which the said quantity $2y^2\dot{y} + 3y^2\dot{y} + \dot{y}^3$, or either of its two first members $2y^2\dot{y}$ and $3y\dot{y}^2$, may not, by increasing the quantity y continually, be made to exceed. Therefore the increment of the residual quantity $y^3 - qy$ (which increment is greater than the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$) will increase continually *ad infinitum*, or so as to become greater than any finite quantity, how great soever. And consequently the residual quantity itself (which receives the said continually-increasing increments) will increase continually from 0 *ad infinitum*, or so as to become greater than any finite quantity, how great soever. Q. E. D.

Obs. 4. Since the compound quantity $y^3 - qy$ increases continually at the same time as y increases; and, when y is equal to $\frac{2\sqrt{q}}{3\sqrt{3}}$, the said compound quantity is equal to ($\frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{2q\sqrt{q}}{\sqrt{3}}$, or $\frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{6q\sqrt{q}}{3\sqrt{3}}$, or) $\frac{2q\sqrt{q}}{3\sqrt{3}}$; it follows that, if y is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity $y^3 - qy$ will be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and, if y is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, the said compound quantity will be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and *à converso*, if the compound quantity $y^3 - qy$, or, its equal, the absolute term r, is greater,

greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of y will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, and, if $y^3 - qy$, or r, is lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of y will be lefs than $\frac{2\sqrt{q}}{\sqrt{3}}$; or if $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, y will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, y will be lefs than $\frac{2\sqrt{q}}{\sqrt{3}}$.

Obs. 5. When r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and confequently (by the laft obfervation) y is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, yy will be greater than $\frac{4q}{3}$, and confequently $\frac{yy}{4}$ will be greater than $\frac{q}{3}$. But $\frac{yy}{4}$ is the fquare of $\frac{y}{2}$. Therefore, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, the fquare of $\frac{y}{2}$, or of half the unknown quantity y, will be greater than $\frac{q}{3}$. But (by Euclid's Elements, Book II, Prop. 5) it is always poffible to divide a line, as y, into two unequal parts of fuch magnitudes that the rectangle under the faid parts fhall be equal to any quantity that is lefs than the fquare of it's half. Therefore, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it is poffible to divide the line, or root, y into two unequal parts of fuch magnitudes that their rectangle, or product, fhall be equal to $\frac{q}{3}$.

This laft obfervation is the foundation of Cardan's rule for the refolution of the cubick equation $y^3 - qy = r$, in the firft cafe of it, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$; the inveftigation of which rule I fhall now proceed to explain by the folution of the following problem.

PROBLEM I.

5. To refolve the cubick equation $y^3 - qy = r$, when the abfolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

SOLUTION.

Since r is fuppofed to be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and confequently (by Obfervation 5) $\frac{yy}{4}$ is greater than $\frac{q}{3}$, it is poffible for y to be divided into two unequal parts of fuch magnitudes that their rectangle, or product, fhall be equal to $\frac{q}{3}$. Now, let it be conceived to be fo divided; and let the greater of its two parts be called v, and the leffer be called z. Then will vz be $= \frac{q}{3}$, and confequently $3vz$ will be $= q$, and $3vz \times \overline{v+z}$ will be $= q \times \overline{v+z}$.

Now,

Now, ſince $v + z$ is $= y$, we ſhall have $y^3 = \overline{v+z}|^3 = v^3 + 3v^2z + 3vz^2 + z^3 = v^3 + z^3 + \overline{3v^2z + 3vz^2} = v^3 + z^3 + 3vz \times \overline{v+z}$; and we ſhall have $qy = q \times \overline{v+z}$. Therefore $y^3 - qy$ will be $= v^3 + z^3 + 3vz \times \overline{v+z} - q \times \overline{v+z}$; that is (becauſe $3vz \times \overline{v+z}$ is equal to $q \times \overline{v+z}$), $y^3 - qy$ will be $= v^3 + z^3$. Therefore the abſolute term r (which is equal to $y^3 - qy$) will be $= v^3 + z^3$.

But, ſince $3vz$ is $= q$, we ſhall have $z = \frac{q}{3v}$, and conſequently $z^3 = \frac{q^3}{27v^3}$. Therefore $v^3 + z^3$ will be $= v^3 + \frac{q^3}{27v^3}$, and conſequently r (which has been ſhewn to be equal $v^3 + z^3$) will be $= v^3 + \frac{q^3}{27v^3}$. Therefore (multiplying both ſides of the equation by v^3) rv^3 will be $= v^6 + \frac{q^3}{27}$; and ſubtracting v^6 from both ſides) $rv^3 - v^6$ will be $= \frac{q^3}{27}$.

But $rv^3 - v^6$ is the product of the multiplication of $r - v^3$ into v^3, which are together equal to r. Therefore (by El. II, 5,) $rv^3 - v^6$ muſt be leſs than the ſquare of half of r, that is, than $\frac{rr}{4}$, and conſequently may be ſubtracted from $\frac{rr}{4}$. Let it be ſo ſubtracted; and, let its equal, $\frac{q^3}{27}$, be alſo ſubtracted from the ſame quantity $\frac{rr}{4}$. And the remainders will be equal to each other; that is, $\frac{rr}{4} - \overline{rv^3 - v^6}$, or $\frac{rr}{4} - rv^3 + v^6$, will be equal to $\frac{rr}{4} - \frac{q^3}{27}$. Therefore the ſquare-root of the trinomial quantity $\frac{rr}{4} - rv^3 + v^6$ will be equal to the ſquare-root of $\frac{rr}{4} - \frac{q^3}{27}$. But the ſquare-root of the trinomial quantity $\frac{rr}{4} - rv^3 + v^6$ is the difference of the ſimple quantities $\frac{r}{2}$ and v^3, that is, either $\frac{r}{2} - v^3$ or $v^3 - \frac{r}{2}$, according as $\frac{r}{2}$ or v^3 is the greater quantity. But it has been already ſhewn that v^3 and z^3 together are equal to r; and v is ſuppoſed to be greater than z, and conſequently v^3 is greater than z^3. Therefore v^3 muſt be greater than the half of $v^3 + z^3$, and conſequently than the half of r, or than $\frac{r}{2}$. Therefore the difference of $\frac{r}{2}$ and v^3 muſt be $v^3 - \frac{r}{2}$, and not $\frac{r}{2} - v^3$; and conſequently $v^3 - \frac{r}{2}$ muſt be the ſquare-root of the trinomial quantity $\frac{rr}{4} - rv^3 + v^6$. We ſhall therefore have $v^3 - \frac{r}{2} = \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$; and conſequently (adding $\frac{r}{2}$ to both ſides) v^3 will be $= \frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and therefore (extracting the cube-roots of both ſides) v will be $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$.

But

But z has been ſhewn above to be $= \frac{q}{3v}$. Therefore $v + z$ will be $(= v + \frac{q}{3v}) = \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$; and conſequently y, or the root of the cubick equation $y^3 - qy = r$ (which is equal to $v + z$) will be $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$.

Q. E. I.

6. This expreſſion of the value of y in the equation $y^3 - qy = r$ may be rendered more ſimple by ſubſtituting in it the ſingle letter e inſtead of $\frac{r}{2}$, and the ſingle letter s inſtead of $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. For then it will be $\sqrt[3]{e + s} + \frac{q}{3\sqrt[3]{e+s}}$. Therefore, if e be put for half the abſolute term r of the cubick equation $y^3 - qy = r$, and ss be put for $\frac{rr}{4} - \frac{q^3}{27}$, or s be put for the ſquare-root of $\frac{rr}{4} - \frac{q^3}{27}$, the root y of the ſaid equation will be equal to $\sqrt[3]{e + s} + \frac{q}{3\sqrt[3]{e+s}}$, or to $\overline{e + s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e+s}|^{\frac{1}{3}}}$.

Q. E. I.

A ſynthetick demonſtration of the truth of the foregoing ſolution.

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7. Since this expreſſion $\overline{e + s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e+s}|^{\frac{1}{3}}}$ is equal to the root y of the equation $y^3 - qy = r$, it is evident that, if it were ſubſtituted inſtead of y in the compound quantity $y^3 - qy$, which forms the left-hand ſide of that equation, it would make the ſaid compound quantity be equal to the abſolute term r. And ſo we ſhall find it will, if we make the ſaid ſubſtitution; which may be done in the manner following.

If we ſuppoſe y to be $= \overline{e + s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e+s}|^{\frac{1}{3}}}$, we ſhall have $y^3 = e + s + 3 \times \overline{e + s}|^{\frac{2}{3}} \times \frac{q}{3 \times \overline{e+s}|^{\frac{1}{3}}} + 3 \times \overline{e + s}|^{\frac{1}{3}} \times \frac{qq}{9 \times \overline{e+s}|^{\frac{2}{3}}} + \frac{q^3}{27 \times \overline{e+s}}$ $= e + s + q \times \overline{e + s}|^{\frac{1}{3}} + \frac{qq}{3 \times \overline{e+s}|^{\frac{1}{3}}} + \frac{q^3}{27 \times \overline{e+s}}$, and qy $(= q \times \overline{\overline{e + s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e+s}|^{\frac{1}{3}}}}) = q \times \overline{e + s}|^{\frac{1}{3}} + \frac{qq}{3 \times \overline{e+s}|^{\frac{1}{3}}}$, and conſequently $y^3 -$

qy

$qy = e + s + q \times \overline{e+s}|^{\frac{1}{3}} + \frac{qq}{3 \times \overline{e+s}|^{\frac{1}{3}}} + \frac{q^3}{27 \times \overline{e+s}} - q \times \overline{e+s}|^{\frac{1}{3}} - \frac{qq}{3 \times \overline{e+s}|^{\frac{1}{3}}} = e + s + \frac{q^3}{27 \times \overline{e+s}} = \frac{27 \times \overline{e+s}|^2 + q^3}{27 \times \overline{e+s}} = \frac{27 \times \overline{ee + 2es + ss} + q^3}{27 \times \overline{e+s}} = \frac{27e^3 + 54es + 27s^2 + q^3}{27e + 27s}$.

But, because ss is $= \frac{rr}{4} - \frac{q^3}{27}$, or $ee - \frac{q^3}{27}$, we shall have $ss + \frac{q^3}{27} = ee$, and $\frac{q^3}{27} = ee - ss$, and consequently $q^3 = 27e^2 - 27s^2$.

Therefore the fraction $\frac{27e^2 + 54es + 27s^2 + q^3}{27e + 27s}$ will be $= \frac{27e^2 + 54es + 27s^2 + 27e^2 - 27s^2}{27e + 27s} = \frac{54e^2 + 54es}{27e + 27s}$; and consequently the compound quantity $y^3 - qy$ (which has been shewn to be equal to the said fraction) will also be equal to the fraction $\frac{54e^2 + 54es}{27e + 27s}$, and therefore $= \frac{2e^2 + 2es}{e+s} = \frac{2e \times \overline{e+s}}{e+s} = 2e = 2 \times \frac{r}{2} = r$. Therefore $\overline{e+s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e+s}|^{\frac{1}{3}}}$ must be equal to the root y of the equation $y^3 - qy = r$.

Q. E. D.

Another expression for the root of the foregoing equation $y^3 - qy = r$.

8. By resuming the solution of the foregoing problem we may find another expression for the root y of the equation $y^3 - qy = r$, to wit, the expression $\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, or $\sqrt[3]{e-s} + \frac{q}{3\sqrt[3]{e-s}}$, or $\overline{e-s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e-s}|^{\frac{1}{3}}}$. This expression may be found in the following manner.

The investigation of the said second expression for the value of the root y of the cubick equation $y^3 - qy = r$.

9. In art. 5 we supposed the line y to be divided into two unequal parts, v and z, of which v was supposed to be the greater; and we first found the value of the greater part v, and then determined that of the lesser part z by its relation to v, which is expressed by the equation $3vz = q$. But we may with the same

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eafe firft determine the value of the leffer part z, and then derive from the faid value of z the value of the greater part v by means of the fame equation $3vz = q$, which expreffes their relation to each other; whereby we fhould obtain the fecond expreffion of the value of $v + z$, or $z + v$, or y, which was fet forth in the foregoing article, to wit, $\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, or $\sqrt[3]{e - s} + \frac{q}{3\sqrt[3]{e - s}}$, or $\overline{e - s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e - s}|^{\frac{1}{3}}}$. This may be done in the manner following.

Since $3vz$ is $= q$, and confequently v is $= \frac{q}{3z}$, and v^3 is $= \frac{q^3}{27z^3}$, and it has been fhewn in art. 5 that r is $= v^3 + z^3$, it follows that r will be $= \frac{q^3}{27z^3}, + z^3$. Therefore rz^3 will be $= \frac{q^3}{27} + z^6$, and $rz^3 - z^6$ will be $= \frac{q^3}{27}$. Therefore, if we fubtract both fides of this equation from $\frac{rr}{4}$ (which is fuppofed to be greater than $\frac{q^3}{27}$, and confequently muft alfo be greater than, its equal, $rz^3 - z^6$) the remainders will be equal to each other; that is, $\frac{rr}{4} - \overline{rz^3 - z^6}$, or $\frac{rr}{4} - rz^3 + z^6$, will be equal to $\frac{rr}{4} - \frac{q^3}{27}$. Therefore the fquare-root of the trinomial quantity $\frac{rr}{4} - rz^3 + z^6$ will be equal to the fquare-root of $\frac{rr}{4} - \frac{q^3}{27}$. But the fquare-root of the trinomial quantity $\frac{rr}{4} - rz^3 + z^6$ is the difference of the fimple quantities $\frac{r}{2}$ and z^3, that is, either $\frac{r}{2} - z^3$ or $z^3 - \frac{r}{2}$, according as $\frac{r}{2}$ or z^3 is the greater quantity. But, becaufe r is equal to $v^3 + z^3$, and v^3 is greater than z^3, it follows that z^3 muft be lefs than one half of $v^3 + z^3$, or than one half of r, or than $\frac{r}{2}$; and confequently the difference between $\frac{r}{2}$ and z^3 will be $\frac{r}{2} - z^3$, and not $z^3 - \frac{r}{2}$. Therefore $\frac{r}{2} - z^3$ will be the fquare-root of the trinomial quantity $\frac{rr}{4} - rz^3 + z^6$, and confequently will be equal to the fquare-root of $\frac{rr}{4} - \frac{q^3}{27}$, or $\frac{r}{2} - z^3$ will be $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore (adding z^3 to both fides) $\frac{r}{2}$ will be $= z^3 + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and (fubtracting $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$ from both fides) z^3 will be $= \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and confequently (extracting the cube-roots of both fides) z will be $= \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$. Therefore $z + v$ will be $= \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + v = \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3z} =$

$\sqrt{}^3$

$\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, or the root y of the equation $y^3 - qy = r$ will be equal to the expression $\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, or to the expression $\sqrt[3]{e-s} + \frac{q}{3\sqrt[3]{e-s}}$, or $\overline{e-s}|^{\frac{1}{3}} + \frac{q}{\overline{e-s}|^{\frac{1}{3}}}$.

Q. E. I.

A synthetick demonstration of the truth of the foregoing expression.

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10. Here again we may demonstrate synthetically, that this expression $\overline{e-s}|^{\frac{1}{3}} + \frac{q}{3\overline{e-s}|^{\frac{1}{3}}}$ is equal to the true value of y in the proposed equation $y^3 - qy = r$, by substituting it for y in the compound quantity $y^3 - qy$, which forms the left-hand side of the said equation. For, if we make this substitution, we shall find that the value of $y^3 - qy$ thence arising will be equal to the absolute term r. This may be shewn in the manner following.

If y is supposed to be equal to $\overline{e-s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e-s}|^{\frac{1}{3}}}$, we shall have y^3 $(= e - s + 3 \times \overline{e-s}|^{\frac{2}{3}} \times \frac{q}{3 \times \overline{e-s}|^{\frac{1}{3}}} + 3 \times \overline{e-s}|^{\frac{1}{3}} \times \frac{qq}{9 \times \overline{e-s}|^{\frac{2}{3}}} + \frac{q^3}{27 \times \overline{e-s}}) = e - s + q \times \overline{e-s}|^{\frac{1}{3}} + \frac{qq}{3 \times \overline{e-s}|^{\frac{1}{3}}} + \frac{q^3}{27 \times \overline{e-s}}$, and qy $(= q \times \overline{\overline{e-s}|^{\frac{1}{3}} + \frac{q}{3 \times \overline{e-s}|^{\frac{1}{3}}}}) = q \times \overline{e-s}|^{\frac{1}{3}} + \frac{qq}{3 \times \overline{e-s}|^{\frac{1}{3}}}$, and consequently $y^3 - qy$ $(= e - s + q \times \overline{e-s}|^{\frac{1}{3}} + \frac{qq}{3 \times \overline{e-s}|^{\frac{1}{3}}} + \frac{q^3}{27 \times \overline{e-s}} - q \times \overline{e-s}|^{\frac{1}{3}} - \frac{qq}{3 \times \overline{e-s}|^{\frac{1}{3}}}) = e - s + \frac{q^3}{27 \times \overline{e-s}} = \frac{27 \times \overline{e-s}|^2 + q^3}{27 \times \overline{e-s}} = \frac{27 \times \overline{e^2 - 2es + s^2} + q^3}{27e - 27s} = \frac{27e^2 - 54es + 27s^2 + q^3}{27e - 27s}$.

But it has been shewn in art. 7 that q^3 is $= 27e^2 - 27s^2$. Therefore the fraction $\frac{27e^2 - 54es + 27s^2 + q^3}{27e - 27s}$ is $= \frac{27e^2 - 54es + 27s^2 + 27e^2 - 27s^2}{27e - 27s} = \frac{54e^2 - 54es}{27e - 27s}$; and consequently the compound quantity $y^3 - qy$ (which has been shewn to be equal to the said fraction) will also be equal to the fraction $\frac{54e^2 - 54es}{27e - 27s} =$

3 M 2 $\frac{54 \times \overline{e^2 - es}}{27 \times \overline{e-s}}$

$\frac{54\times\overline{e^2-es}}{27\times\overline{e-s}} = 2\times\frac{e^2-es}{e-s} = 2\times\frac{e\times\overline{e-s}}{e-s} = 2\times e = 2\times\frac{r}{2} = r$; that is, the value of the compound quantity $y^3 - qy$ arifing from the fubftitution of the expreffion $\overline{e-s}\big|^{\frac{1}{3}} + \frac{q}{3\times\overline{e-s}\big|^{\frac{1}{3}}}$ in its terms inftead of y, is equal to the abfolute term r of the cubick equation $y^3 - qy = r$. Therefore the faid expreffion $\overline{e-s}\big|^{\frac{1}{3}} + \frac{q}{3\times\overline{e-s}\big|^{\frac{1}{3}}}$ muft be equal to the value of y in the faid equation.

Q. E. D.

A third expreffion for the root of the foregoing equation $y^3 - qy = r$.

11. We may alfo, by refuming the folution of the foregoing problem contained in art. 5, find a third expreffion for the root of this equation $y^3 - qy = r$, to wit, the expreffion $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, or $\sqrt[3]{e+s} + \sqrt[3]{e-s}$, or $\overline{e+s}\big|^{\frac{1}{3}} + \overline{e-s}\big|^{\frac{1}{3}}$. This expreffion may be found in the following manner.

The inveftigation of the faid third expreffion for the value of the root y of the cubick equation $y^3 - qy = r$.

12. Since $v^3 + z^3$ is $= r$, it follows that z^3 will be $= r - v^3$. But v^3 is fhewn in art. 5 to be $= \frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore $r - v^3$ will be $= r - \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}} = \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Confequently z^3 (which is equal to $r - v^3$) will be $= \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore v will be $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, and z will be $= \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, and confequently $v + z$, or y, will be $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, or $\sqrt[3]{e+s} + \sqrt[3]{e-s}$, or $\overline{e+s}\big|^{\frac{1}{3}} + \overline{e-s}\big|^{\frac{1}{3}}$.

Q. E. I.

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A ſynthetick demonſtration of the truth of the foregoing expreſſion.

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13. Here again we may demonſtrate ſynthetically that this expreſſion $\overline{e+s}|^{\frac{1}{3}} + \overline{e-s}|^{\frac{1}{3}}$ is equal to the true value of y in the propoſed equation $y^3 - qy = r$, by ſubſtituting it for y in the compound quantity $y^3 - qy$, which forms the firſt, or left-hand, ſide of the ſaid equation. For, if we make this ſubſtitution, we ſhall find that the value of $y^3 - qy$ thence ariſing will be equal to the abſolute term r of the ſaid equation. This may be ſhewn in the following manner.

If y is ſuppoſed to be $= \overline{e+s}|^{\frac{1}{3}} + \overline{e-s}|^{\frac{1}{3}}$, we ſhall have y^3 ($= e + s + 3 \times \overline{e+s}|^{\frac{2}{3}} \times \overline{e-s}|^{\frac{1}{3}} + 3 \times \overline{e+s}|^{\frac{1}{3}} \times \overline{e-s}|^{\frac{2}{3}} + e - s = 2e + 3 \times \overline{e+s}|^{\frac{2}{3}} \times \overline{e-s}|^{\frac{1}{3}} + 3 \times \overline{e+s}|^{\frac{1}{3}} \times \overline{e-s}|^{\frac{2}{3}} = 2e + 3 \times \overline{e+s}|^{\frac{1}{3}} \times \overline{e+s}|^{\frac{1}{3}} \times \overline{e-s}|^{\frac{1}{3}} + 3 \times \overline{e-s}|^{\frac{1}{3}} \times \overline{e+s}|^{\frac{1}{3}} \times \overline{e-s}|^{\frac{1}{3}}$) $= 2e + 3 \times \overline{e+s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}} + 3 \times \overline{e-s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}}$, and qy ($= q \times \overline{\overline{e+s}|^{\frac{1}{3}} + \overline{e-s}|^{\frac{1}{3}}}$) $= q \times \overline{e+s}|^{\frac{1}{3}} + q \times \overline{e-s}|^{\frac{1}{3}}$, and conſequently $y^3 - qy = 2e + 3 \times \overline{e+s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}} + 3 \times \overline{e-s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}} - q \times \overline{e+s}|^{\frac{1}{3}} - q \times \overline{e-s}|^{\frac{1}{3}}$.

But it has been ſhewn in art. 7 that q^3 is $= 27e^2 - 27s^2 = 27 \times \overline{ee-ss}$. Therefore q will be $= 3 \times \sqrt[3]{ee-ss}$, or $3 \times \overline{ee-ss}|^{\frac{1}{3}}$. Conſequently $q \times \overline{e+s}|^{\frac{1}{3}}$ will be $= 3 \times \overline{ee-ss}|^{\frac{1}{3}} \times \overline{e+s}|^{\frac{1}{3}}$, and $q \times \overline{e-s}|^{\frac{1}{3}}$ will be $= 3 \times \overline{ee-ss}|^{\frac{1}{3}} \times \overline{e-s}|^{\frac{1}{3}}$. Therefore $y^3 - qy$ (which is equal to $2e + 3 \times \overline{e+s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}} + 3 \times \overline{e-s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}} - q \times \overline{e+s}|^{\frac{1}{3}} - q \times \overline{e-s}|^{\frac{1}{3}}$) will be $= 2e + 3 \times \overline{e+s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}} + 3 \times \overline{e-s}|^{\frac{1}{3}} \times \overline{ee-ss}|^{\frac{1}{3}} - 3 \times \overline{ee-ss}|^{\frac{1}{3}} \times \overline{e+s}|^{\frac{1}{3}} - 3 \times \overline{ee-ss}|^{\frac{1}{3}} \times \overline{e-s}|^{\frac{1}{3}} = 2e = 2 \times \frac{r}{2} = r$; that is, the value of the compound quantity $y^3 - qy$ ariſing from the ſubſtitution of the expreſſion $\overline{e+s}|^{\frac{1}{3}} + \overline{e-s}|^{\frac{1}{3}}$ in its terms inſtead of y, is equal to the abſolute term r of the cubick equation $y^3 - qy = r$, and

consequently the said expression $\overline{e+s}|^{\frac{1}{3}} + \overline{e-s}|^{\frac{1}{3}}$ must be equal to the root y of the said equation. Q. E. D.

14. I do not remember to have seen these substitutions, or synthetick demonstrations of the expressions given by the foregoing rule of Cardan for the root of the cubick equation $y^3 - qy = r$, in any book of Algebra.

15. I will now, by way of illustration of the expressions that have been above investigated, subjoin an example of the resolution of a numeral equation of the foregoing form $y^3 - qy = r$ by each of the three expressions above-mentioned, by which it will appear that they will all three bring out the same number for the root of the proposed equation.

An example of the resolution of a cubick equation of the foregoing form, $y^3 - qy = r$, by means of each of the three foregoing expressions.

16. Let it be required to find the value of y in the equation $y^3 - 3y = 18$.

In this equation q is $= 3$, and r is $= 18$. Therefore $\sqrt{q}$ is $= \sqrt{3}$, and $\frac{2q\sqrt{q}}{3\sqrt{3}}$ is $= \frac{2 \times 3 \times \sqrt{3}}{3\sqrt{3}} = 2$, which is much less than 18, or r. Therefore this equation comes under the first case of the general equation $y^3 - qy = r$, and consequently may be resolved by Cardan's rule, or by either of the three foregoing expressions.

The resolution of the equation $y^3 - 3y = 18$ by means of the first of the three foregoing expressions, to wit, the expression $\sqrt[3]{e+s} + \frac{q}{3\sqrt[3]{e+s}}$.

17. Now, since q is $= 3$, $\frac{q}{3}$ will be $(= \frac{3}{3}) = 1$, and consequently $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, will also be $= 1$. And, since r is $= 18$, we shall have $\frac{r}{2}$, or e, $= 9$, and $\frac{rr}{4}$, or ee, $= 81$, and consequently $\frac{rr}{4} - \frac{q^3}{27}$ $(= 81 - 1) = 80$; that is, ss will be $= 80$. Therefore s will be $(= \sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5}) = 4\sqrt{5}$, and $e + s$ will be $= 9 + 4\sqrt{5} = \frac{9 \times 8}{8} + \frac{8 \times 4 \times \sqrt{5}}{8} = \frac{9 \times 8 + 8 \times 4 \times \sqrt{5}}{8} = \frac{72 + 32\sqrt{5}}{8} = \frac{27 + 27\sqrt{5} + 45 + 5\sqrt{5}}{8} =$

$$\frac{3|^3 + 3 \times 9\sqrt{5} + 3 \times 3 \times 5 + 5\sqrt{5}}{8}$$

$\frac{\overline{3}|^3 + 3 \times 9\sqrt{5} + 3 \times 3 \times 5 + 5\sqrt{5}}{8} = \frac{\overline{3}|^3 + 3 \times \overline{3}|^2 \times \sqrt{5} + 3 \times 3 \times 5 + 5\sqrt{5}}{8} = \frac{\overline{3+\sqrt{5}}|^3}{\overline{2}|^3}$, and consequently $\sqrt{}^3\,\overline{e+s}$ will be $= \frac{3+\sqrt{5}}{2}$. Therefore $3 \times \sqrt{}^3\,\overline{e+s}$ will be $= 3 \times \frac{3+\sqrt{5}}{2}$, and $\frac{q}{3\sqrt{}^3\,\overline{e+s}}$ will be $= \frac{3}{3 \times \frac{3+\sqrt{5}}{2}} = \frac{1}{\frac{3+\sqrt{5}}{2}} = 1 \times \frac{2}{3+\sqrt{5}} = \frac{2}{3+\sqrt{5}}$, and $\sqrt{}^3\,\overline{e+s} + \frac{q}{3 \times \sqrt{}^3\,\overline{e+s}}$ will be $= \frac{3+\sqrt{5}}{2} + \frac{2}{3+\sqrt{5}}$ $= \frac{\overline{3+\sqrt{5}} \times \overline{3+\sqrt{5}}}{2 \times \overline{3+\sqrt{5}}} + \frac{2 \times 2}{2 \times \overline{3+\sqrt{5}}} = \frac{9 + 6\sqrt{5} + 5}{2 \times \overline{3+\sqrt{5}}} + \frac{4}{2 \times \overline{3+\sqrt{5}}} = \frac{14 + 6\sqrt{5}}{2 \times \overline{3+\sqrt{5}}}$ $+ \frac{4}{2 \times \overline{3+\sqrt{5}}} = \frac{18 + 6\sqrt{5}}{2 \times \overline{3+\sqrt{5}}} = \frac{9 + 3\sqrt{5}}{3+\sqrt{5}} = \frac{3 \times \overline{3+\sqrt{5}}}{3+\sqrt{5}} = 3$. Therefore 3 is the value of y in the equation $y^3 - 3y = 18$ obtained by means of the expression $\sqrt{}^3\,\overline{e+s} + \frac{q}{3\sqrt{}^3\,\overline{e+s}}$.

Q. E. I.

18. And, upon trial, we shall find 3 to be the true value of y in the said equation $y^3 - 3y = 18$. For, if y be supposed to be $= 3$, we shall have $y^3 = 27$, and $3y = 3 \times 3 = 9$, and consequently $y^3 - 3y = 27 - 9 = 18$, or the absolute term of the proposed equation $y^3 - 3y = 18$. Therefore 3 is the true value of y in the said equation.

Q. E. D.

And thus we see that the first of the three foregoing expressions, to wit, the expression $\sqrt{}^3\,\overline{e+s} + \frac{q}{3\sqrt{3}\,\overline{e+s}}$, has given us the true value of y in the proposed equation $y^3 - 3y = 18$.

The resolution of the same equation $y^3 - 3y = 18$ by means of the second of the three foregoing expressions, to wit, the expression $\sqrt{}^3\,\overline{e-s} + \frac{q}{3\sqrt{}^3\,\overline{e-s}}$.

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19. Now, since r is $= 18$, and consequently $\frac{r}{2}$, or e, is $= 9$, and s has been shewn to be $= \sqrt{80}$, or $4\sqrt{5}$, we shall have $e - s = 9 - 4\sqrt{5} = \frac{72 - 32\sqrt{5}}{8}$ $= \frac{27 - 27\sqrt{5} + 45 - 5\sqrt{5}}{8}$, and consequently $\sqrt{}^3\,\overline{e-s} = \frac{3-\sqrt{5}}{2}$. Therefore $\frac{q}{3\sqrt{}^3\,\overline{e-s}}$ will be $= \frac{3}{3 \times \frac{3-\sqrt{5}}{2}} = \frac{1}{\frac{3-\sqrt{5}}{2}} = \frac{2}{3-\sqrt{5}}$, and $\sqrt{}^3\,\overline{e-s} + \frac{q}{3\sqrt{}^3\,\overline{e-s}}$ will be $= \frac{3-\sqrt{5}}{2} + \frac{2}{3-\sqrt{5}} = \frac{\overline{3-\sqrt{5}} \times \overline{3-\sqrt{5}}}{2 \times \overline{3-\sqrt{5}}} + \frac{2 \times 2}{2 \times \overline{3-\sqrt{5}}} = \frac{9 - 6\sqrt{5} + 6}{2 \times \overline{3-\sqrt{5}}}$

$\frac{9-6\sqrt{5}+5}{2\times\overline{3-\sqrt{5}}} + \frac{4}{2\times\overline{3-\sqrt{5}}} = \frac{14-6\sqrt{5}}{2\times\overline{3-\sqrt{5}}} + \frac{4}{2\times\overline{3-\sqrt{5}}} = \frac{18-6\sqrt{5}}{2\times\overline{3-\sqrt{5}}} = \frac{9-3\sqrt{5}}{3-\sqrt{5}} = \frac{3\times\overline{3-\sqrt{5}}}{3-\sqrt{5}} = 3$. Therefore, according to this second expression $\sqrt[3]{e-s} + \frac{q}{3\sqrt[3]{e-s}}$, the root of the cubick equation $y^3 - 3y = 18$ is 3; as it was found to be by the first expression $\sqrt[3]{e+s} + \frac{q}{3\sqrt[3]{e+s}}$.

Q. E. I.

The resolution of the same equation $y^3 - 3y = 18$ by means of the third, or last, of the three foregoing expressions, to wit, the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$.

20. Since $\sqrt[3]{e+s}$ has been shewn in art. 17 to be $= \frac{3+\sqrt{5}}{2}$, and $\sqrt[3]{e-s}$ has been shewn in art. 19 to be $= \frac{3-\sqrt{5}}{2}$, we shall have $\sqrt[3]{e+s} + \sqrt[3]{e-s} = \frac{3+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2} = \frac{6}{2} = 3$. Therefore by this expression, as well as by both the former, the value of y in the equation $y^3 - 3y = 18$ comes out to be 3.

A SCHOLIUM.

21. I have dwelt the longer on the foregoing explanation of Cardan's rule for resolving the cubick equation $y^3 - qy = r$ in its first case, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, because I have observed that it has been delivered by many writers of Algebra with an uncommon degree of obscurity*, and has been made the subject of much mysterious and fantastick reasoning, or rather discoursing, concerning negative and impossible quantities. But now, I hope, it will be found intelligible, and even easy, to every reader who is acquainted with the first rudiments of Algebra.

22. This method of resolving cubick equations of this form $y^3 - qy = r$, in the first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, or r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and a similar method of resolving cubick equations of the first form, $y^3 + qy = r$, in

* See Newton's *Arithmetica Universalis*, the 2d edition, in the year 1722, pages 279, 280, 281; and *Mac Laurin's Algebra*, Part II. chap. viii. pages 223, 224, 225, and Part I. chap. xiv. the Supplement, pages 127, 128, 129, 130; and *Clairaut's Elémens d'Algèbre*, pages 282, 283, ... 297, and *Saunderson's Algebra*, Vol. II. pages 692, 693, 694, 695, ... 707.

all

all cases of that form, or whatever may be the relative magnitudes of the co-efficient q and the absolute term r, are usually known by the name of *Cardan's Rules*, because they were first published by him in his Treatise of Algebra, intitled, *Ars magna, quam vulgò* COSSAM *vocant, seu* REGULAS ALGEBRAICAS, in A. D. 1545, and not because they were of his invention. For the rule for resolving cubick equations of the first form, $y^3 + qy = r$ was first discovered (as Cardan himself informs us) by one *Scipio Ferreus*, or *Ferreo*, of *Bononia*, or *Bologna*, in Italy, about 30 years before the publication of Cardan's treatise above mentioned; and the other rule for resolving cubick equations of the second form $y^3 - qy = r$, in the first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$ (and which has been the subject of the foregoing pages) was invented by another Italian named *Nicholas Tartaglia*, or *Tartalea*, who was Professor of Mathematicks at Venice, and distinguished himself very much by his ingenious discoveries. He was a cotemporary and a friend of Cardan, and communicated this invention to him under a strict promise that he would keep it secret; and when Cardan afterwards published it in the treatise above mentioned, Tartalea complained bitterly of his breach of promise, and would never afterwards be reconciled to him. He died in A. D. 1557. See Montucla's *Histoire des Mathematiques*, Vol. I. pages 462, and 479, 480, 481.

End of the investigation and illustration of Cardan's Rule for resolving the cubick equation $y^3 - qy = r$ in the first case of it, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

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23. I shall now proceed to convert the last of the three foregoing expressions of the value of the root y in the cubick equation $y^3 - qy = r$, to wit, the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$, into an expression involving in it an infinite series of terms; which may be done by means of Sir Isaac Newton's two theorems for finding the root of a binomial quantity, as $1 + x$, and the root of a residual quantity, as $1 - x$; both which theorems have been demonstrated in some of the preceeding discourses contained in this volume of tracts.

PROBLEM II.

24. To convert the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ (which has been shewn to be equal to the root y of the equation $y^3 - qy = r$ in the first case of that equation, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) into an expression containing an infinite converging series, by means of Sir Isaac Newton's binomial and residual theorems in the case of roots.

SOLUTION.

Since $e + s$ is $= e \times \overline{1 + \frac{s}{e}}$, and $e - s$ is $= e \times \overline{1 - \frac{s}{e}}$, it follows that $\sqrt[3]{e + s}$ will be $= \sqrt[3]{e} \times \sqrt[3]{1 + \frac{s}{e}}$, and $\sqrt[3]{e - s}$ will be $= \sqrt[3]{e} \times \sqrt[3]{1 - \frac{s}{e}}$, and consequently $\sqrt[3]{e + s} + \sqrt[3]{e - s}$ will be $= \sqrt[3]{e} \times \sqrt[3]{1 + \frac{e}{s}} + \sqrt[3]{e} \times \sqrt[3]{1 - \frac{s}{e}} = \sqrt[3]{e} \times \overline{\sqrt[3]{1 + \frac{s}{e}} + \sqrt[3]{1 - \frac{s}{e}}}$. But, by the binomial theorem in the case of roots, $\sqrt[3]{1 + \frac{s}{e}}$ is equal to the infinite series $1 + \frac{1}{3} A \frac{s}{e} - \frac{2}{6} B \frac{s^2}{e^2} + \frac{5}{9} C \frac{s^3}{e^3} - \frac{8}{12} D \frac{s^4}{e^4} + \frac{11}{15} E \frac{s^5}{e^5} - \frac{14}{18} F \frac{s^6}{e^6} + \frac{17}{21} G \frac{s^7}{e^7} - \frac{20}{24} H \frac{s^8}{e^8} + \frac{23}{27} I \frac{s^9}{e^9} - \frac{26}{30} K \frac{s^{10}}{e^{10}} + \frac{29}{33} L \frac{s^{11}}{e^{11}} - \frac{32}{36} M \frac{s^{12}}{e^{12}} + \frac{35}{39} N \frac{s^{13}}{e^{13}} - \frac{38}{42} O \frac{s^{14}}{e^{14}} + \frac{41}{45} P \frac{s^{15}}{e^{15}} - \frac{44}{48} Q \frac{s^{16}}{e^{16}} + \frac{47}{51} R \frac{s^{17}}{e^{17}} - \frac{50}{54} S \frac{s^{18}}{e^{18}}$ + &c *ad infinitum*, or (if, for the sake of brevity, we omit the several generating fractions $\frac{1}{3}, \frac{2}{6}, \frac{5}{9}, \frac{8}{12}, \frac{11}{15}, \frac{14}{18}, \frac{17}{21}$, &c) to the series $1 + \frac{Bs}{e} - \frac{Cs^2}{e^2} + \frac{Ds^3}{e^3} - \frac{Es^4}{e^4} + \frac{Fs^5}{e^5} - \frac{Gs^6}{e^6} + \frac{Hs^7}{e^7} - \frac{Is^8}{e^8} + \frac{Ks^9}{e^9} - \frac{Ls^{10}}{e^{10}} + \frac{Ms^{11}}{e^{11}} - \frac{Ns^{12}}{e^{12}} + \frac{Os^{13}}{e^{13}} - \frac{Ps^{14}}{e^{14}} + \frac{Qs^{15}}{e^{15}} - \frac{Rs^{16}}{e^{16}} + \frac{Ss^{17}}{e^{17}} - \frac{Ts^{18}}{e^{18}}$ + &c *ad infinitum*. And, by the residual theorem in the case of roots, $\sqrt[3]{1 - \frac{s}{e}}$ is equal to the series $1 - \frac{1}{3} A \frac{s}{e} - \frac{2}{6} B \frac{s^2}{e^2} - \frac{5}{9} C \frac{s^3}{e^3} - \frac{8}{12} D \frac{s^4}{e^4} - \frac{11}{15} E \frac{s^5}{e^5} - \frac{14}{18} F \frac{s^6}{e^6} - \frac{17}{21} G \frac{s^7}{e^7} - \frac{20}{24} H \frac{s^8}{e^8} - \frac{23}{27} I \frac{s^9}{e^9} - \frac{26}{30} K \frac{s^{10}}{e^{10}} - \frac{29}{33} L \frac{s^{11}}{e^{11}} - \frac{32}{36} M \frac{s^{12}}{e^{12}} - \frac{35}{39} N \frac{s^{13}}{e^{13}} - \frac{38}{42} O \frac{s^{14}}{e^{14}} - \frac{41}{45} P \frac{s^{15}}{e^{15}} - \frac{44}{48} Q \frac{s^{16}}{e^{16}} - \frac{47}{51} R \frac{s^{17}}{e^{17}} - \frac{50}{54} S \frac{s^{18}}{e^{18}}$ — &c *ad infinitum*, or to the series $1 - \frac{Bs}{e} - \frac{Cs^2}{e^2} - \frac{Ds^3}{e^3} - \frac{Es^4}{e^4} - \frac{Fs^5}{e^5} - \frac{Gs^6}{e^6} - \frac{Hs^7}{e^7} - \frac{Is^8}{e^8} - \frac{Ks^9}{e^9} - \frac{Ls^{10}}{e^{10}} - \frac{Ms^{11}}{e^{11}} - \frac{Ns^{12}}{e^{12}} - \frac{Os^{13}}{e^{13}} - \frac{Ps^{14}}{e^{14}} - \frac{Qs^{15}}{e^{15}} - \frac{Rs^{16}}{e^{16}} - \frac{Ss^{17}}{e^{17}} - \frac{Ts^{18}}{e^{18}}$ — &c *ad infinitum*, which consists of the very same terms as the series which is equal to $\sqrt[3]{1 + \frac{s}{e}}$, but with the sign — prefixed to every term after the first term 1 instead of every other term.

Now, if these two serieses (which are equal to $\sqrt[3]{1 + \frac{s}{e}}$ and $\sqrt[3]{1 - \frac{s}{e}}$) be added together their sum will be the series $2 - \frac{2Cs^2}{e^2} - \frac{2Es^4}{e^4} - \frac{2Gs^6}{e^6} - \frac{2Is^8}{e^8} - \frac{2Ls^{10}}{e^{10}} - \frac{2Ns^{12}}{e^{12}} - \frac{2Ps^{14}}{e^{14}} - \frac{2Rs^{16}}{e^{16}} - \frac{2Ts^{18}}{e^{18}}$ — &c *ad infinitum*, in which all the terms, after the first term, 2, are marked with the sign —, or subtracted from the said first term. Therefore $\sqrt[3]{1 + \frac{s}{e}} + \sqrt[3]{1 - \frac{s}{e}}$ will be equal to the said

said series $2 - \frac{2Css}{e^2} - \frac{2Es^4}{e^4} - \frac{2Gs^6}{e^6} - \frac{2Is^8}{e^8} - \frac{2Ls^{10}}{e^{10}} - \frac{2Ns^{12}}{e^{12}} - \frac{2Ps^{14}}{e^{14}} - \frac{2Rs^{16}}{e^{16}}$ $- \frac{2Ts^{18}}{e^{18}} -$ &c *ad infinitum.* Therefore $\sqrt[3]{e} \times \overline{\sqrt[3]{1 + \frac{s}{e}} + \sqrt[3]{1 - \frac{s}{e}}}$ will be $= \sqrt[3]{e} \times$ the said series $2 - \frac{2Cs^2}{e^2} - \frac{2Es^4}{e^4} - \frac{2Gs^6}{e^6} - \frac{2Is^8}{e^8} - \frac{2Ls^{10}}{e^{10}}$ $- \frac{2Ns^{12}}{e^{12}} - \frac{2Ps^{14}}{e^{14}} - \frac{2Rs^{16}}{e^{16}} - \frac{2Ts^{18}}{e^{18}} -$ &c *ad infinitum* $= 2\sqrt[3]{e} \times$ the series $1 - \frac{Cs^2}{e^2} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} -$ &c *ad infinitum.* Therefore the expression $\sqrt[3]{e + s} + \sqrt[3]{e - s}$ (which is $= \sqrt[3]{e} \times \overline{\sqrt[3]{1 + \frac{s}{e}} + \sqrt[3]{1 - \frac{s}{e}}}$) will be equal to $2\sqrt[3]{e} \times$ the series $1 - \frac{Cs^2}{e^2}$ $- \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} -$ &c *ad infinitum.*

Q. E. F.

25. It follows from the foregoing article, that the root y of the cubick equation $y^3 - qy = r$ (which has been shewn above to be equal to the expression $\sqrt[3]{e + s} + \sqrt[3]{e - s}$) will be equal to $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Cs^2}{e^2}$ $- \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} -$ &c *ad infinitum.*

26. This series will always be a converging one; because $\frac{rr}{4} - \frac{q^3}{27}$, or ss, is always less than $\frac{rr}{4}$, or ee, and consequently the fractions $\frac{ss}{ee}$, $\frac{s^4}{e^4}$, $\frac{s^6}{e^6}$, $\frac{s^8}{e^8}$, $\frac{s^{10}}{e^{10}}$, $\frac{s^{12}}{e^{12}}$, $\frac{s^{14}}{e^{14}}$, $\frac{s^{16}}{e^{16}}$, $\frac{s^{18}}{e^{18}}$, &c (which are the literal parts of the terms of the said series), will form a decreasing progression, and therefore the terms themselves, which are produced by the multiplication of the said fractions into the numeral co-efficients C, E, G, I, L, N, P, R, T, &c (which likewise are a decreasing progression), will also form a decreasing progression.

27. If we compute the several numeral co-efficients of the terms of the series $1 + \frac{Bs}{e} - \frac{Cs^2}{e^2} + \frac{Ds^3}{e^3} - \frac{Es^4}{e^4} + \frac{Fs^5}{e^5} - \frac{Gs^6}{e^6} + \frac{Hs^7}{e^7} - \frac{Is^8}{e^8} + \frac{Ks^9}{e^9} - \frac{Ls^{10}}{e^{10}} + \frac{Ms^{11}}{e^{11}} -$ $\frac{Ns^{12}}{e^{12}} + \frac{Os^{13}}{e^{13}} - \frac{Ps^{14}}{e^{14}} + \frac{Qs^{15}}{e^{15}} - \frac{Rs^{16}}{e^{16}} + \frac{Ss^{17}}{e^{17}} - \frac{Ts^{18}}{e^{18}} +$ &c *ad infinitum,* which is equal to $\sqrt[3]{1 + \frac{s}{e}}$, we shall find them to be as follows, to wit:

$A = 1$,

and B $(= \frac{1}{3} A = \frac{1}{3} \times 1) = \frac{1}{3}$,

and C $(= \frac{2}{6} B = \frac{2}{6} \times \frac{1}{3} = \frac{1}{3} \times \frac{1}{3}) = \frac{1}{9}$,

and D $(= \frac{5}{9} C = \frac{5}{9} \times \frac{1}{9}) = \frac{5}{81}$,

 and

and E $\left(= \frac{8}{12} D = \frac{8}{12} \times \frac{5}{81} = \frac{2}{3} \times \frac{5}{81}\right) = \frac{10}{243}$,

and F $\left(= \frac{11}{15} E = \frac{11}{15} \times \frac{10}{243} = \frac{11}{3 \times 5} \times \frac{2 \times 5}{243} = \frac{11 \times 2}{3 \times 243}\right) = \frac{22}{729}$,

and G $\left(= \frac{14}{18} F = \frac{14}{18} \times \frac{22}{729} = \frac{7}{9} \times \frac{22}{729}\right) = \frac{154}{6561}$,

and H $\left(= \frac{17}{21} G = \frac{17}{21} \times \frac{154}{6561} = \frac{17}{3 \times 7} \times \frac{7 \times 22}{6561} = \frac{17 \times 22}{3 \times 6561}\right) = \frac{374}{19,683}$,

and I $\left(= \frac{20}{24} H = \frac{20}{24} \times \frac{374}{19,683} = \frac{5}{6} \times \frac{374}{19,683} = \frac{5}{3 \times 2} \times \frac{187 \times 2}{19,683} = \frac{5 \times 187}{3 \times 19,683}\right)$
$= \frac{935}{59,049}$,

and K $\left(= \frac{23}{27} I = \frac{23}{27} \times \frac{935}{59,049}\right) = \frac{21,505}{1,594,323}$,

and L $\left(= \frac{26}{30} K = \frac{26}{30} \times \frac{21,505}{1,594,323} = \frac{13}{15} \times \frac{21,505}{1,594,323} = \frac{13}{3 \times 5} \times \frac{5 \times 4301}{1,594,323} = \right.$
$\left. \frac{13 \times 4301}{3 \times 1,594,323}\right) = \frac{55,913}{4,782,969}$,

and M $\left(= \frac{29}{33} L = \frac{29}{33} \times \frac{55,913}{4,782,969}\right) = \frac{147,407}{14,348,907}$,

and N $\left(= \frac{32}{36} M = \frac{32}{36} \times \frac{147,407}{14,348,907} = \frac{8}{9} \times \frac{147,407}{14,348,907}\right) = \frac{1,179,256}{129,140,163}$,

and O $\left(= \frac{35}{39} N = \frac{35}{39} \times \frac{1,179,256}{129,140,163} = \frac{35}{3 \times 13} \times \frac{13 \times 90,712}{129,140,163} = \frac{35 \times 90,712}{3 \times 129,140,163}\right)$
$= \frac{3,174,920}{387,420,489}$,

and P $\left(= \frac{38}{42} O = \frac{38}{42} \times \frac{3,174,920}{387,420,489} = \frac{19}{21} \times \frac{3,174,920}{387,420,489} = \frac{19}{3 \times 7} \times \frac{3,174,920}{387,420,489}\right.$
$\left. = \frac{19}{3 \times 7} \times \frac{7 \times 453,560}{387,420,489} = \frac{19 \times 453,560}{3 \times 387,420,489}\right) = \frac{8,617,640}{1,162,261,467}$.

and Q $\left(= \frac{41}{45} \times P = \frac{41}{45} \times \frac{8,617,640}{1,162,261,467} = \frac{41}{5 \times 9} \times \frac{8,617,640}{1,162,261,467} = \frac{41}{5 \times 9} \times \right.$
$\left. \frac{5 \times 1,723,528}{1,162,261,467} = \frac{41 \times 1,723,528}{9 \times 1,162,261,467}\right) = \frac{70,664,648}{10,460,353,203}$,

and R $\left(= \frac{44}{48} Q = \frac{44}{48} \times \frac{70,644,648}{10,460,353,203} = \frac{11}{12} \times \frac{70,644,648}{10,460,353,203} = \frac{11}{3 \times 4} \times \right.$
$\left. \frac{4 \times 17,666,162}{10,460,353,203} = \frac{11 \times 17,666,162}{3 \times 10,460,353,203}\right) = \frac{194,327,782}{31,381,059,609}$,

and S $\left(= \frac{47}{51} R = \frac{47}{51} \times \frac{194,327,782}{31,381,059,609} = \frac{47}{3 \times 17} \times \frac{17 \times 11,431,046}{31,381,059,609} = \right.$
$\left. \frac{47 \times 11,431,046}{3 \times 31,381,059,609}\right) = \frac{537,259,162}{94,143,178,827}$,

and T $\left(= \frac{50}{54} S = \frac{50}{54} \times \frac{537,259,162}{94,143,178,827} = \frac{25}{27} \times \frac{537,259,162}{94,143,178,827}\right) =$
$\frac{13,431,479,050}{2,541,865,828,329}$.

Note,

Note, these values of the co-efficients B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, and T, are expressed in the smallest possible numbers.

28. It follows from art. 25 and 27 that the root y of the equation $y^3 - qy = r$ is equal to the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{ss}{9e^2} - \frac{10s^4}{243e^4} - \frac{154s^6}{6561e^6} - \frac{935s^8}{59,049e^8} - \frac{55,913s^{10}}{4,782,969e^{10}} - \frac{1,179,256s^{12}}{129,140,163e^{12}} - \frac{8,617,640s^{14}}{1,162,261,467e^{14}} - \frac{194,327,782s^{16}}{31,381,059,609e^{16}} - \frac{13,431,479,050s^{18}}{2,541,865,828,329e^{18}}$ — &c *ad infinitum.*

29. It has been observed in art. 26 that the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}}$ — &c *ad infinitum* will always converge; from whence it follows that the expression $2\sqrt[3]{e} \times$ the said series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}}$ — &c *ad infinitum* will always truly exhibit the value of the root y of the cubick equation $y^3 - qy = r$ in the aforesaid first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$. And, when ss is considerably less than ee, or $\frac{rr}{4} - \frac{q^3}{27}$ is considerably less than $\frac{rr}{4}$, or $\frac{rr}{4}$ is very little greater than $\frac{q^3}{27}$, the convergency of the terms of this series will be sufficient to make it useful. But in other cases, or when $\frac{rr}{4}$ is much greater than $\frac{q^3}{27}$ (as, for example, when $\frac{rr}{4}$ is triple, or quadruple, or quintuple, of $\frac{q^3}{27}$, or of some still greater magnitude), the terms of this series will decrease so slowly as to render it very unfit for practice. And, indeed, in the most favourable cases, this expression of the value of the root y of the equation $y^3 - qy = r$ will be less convenient in practice than the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$, from which it is derived. But, though its merit in a practical view be but small, yet, as it is the foundation of the method, which is here intended to be explained, of extending Cardan's rule to the second case of the equation $y^3 - qy = r$, I shall now proceed to illustrate the truth of it by applying it to the resolution of a single numeral equation of the foregoing form $y^3 - qy = r$, in the first case of the said equation, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, in which I have taken care to choose such numbers for q and r as shall make $\frac{rr}{4}$ be but little greater than $\frac{q^3}{27}$, and consequently shall give us only a small quantity for the value of the fraction $\frac{ss}{ee}$, by the continual multiplication of which the terms of the above series are generated.

An

An example of the resolution of a cubick equation of the foregoing form $y^3 - qy = r$, in the first case of it, (or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) by means of the expression $2\sqrt{^3}\overline{|e} \times$ *the infinite series* $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} -$ &c *ad infinitum, or* $2\sqrt{^3}\overline{|e} \times$ *the infinite series* $1 - \frac{1}{9} \times \frac{ss}{ee} - \frac{10}{243} \times \frac{s^4}{e^4} - \frac{154}{6561} \times \frac{s^6}{e^6} - \frac{935}{59,049} \times \frac{s^8}{e^8} - \frac{55,913}{4,782,969} \times \frac{s^{10}}{e^{10}} - \frac{1,179,256}{129,140,163} \times \frac{s^{12}}{e^{12}} - \frac{8,617,640}{1,162,261,467} \times \frac{s^{14}}{e^{14}} - \frac{194,327,782}{31,381,059,609} \times \frac{s^{16}}{e^{16}} - \frac{13,431,479,050}{2,541,865,828,329} \times \frac{s^{18}}{e^{18}}$ — *&c ad infinitum.*

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30. Let it be required to resolve the equation $y^3 - 300y = 2108$ by means of this expression.

Here q is $= 300$, and r is $= 2108$. Therefore $\frac{q}{3}$ is $= 100$, and $\frac{\sqrt{q}}{\sqrt{3}}$ is $(= \sqrt{\frac{q}{3}} = \sqrt{100}) = 10$, and consequently $\frac{q\sqrt{q}}{3\sqrt{3}}$ is $(= 100 \times 10) = 1000$, and $\frac{2q\sqrt{q}}{3\sqrt{3}}$ is $(= 2 \times 1000) = 2000$, which is less than 2108, or r. Therefore this equation comes under the first case of the general equation $y^3 - qy = r$, and consequently may be resolved either by one of the three expressions obtained by Cardan's second rule above explained, or by the foregoing transcendental expression $2\sqrt{^3}\overline{|e} \times$ the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c *ad infinitum*, which was derived from the third, or last, of them.

31. Now, since r is $= 2108$, we shall have $\frac{r}{2}$, or e, $= 1054$, and $\frac{rr}{4}$, or ee, $(= \overline{1054}|^2) = 1,110,916$. And, since q is $= 300$, we shall have $\frac{q}{3} = 100$, and $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, $(= \overline{100}|^3) = 1000,000$, and consequently ss, or $\frac{rr}{4} - \frac{q^3}{27}$ $(= 1,110,916 - 1000,000) = 110,916$. Therefore the fraction $\frac{ss}{ee}$ will be $(= \frac{110,916}{1,110,916}) = 0.099,841,932,2$;

and $\frac{s^4}{e^4}$ will be $(= \overline{0.099,841,932,2}|^2) = 0.009,968,411,4$;

and $\frac{s^6}{e^6}$ will be $(= \frac{s^4}{e^4} \times \frac{ss}{ee} = 0.009,968,411,4 \times 0.099,841,932,2) = 0.000,995,265,4$;

and

and $\frac{s^8}{e^8}$ will be $(= \frac{s^6}{e^6} \times \frac{s^2}{e^2} = 0.000,995,265,4 \times 0.099,841,932,2) =$ 0.000,099,369,2;

and $\frac{s^{10}}{e^{10}}$ will be $(= \frac{s^8}{e^8} \times \frac{s^2}{e^2} = 0.000,099,369,2 \times 0.099,841,932,2) =$ 0.000,009,921,2;

and $\frac{s^{12}}{e^{12}}$ will be $(= \frac{s^{10}}{e^{10}} \times \frac{s^2}{e^2} = 0.000,009,921,2 \times 0.099,841,932,2) =$ 0.000,000,990,5;

and $\frac{s^{14}}{e^{14}}$ will be $(= \frac{s^{12}}{e^{12}} \times \frac{s^2}{e^2} = 0.000,000,990,5 \times 0.099,841,932,2) =$ 0.000,000,098,8;

and $\frac{s^{16}}{e^{16}}$ will be $(= \frac{s^{14}}{e^{14}} \times \frac{s^2}{e^2} = 0.000,000,098,8 \times 0.099,841,932,2) =$ 0.000,000,009,8;

and $\frac{s^{18}}{e^{18}}$ will be $(= \frac{s^{16}}{e^{16}} \times \frac{s^2}{e^2} = 0.000,000,009,8 \times 0.099,841,932,2) =$ 0.000,000,000,0;

And confequently $\frac{C s^2}{e^2}$ will be $(= C \times 0.099,841,932,2 = \frac{1}{9} \times 0.099,841,932,2 = \frac{0.099,841,932,2}{9}) = 0.011,093,548,0$;

and $\frac{E s^4}{e^4}$ will be $(= E \times 0.009,968,411,4 = \frac{10}{243} \times 0.009,968,411,4 = \frac{10 \times 0.009,968,411,4}{243} = \frac{0.099,684,114,0}{223}) = 0.000,410,222,6$;

and $\frac{G s^6}{e^6}$ will be $(= G \times 0.000,995,265,4 = \frac{154}{6561} \times 0.000,995,265,4 = \frac{154 \times 0.000,995,265,4}{6561} = \frac{0.153,270,871,6}{6561} (= 0.000,023,360,9$;

and $\frac{I s^8}{e^8}$ will be $(= I \times 0.000,099,369,2 = \frac{935}{59,049} \times 0.000,099,369,2 = \frac{935 \times 0.000,099,369,2}{59,049} = \frac{0.092,910.202,0}{59,049}) = 0.000,001,573,4$;

and $\frac{L s^{10}}{e^{10}}$ will be $(= L \times 0.000,009,921,2 = \frac{55,913}{4,782,969} \times 0.000,009,921,2 = \frac{55,913 \times 0.000,009,921,2}{4,782,969} = \frac{0.554,724,055,6}{4,782,969}) 0.000,000,115,9$;

and $\frac{N s^{12}}{e^{12}}$ will be $(= N \times 0.000,000,990,5 = \frac{1,179,256}{129,140,163} \times 0.000,000,990,5 = \frac{1,179,256 \times 0.000,000,990,5}{129,140,163} = \frac{1.168,053,068,0}{129,140,163}) = 0.000,000,009,0$;

and $\frac{P s^{14}}{e^{14}}$ will be $(= P \times 0.000,000,098,8 = \frac{8,617,640}{1,162,261,467} \times 0.000,000,098,8 = \frac{8,617,640 \times 0.000,000,098,8}{1,162,261,467} = \frac{0.851,422,832,0}{1,162,261,467}) = 0.000,000,000,7$;

and $\frac{R s^{16}}{e^{16}}$ will be $(= R \times 0.000,000,009,8 = \frac{194,327,782}{31,381,059,609} \times 0.000,000,009,8 = \frac{194,327,782 \times 0.000,000,009,8}{31,381,059,609} = \frac{1.904,412,263,6}{31,381,059,609}) = 0.000,000,000,0.$

Therefore

Therefore $\frac{Css}{ee} + \frac{Es^4}{e^4} + \frac{Gs^6}{e^6} + \frac{Is^8}{e^8} + \frac{Ls^{10}}{e^{10}} + \frac{Ns^{12}}{e^{12}} + \frac{Ps^{14}}{e^{14}} + \frac{Rs^{16}}{e^{16}}$ + &c will be = 0.011,093,548,0 + 0.000,410,222,6 + 0.000,023,360,9 + 0.000,001, 573,4 + 0.000,000,115,9 + 0.000,000,009,0 + 0.000,000,000,7 + 0.000, 000,000,0 = 0.011,528,830,5 + &c; and consequently the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}}$ — &c will be = 1.000,000, 000,0 — 0.011,528,830,5 &c = 0.988,471,169,5 — &c.

Further, since e is = 1054, we shall have $\sqrt[3]{e}$ (= $\sqrt[3]{1054}$) = 10.176, 853,833,7, and consequently $2\sqrt[3]{e}$ (= 2 × 10.176,853,833,7) = 20.353, 707,667,4. Therefore $2\sqrt[3]{e}$ × the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}}$ — &c *ad infinitum* will be = 20.353,707,667,4 × 0.988,471,169,5 = 20.119,053,221,6. Therefore the root of the proposed equation $y^3 - 300y = 2108$ is = 20.119,053,221,6. Q. E. I.

32. This value of y is true to nine places of figures. For its true value is somewhat greater than 20.119,053,2, as will appear by substituting 20.119, 053,2 instead of y in the compound quantity $y^3 - 300y$. For, if we suppose y to be = 20.119,053,2, we shall have y^2 = 404.776,301,664,430,24, and y^3 = 8143.715,947,285,920,546,248,768, and $300y$ (= 300 × 20.119,053,2) = 6035.715,960,0, and consequently $y^3 - 300y$ (= 8143.715,947,285,920, 546,248,768 — 6035.715,960,0) = 2107.999,987,285,920,546,248,768, which is somewhat less than the absolute term, 2108, of the proposed equation $y^3 - 300y = 2108$; and consequently 20.119,053,2 must be somewhat less than the true value of y in the said equation. And, if we were to prosecute the value of y somewhat further by means of Mr. Raphson's method of approximation, by supposing y to be = 20.119,053,2 + z, and substituting this quantity instead of y in the equation $y^3 - 300y = 2108$, and resolving the new equation that would result from such substitution, as if it was a simple equation, (to wit, by omitting all the terms that involve either the square, or cube, of z) we should find z to be equal to the fraction $\frac{0.000,012,714,079,453,751,232}{914.328,904,993,290,72}$, or to 0.000, 000,013. And consequently the first eleven figures of 20.119,053,2 + z, or of the true value of y in the proposed equation $y^3 - 300y = 2108$, would be 20.119,053,2 + 0.000,000,013, or 20.119,053,213.

33. It appears therefore from the foregoing example, that this expression $2\sqrt[3]{e}$ × the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}}$ — &c does truly exhibit the root y of the cubick equation $y^3 - qy = r$ in that case of it which falls under Cardan's second rule above explained, or in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

Of the second case of the cubick equation $y^3 - qy = r$, in which the absolute term r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$.

34. We must now proceed to consider the other case of the cubick equation $y^3 - qy = r$, in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$. This case (as we have already seen) cannot be resolved by the aforesaid rule of Cardan, because it is impossible in this case to divide the line, or root, y into two such parts v and z, that the product, or rectangle, under the said parts, to wit, the product vz, shall be equal to $\frac{q}{3}$, and consequently that $3vz$ shall be $= q$, and $3vz \times \overline{v+z}$ shall be $= q \times \overline{v+z}$, which is a fundamental step in the solution of Problem 1 given above in art 5. And on this account this case of the equation $y^3 - qy = r$ has obtained amongst Algebräists the name of *the irreducible case*; and particularly it is often so denominated by the French writers of Algebra. Monsieur Montucla in his *Histoire des Mathématiques*, Tom. I. page 482, speaks of it in these words: *On doit à Cardan la remarque de la limitation d'un cas des équations cubiques, où il arrive que l'extraction de la racine quarrée qui entre dans la formule, n'est pas possible. C'est ce que nous appellons* le cas irréductible; *dont la difficulté a donné et donne encore la torture aux Analystes.* It may, however, be resolved by means of a certain transcendental expression (or expression containing an infinite series of terms) which bears a great resemblance to the foregoing transcendental expression which we have shewn to be equal to the value of y in the first case of the equation $y^3 - qy = r$, to wit, the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Cs^2}{e^2} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}}$ — &c, and which was derived from the finite expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ by the help of Sir Isaac Newton's binomial and residual theorems. To assign such a transcendental expression, and to demonstrate that it will be equal to the root y of the cubick equation $y^3 - qy = r$ in the second case of it, or when r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, with a certain limitation which we shall mention presently, is the chief object of the remaining part of this discourse.

35. In order to preserve the two cases of the cubick equation $y^3 - qy = r$ (in the first of which the absolute term r is supposed to be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the second of which it is supposed to be less than the said quantity), distinct from each other, it will be convenient to denote the root of it in these two cases, and likewise the absolute term of the equation, by different letters. I shall therefore henceforward denote the root of this equation in the first case of it (or when the absolute term is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$), by the letter y, and in the

the ſecond caſe by the letter x, and ſhall denote the abſolute term of the equation in the firſt caſe by the letter r (as in the foregoing articles) and in the ſecond caſe by the letter t; ſo that the two caſes of the equation $y^3 - qy = r$ will now be expreſſed by the two ſeparate equations $y^3 - qy = r$ and $x^3 - qx = t$, in the former of which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the latter of which t is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the letter q being ſuppoſed to denote the ſame quantity in both equations. And I ſhall denote $\frac{r}{2}$ (as before) by the letter e, and $\frac{rr}{4} - \frac{q^3}{27}$ (as before) by the letters ss, but ſhall put the letter g for $\frac{t}{2}$, and the letters zz for $\frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$. With this notation the propoſition which I ſhall now endeavour to demonſtrate, will be as follows.

PROP. 3. A THEOREM.

36. If in the foregoing tranſcendental expreſſion, to wit, $2\sqrt[3]{e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} -$ &c *ad infinitum* (which has been ſhewn to be equal to the root y of the cubick equation $y^3 - qy = r$), we make the following changes, to wit, firſt, inſert the letter g every where inſtead of the letter e, and, ſecondly, inſert the letter z every where inſtead of the letter s, and, 3dly, change the ſign $-$ into the ſign $+$ in the ſecond, and fourth, and ſixth, and eighth, and every following even term of the infinite ſeries contained in the ſaid expreſſion, the new tranſcendental expreſſion that will be thereby obtained, to wit, the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c *ad infinitum* will be equal to the root x of the cubick equation $x^3 - qx = t$; provided that the abſolute term t of this equation (though it be leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) be greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{tt}{4}$, or gg, (though it be leſs than $\frac{q^3}{27}$) be greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$.

37. This limitation is neceſſary to the end that the ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c (which forms a part of the expreſſion of the value of the root x) may be a converging ſeries. For if t is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is leſs than $\frac{q^3}{54}$, the compound quantity $\frac{q^3}{27} - \frac{tt}{4}$ will be greater than $\frac{q^3}{27} - \frac{q^3}{54}$, or than $\frac{2q^3}{54} - \frac{q^3}{54}$, or than $\frac{q^3}{54}$, and, *à fortiori*, greater than

than $\frac{tt}{4}$; that is, zz will be greater than gg; and consequently the terms of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c will diverge, instead of converging, and the said expression $2\sqrt{}^3 g$ $\times$ the said series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}}$ $- \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c, will consequently become useless. But, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and yet greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, and yet greater than $\frac{q^3}{54}$, the terms of the aforesaid series will converge, or decrease, and the aforesaid transcendental expression will be equal to the root x of the equation $x^3 - qx = t$.

38. Now, in order to demonstrate this proposition, it will be necessary to make some further observations on the former transcendental expression $2\sqrt{}^3 \lceil e \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} -$ &c *ad infinitum*, which has been shewn to be equal to the root of the equation $y^3 - qy = r$.

Observations on the expression $2\sqrt{}^3 \lceil e \times$ *the infinite series* $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c ad infinitum, *which is equal to the root of the equation* $y^3 - qy = r$.

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39. Since the expression $2\sqrt{}^3 \lceil e \times$ the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8}$ $- \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum*, is equal to the root of the cubick equation $y^3 - qy = r$, it follows that, if we were, first, to raise the said expression to its cube by multiplying it twice into itself, and then were to multiply it into the co-efficient q, the said cube would be greater than the said product, and their difference would be equal to r, to whatever number of terms the said series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c may be continued. For, if this difference were not equal to r, it would not be true that the said expression $2\sqrt{}^3 \lceil e \times$ the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum* was equal to the root of the equation $y^3 - qy = r$. These multiplications may be performed in the manner following.

40. The

40. The cube of the expreffion $2\sqrt[3]{e} \times$ the feries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum* is equal to the product of the multiplication of the cube of $2\sqrt[3]{e}$ into the cube of the faid feries, that is, to the product of the multiplication of $8e$ into the cube of the faid feries. We muft therefore raife the faid feries to its cube by multiplying it twice into itfelf; which may be done as follows.

The multiplication of the feries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ *&c* ad infinitum *twice into itfelf in order to obtain its cube.*

$$
\begin{array}{cccccccc}
1 & -\frac{Css}{ee} & -\frac{Es^4}{e^4} & -\frac{Gs^6}{e^6} & -\frac{Is^8}{e^8} & -\frac{Ls^{10}}{e^{10}} & -\frac{Ns^{12}}{e^{12}} & -\&c \\
1 & -\frac{Css}{ee} & -\frac{Es^4}{e^4} & -\frac{Gs^6}{e^6} & -\frac{Is^8}{e^8} & -\frac{Ls^{10}}{e^{10}} & -\frac{Ns^{12}}{e^{12}} & -\&c \\
\hline
1 & -\frac{Css}{ee} & -\frac{Es^4}{e^4} & -\frac{Gs^6}{e^6} & -\frac{Is^8}{e^8} & -\frac{Ls^{10}}{e^{10}} & -\frac{Ns^{12}}{e^{12}} & -\&c \\
 & -\frac{Css}{ee} & +\frac{C^2s^4}{e^4} & +\frac{CEs^6}{e^6} & +\frac{CGs^8}{e^8} & +\frac{CIs^{10}}{e^{10}} & +\frac{CLs^{12}}{e^{12}} & +\&c \\
 & & -\frac{Es^4}{e^4} & +\frac{CEs^6}{e^6} & +\frac{E^2s^8}{e^8} & +\frac{EGs^{10}}{e^{10}} & +\frac{EIs^{12}}{e^{12}} & +\&c \\
 & & & -\frac{Gs^6}{e^6} & +\frac{CGs^8}{e^8} & +\frac{EGs^{10}}{e^{10}} & +\frac{G^2s^{12}}{e^{12}} & +\&c \\
 & & & & -\frac{Is^8}{e^8} & +\frac{CIs^{10}}{e^{10}} & +\frac{EIs^{12}}{e^{12}} & +\&c \\
 & & & & & -\frac{Ls^{10}}{e^{10}} & +\frac{CLs^{12}}{e^{12}} & +\&c \\
 & & & & & & -\frac{Ns^{12}}{e^{12}} & +\&c \\
\hline
1 & -\frac{2Css}{ee} & -\frac{2Es^4}{e^4} & -\frac{2Gs^6}{e^6} & -\frac{2Is^8}{e^8} & -\frac{2Ls^{10}}{e^{10}} & -\frac{2Ns^{12}}{e^{12}} & -\&c \\
 & & +\frac{C^2s^4}{e^4} & +\frac{2CEs^6}{e^6} & +\frac{2CGs^8}{e^8} & +\frac{2CIs^{10}}{e^{10}} & +\frac{2CLs^{12}}{e^{12}} & +\&c \\
 & & & & +\frac{E^2s^8}{e^8} & +\frac{2EGs^{10}}{e^{10}} & +\frac{2EIs^{12}}{e^{12}} & +\&c \\
 & & & & & & +\frac{G^2s^{12}}{e^{12}} & +\&c \\
\hline
1 & -\frac{Css}{ee} & -\frac{Es^4}{e^4} & -\frac{Gs^6}{e^6} & -\frac{Is^8}{e^8} & -\frac{Ls^{10}}{e^{10}} & -\frac{Ns^{12}}{e^{12}} & -\&c \\
\hline
1 & -\frac{2Css}{ee} & -\frac{2Es^4}{e^4} & -\frac{2Gs^6}{e^6} & -\frac{2Is^8}{e^8} & -\frac{2Ls^{10}}{e^{10}} & -\frac{2Ns^{12}}{e^{12}} & -\&c
\end{array}
$$

+

$$
\begin{array}{cccccccc}
 & & +\frac{C^2s^4}{e^4} & +\frac{2CEs^6}{e^6} & +\frac{2CGs^8}{e^8} & +\frac{2CIs^{10}}{e^{10}} & +\frac{2CLs^{12}}{e^{12}} & +\&c \\
 & & & & +\frac{E^2s^8}{e^8} & +\frac{2EGs^{10}}{e^{10}} & +\frac{2EIs^{12}}{e^{12}} & +\&c \\
 & & & & & & +\frac{G^2s^{12}}{e^{12}} & +\&c \\
 & -\frac{Css}{ee} & +\frac{2C^2s^4}{e^4} & +\frac{2CEs^6}{e^6} & +\frac{2CGs^8}{e^8} & +\frac{2CIs^{10}}{e^{10}} & +\frac{2CLs^{12}}{e^{12}} & +\&c \\
 & & & -\frac{C^3s^6}{e^6} & -\frac{2C^2Es^8}{e^8} & -\frac{2C^2Gs^{10}}{e^{10}} & -\frac{2C^2Is^{12}}{e^{12}} & -\&c \\
 & & & & & -\frac{CE^2s^{10}}{e^{10}} & -\frac{2CEGs^{12}}{e^{12}} & -\&c \\
 & & -\frac{Es^4}{e^4} & +\frac{2CEs^6}{e^6} & +\frac{2E^2s^8}{e^8} & +\frac{2EGs^{10}}{e^{10}} & +\frac{2EIs^{12}}{e^{12}} & +\&c \\
 & & & & -\frac{C^2Es^8}{e^8} & -\frac{2CE^2s^{10}}{e^{10}} & -\frac{2CEGs^{12}}{e^{12}} & -\&c \\
 & & & & & & -\frac{E^3s^{12}}{e^{12}} & -\&c \\
 & & & -\frac{Gs^6}{e^6} & +\frac{2CGs^8}{e^8} & +\frac{2EGs^{10}}{e^{10}} & +\frac{2G^2s^{12}}{e^{12}} & +\&c \\
 & & & & & -\frac{C^2Gs^{10}}{e^{10}} & -\frac{2CEGs^{12}}{e^{12}} & -\&c \\
 & & & & -\frac{Is^8}{e^8} & +\frac{2CIs^{10}}{e^{10}} & +\frac{2EIs^{12}}{s^{12}} & +\&c \\
 & & & & & & -\frac{C^2Is^{12}}{e^{12}} & -\&c \\
 & & & & & -\frac{Ls^{10}}{e^{10}} & +\frac{2CLs^{12}}{e^{12}} & +\&c \\
 & & & & & & -\frac{Ns^{12}}{e^{12}} & +\&c \\
\hline
1 & -\frac{3Css}{ee} & -\frac{3Es^4}{e^4} & -\frac{3Gs^6}{e^6} & -\frac{3Is^8}{e^8} & -\frac{3Ls^{10}}{e^{10}} & -\frac{3Ns^{12}}{e^{12}} & -\&c \\
 & & +\frac{3C^2s^4}{e^4} & +\frac{6CEs^6}{e^6} & +\frac{6CGs^8}{e^8} & +\frac{6CIs^{10}}{e^{10}} & +\frac{6CLs^{12}}{e^{12}} & +\&c \\
 & & & -\frac{C^3s^6}{e^6} & +\frac{3E^2s^8}{e^8} & +\frac{6EGs^{10}}{e^{10}} & +\frac{6EIs^{12}}{e^{12}} & +\&c \\
 & & & & -\frac{3C^2Es^8}{e^8} & +\frac{3C^2Gs^{10}}{e^{10}} & +\frac{3Gs^{12}}{e^{12}} & +\&c \\
 & & & & & -\frac{3CE^2s^{10}}{e^{10}} & -\frac{3C^2Is^{12}}{e^{12}} & -\&c \\
 & & & & & & -\frac{6CEGs^{12}}{e^{12}} & -\&c \\
 & & & & & & -\frac{E^3s^{12}}{e^{12}} & -\&c.
\end{array}
$$

This laſt compound series (which, for the ſake of brevity, we will denote by the Greek capital letter Γ) is the cube of the ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum*. Therefore $8e \times$ the cube of the ſeries $1 - \frac{Css}{ee}$

$\frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c *ad infinitum*, will be equal to $8e \times$ the said last compound series, or to $8e \times$ the compound series Γ; and consequently y^3, or the cube of the root y of the equation $y^3 - qy = r$, will be equal to $8e \times$ the compound series Γ.

41. If the foregoing compound series Γ be actually multiplied into $8e$, the product thence arising will be the following compound series, to wit,

$$\begin{array}{l} 8e - \frac{24Css}{e} - \frac{24Es^4}{e^3} - \frac{24Gs^6}{e^5} - \frac{24Is^8}{e^7} - \frac{24Ls^{10}}{e^9} - \frac{24Ns^{12}}{e^{11}} - \&c \\ \qquad + \frac{24C^2s^4}{e^3} + \frac{48CEs^6}{e^5} + \frac{48CGs^8}{e^7} + \frac{48CIs^{10}}{e^9} + \frac{48CLs^{12}}{e^{11}} + \&c \\ \qquad\qquad - \frac{8C^3s^6}{e^5} + \frac{24E^2s^8}{e^7} + \frac{48EGs^{10}}{e^9} + \frac{48EIs^{12}}{e^{11}} + \&c \\ \qquad\qquad\qquad - \frac{24C^2Es^8}{e^7} - \frac{24C^2Gs^{10}}{e^9} + \frac{24G^2s^{12}}{e^{11}} + \&c \\ \qquad\qquad\qquad\qquad - \frac{24CE^2s^{10}}{e^9} - \frac{24C^2Is^{12}}{e^{11}} - \&c \\ \qquad\qquad\qquad\qquad\qquad - \frac{48CEGs^{12}}{e^{11}} - \&c \\ \qquad\qquad\qquad\qquad\qquad - \frac{8E^3s^{12}}{e^{11}} - \&c, \end{array}$$

which, for the sake of brevity, we will denote by the capital letter Δ. Then will y^3 be $=$ to the compound series Δ.

42. We must now multiply the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c *ad infinitum* (which is equal to the root y of the cubick equation $y^3 - qy = r$) into the co-efficient q.

Now, since ss is $= \frac{rr}{4} - \frac{q^3}{27}$, we shall have $ss + \frac{q^3}{27} = \frac{rr}{4}$, and $\frac{q^3}{27} = \frac{rr}{4} - ss = ee - ss$, and consequently $q^3 = 27 \times \overline{ee - ss} = 27 \times ee \times \overline{1 - \frac{ss}{ee}}$, and $q = 3 \times \sqrt[3]{ee} \times \sqrt[3]{1 - \frac{ss}{ee}} = 3 \times e^{\frac{2}{3}} \times \overline{1 - \frac{ss}{ee}}\Big|^{\frac{1}{3}} =$ (by the residual theorem in the case of roots) $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} - \frac{Hs^{14}}{e^{14}} - \frac{Is^{16}}{e^{16}} - \frac{Ks^{18}}{e^{18}}$ — &c *ad infinitum*. Therefore $q \times$ the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c will be equal to $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} - \frac{Hs^{14}}{e^{14}} - \frac{Is^{16}}{e^{16}} - \frac{Ks^{18}}{e^{18}}$ — &c *ad infinitum* $\times$ the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} -$

$\frac{Ns^{12}}{e^{12}}$

$\frac{Ns^{12}}{e^{12}}$ — &c *ad infinitum* = $6e$ × the infinite series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}}$ — &c *ad infinitum* × the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c *ad infinitum*. We must therefore multiply these two serieses together; which may be done as follows.

The multiplication of the infinite series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}}$ — *&c into the infinite series* $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — *&c.*

$$
\begin{array}{rrrrrrrl}
1 & - \frac{Bss}{ee} & - \frac{Cs^4}{e^4} & - \frac{Ds^6}{e^6} & - \frac{Es^8}{e^8} & - \frac{Fs^{10}}{e^{10}} & - \frac{Gs^{12}}{e^{12}} & - \&c \\
1 & - \frac{Css}{ee} & - \frac{Es^4}{e^4} & - \frac{Gs^6}{e^6} & - \frac{Is^8}{e^8} & - \frac{Ls^{10}}{e^{10}} & - \frac{Ns^{12}}{e^{12}} & - \&c \\
\hline
1 & - \frac{Bss}{ee} & - \frac{Cs^4}{e^4} & - \frac{Ds^6}{e^6} & - \frac{Es^8}{e^8} & - \frac{Fs^{10}}{e^{10}} & - \frac{Gs^{12}}{e^{12}} & - \&c \\
 & - \frac{Css}{ee} & + \frac{BCs^4}{e^4} & + \frac{C^2s^6}{e^6} & + \frac{CDs^8}{e^8} & + \frac{CEs^{10}}{e^{10}} & + \frac{CFs^{12}}{e^{12}} & + \&c \\
 & & - \frac{Es^4}{e^4} & + \frac{BEs^6}{e^6} & + \frac{CEs^8}{e^8} & + \frac{DEs^{10}}{e^{10}} & + \frac{E^2s^{12}}{e^{12}} & + \&c \\
 & & & - \frac{Gs^6}{e^6} & + \frac{BGs^8}{e^8} & + \frac{CGs^{10}}{e^{10}} & + \frac{DGs^{12}}{e^{12}} & + \&c \\
 & & & & - \frac{Is^8}{e^8} & + \frac{BIs^{10}}{e^{10}} & + \frac{CIs^{12}}{e^{12}} & + \&c \\
 & & & & & - \frac{Ls^{10}}{e^{10}} & + \frac{BLs^{12}}{e^{12}} & + \&c \\
 & & & & & & - \frac{Ns^{12}}{e^{12}} & + \&c.
\end{array}
$$

Therefore the product of the multiplication of q into the expression $2\sqrt{^3}\,e$ × the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{10}}{e^{10}}$ — &c *ad infinitum* (which is equal to the root of the equation $y^3 - qy = r$) will be equal to $6e$ × the compound series just now obtained, to wit, the series

1 —

$$
\begin{array}{l}
1 - \frac{\text{B}ss}{ee} - \frac{\text{C}s^4}{e^4} - \frac{\text{D}s^6}{e^6} - \frac{\text{E}s^8}{e^8} - \frac{\text{F}s^{10}}{e^{10}} - \frac{\text{G}s^{12}}{e^{12}} - \&c \\
\quad - \frac{\text{C}ss}{ee} + \frac{\text{BC}s^4}{e^4} + \frac{\text{C}^2s^6}{e^6} + \frac{\text{CD}s^8}{e^8} + \frac{\text{CE}s^{10}}{e^{10}} + \frac{\text{CF}s^{12}}{e^{12}} + \&c \\
\qquad\quad - \frac{\text{E}s^4}{e^4} + \frac{\text{BE}s^6}{e^6} + \frac{\text{CE}s^8}{e^8} + \frac{\text{DE}s^{10}}{e^{10}} + \frac{\text{E}^2s^{12}}{e^{12}} + \&c \\
\qquad\qquad\qquad - \frac{\text{G}s^6}{e^6} + \frac{\text{BG}s^8}{e^8} + \frac{\text{CG}s^{10}}{e^{10}} + \frac{\text{DG}s^{12}}{e^{12}} + \&c \\
\qquad\qquad\qquad\qquad\quad - \frac{\text{I}s^8}{e^8} + \frac{\text{BI}s^{10}}{e^{10}} + \frac{\text{CI}s^{12}}{e^{12}} + \&c \\
\qquad\qquad\qquad\qquad\qquad\qquad - \frac{\text{L}s^{10}}{e^{10}} + \frac{\text{BL}s^{12}}{e^{12}} + \&c \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \frac{\text{N}s^{12}}{e^{12}} + \&c.
\end{array}
$$

Or if, for the ſake of brevity, we denote this compound ſeries by the Greek capital letter Λ, the ſaid product will be equal to $6e \times$ into the compound ſeries Λ. And conſequently the product qy (which is equal to the ſaid product) will alſo be equal to $6e \times$ the ſaid compound ſeries Λ.

43. If the foregoing compound ſeries Λ be actually multiplied into $6e$, the product thence ariſing will be the following compound ſeries, to wit,

$$
\begin{array}{l}
6e - \frac{6\text{B}ss}{e} - \frac{6\text{C}s^4}{e^3} - \frac{6\text{D}s^6}{e^5} - \frac{6\text{E}s^8}{e^7} - \frac{6\text{F}s^{10}}{e^9} - \frac{6\text{G}s^{12}}{e^{11}} - \&c \\
\quad - \frac{6\text{C}ss}{e} + \frac{6\text{BC}s^4}{e^3} + \frac{6\text{C}^2s^6}{e^5} + \frac{6\text{CD}s^8}{e^7} + \frac{6\text{CE}s^{10}}{e^9} + \frac{6\text{CF}s^{12}}{e^{11}} + \&c \\
\qquad\quad - \frac{6\text{E}s^4}{e^3} + \frac{6\text{BE}s^6}{e^5} + \frac{6\text{CE}s^8}{e^7} + \frac{6\text{DE}s^{10}}{e^9} + \frac{6\text{E}^2s^{12}}{e^{11}} + \&c \\
\qquad\qquad\qquad - \frac{6\text{G}s^6}{e^5} + \frac{6\text{BG}s^8}{e^7} + \frac{6\text{CG}s^{10}}{e^9} + \frac{6\text{DG}s^{12}}{e^{11}} + \&c \\
\qquad\qquad\qquad\qquad\quad - \frac{6\text{I}s^8}{e^7} + \frac{6\text{BI}s^{10}}{e^9} + \frac{6\text{CI}s^{12}}{e^{11}} + \&c \\
\qquad\qquad\qquad\qquad\qquad\qquad - \frac{6\text{CL}s^{10}}{e^9} + \frac{6\text{BL}s^{12}}{e^{11}} + \&c \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \frac{6\text{N}s^{12}}{e^{11}} + \&c
\end{array}
$$

which for the ſake of brevity, we will denote by the Greek capital letter Π. Then will qy be equal to the compound ſeries Π.

44. Since the compound ſeries Δ, obtained in art. 41, is equal to y^3, and the compound ſeries Π, obtained in the laſt foregoing article, 43, is equal to the product qy (which is leſs than y^3 by the difference r, which is the abſolute term of the equation $y^3 - qy = r$) it follows that the compound ſeries Π will be leſs than the compound ſeries Δ, and that their difference will be equal to the difference of y^3 and qy, or to the abſolute term r of the equation $y^3 - qy = r$; that is $\Delta - \Pi$ will be $= r$. But the exceſs of the firſt term of the compound ſeries Δ above the firſt term of the compound ſeries Π is alſo equal to r; the firſt term of the compound ſeries Δ being $8e$, and the firſt term of the compound ſeries

ſeries Π being $6e$, and their difference conſequently being equal to $8e - 6e$, or $2e$, or $2 \times \frac{r}{2}$, or r. Therefore the exceſs of the firſt term of the compound ſeries Δ above the firſt term of the compound ſeries Π is equal to the exceſs of the whole compound ſeries Δ above the whole compound ſeries Π. Therefore the difference between the whole compound ſeries Δ and its firſt term $8e$ will be equal to the difference between the whole compound ſeries Π and its firſt term $6e$; that is (becauſe each of theſe compound ſerieſes Δ and Π is leſs than its own firſt term, the ſecond terms in both ſerieſes, to wit, the term $-\frac{24Css}{e}$ and the term $-\frac{6Bss}{e} - \frac{6Css}{e}$, being marked with the ſign —, or ſubtracted from the firſt terms $8e$ and $6e$) the exceſs of the firſt term, $8e$, of the compound ſeries Δ above the whole of the ſaid compound ſeries will be equal to the exceſs of the firſt term, $6e$, of the compound ſeries Π above the whole of the ſaid compound ſeries. But the exceſs of the firſt term, $8e$, of the compound ſeries Δ above the whole of the ſaid ſeries is equal to a compound ſeries conſiſting of all the terms of the ſaid compound ſeries Δ, except its firſt term $8e$, with the ſigns of the terms every where changed into their contraries, that is, to the following compound ſeries, to wit,

$$\begin{array}{llllllll}
\frac{24Css}{e} & + \frac{24Es^4}{e^3} & + \frac{24Gs^6}{e^5} & + \frac{24Is^8}{e^7} & + \frac{24Ls^{10}}{e^9} & + \frac{24Ns^{12}}{e^{11}} & + \&c \\
 & - \frac{24C^2s^4}{e^3} & - \frac{48CEs^6}{e^5} & - \frac{48CGs^8}{e^7} & - \frac{48CIs^{10}}{e^9} & - \frac{48CLs^{12}}{e^{11}} & - \&c \\
 & & + \frac{8C^3s^6}{e^5} & - \frac{24E^2s^8}{e^7} & - \frac{48EGs^{10}}{e^9} & - \frac{48EIs^{12}}{e^{11}} & - \&c \\
 & & & + \frac{24C^2Es^8}{e^7} & + \frac{24C^2Gs^{10}}{e^9} & - \frac{24G^2s^{12}}{e^{11}} & - \&c \\
 & & & & + \frac{24CE^2s^{10}}{e^9} & + \frac{24C^2Is^{12}}{e^{11}} & + \&c \\
 & & & & & + \frac{48CEGs^{12}}{e^{11}} & + \&c \\
 & & & & & + \frac{8E^3s^{12}}{e^{11}} & + \&c;
\end{array}$$

and the exceſs of the firſt term, $6e$, of the compound ſeries Π above the whole of the ſaid ſeries is equal to a compound ſeries conſiſting of all the terms of the ſaid compound ſeries Π, except its firſt term $6e$, with the ſigns of the terms every where changed into their contraries, that is, to the following compound ſeries, to wit,

$$\begin{array}{rcccccccccccl}
 & \frac{6\text{B}ss}{e} & + & \frac{6\text{C}s^4}{e^3} & + & \frac{6\text{D}s^6}{e^5} & + & \frac{6\text{E}s^8}{e^7} & + & \frac{6\text{F}s^{10}}{e^9} & + & \frac{6\text{G}s^{12}}{e^{11}} & + \text{\&c} \\
+ & \frac{6\text{C}ss}{e} & - & \frac{6\text{BC}s^4}{e^3} & - & \frac{6\text{C}^2s^6}{e^5} & - & \frac{6\text{CD}s^8}{e^7} & - & \frac{6\text{CE}s^{10}}{e^9} & - & \frac{6\text{CF}s^{12}}{e^{11}} & - \text{\&c} \\
 & & + & \frac{6\text{E}s^4}{e^3} & - & \frac{6\text{BE}s^6}{e^5} & - & \frac{6\text{CE}s^8}{e^7} & - & \frac{6\text{DE}s^{10}}{e^9} & - & \frac{6\text{E}^2s^{12}}{e^{11}} & - \text{\&c} \\
 & & & & + & \frac{6\text{G}s^6}{e^5} & - & \frac{6\text{EG}s^8}{e^7} & - & \frac{6\text{CG}s^{10}}{e^9} & - & \frac{6\text{DG}s^{12}}{e^{11}} & - \text{\&c} \\
 & & & & & & + & \frac{6\text{I}s^8}{e^7} & - & \frac{6\text{BI}s^{10}}{e^9} & - & \frac{6\text{CI}s^{12}}{e^{11}} & - \text{\&c} \\
 & & & & & & & & + & \frac{6\text{L}s^{10}}{e^9} & - & \frac{6\text{BL}s^{12}}{e^{11}} & - \text{\&c} \\
 & & & & & & & & & & + & \frac{6\text{N}s^{12}}{e^{11}} & - \text{\&c.}
\end{array}$$

Therefore, if, for the ſake of brevity, we call the former of theſe two laſt-mentioned compound ſerieſes $8e - \Delta$, and the latter $6e - \Pi$, we ſhall have the compound ſeries $8e - \Delta =$ the compound ſeries $6e - \Pi$.

Of the equality between each ſeparate term of the compound ſeries $8e - \Delta$, *and the correſponding term of the compound ſeries* $6e - \Pi$.

45. In the foregoing article it has been ſhewn that the whole compound ſeries $8e - \Delta$ is equal to the whole compound ſeries $6e - \Pi$. But it is alſo true that each ſeparate term of the former compound ſeries (reckoning all the quantities in it that involve the ſame power of ss as one term) will be equal to the correſponding term of the latter compound ſeries. Of this equality we will firſt give ſome examples in ſome of the firſt terms of theſe two compound ſerieſes, and then will give a general proof that the ſame equality muſt alſo take place in all the following terms of the ſaid ſerieſes, to whatever number of terms they may be continued.

Examples of the ſaid equality in the firſt ſix terms of the ſaid ſerieſes.

46. In the firſt place, then, the firſt term of the compound ſeries $8e - \Delta$ is $\frac{24\text{C}ss}{e}$, or (becauſe C is $= \frac{1}{9}$), $24 \times \frac{1}{9} \times \frac{ss}{e} = 8 \times 3 \times \frac{1}{9} \times \frac{ss}{e} = \frac{8ss}{3e}$. And the firſt term of the compound ſeries $6e - \Pi$ is $\frac{6\text{B}ss}{e} + \frac{6\text{C}ss}{e} = 6 \times \frac{1}{3} \times \frac{ss}{e}$

$\frac{ss}{e} + 6 \times \frac{1}{9} \times \frac{ss}{e} = \frac{6ss}{3e} + \frac{2ss}{3e} = \frac{8ss}{3e}$. Therefore $\frac{6\text{B}ss}{e} + \frac{6\text{C}ss}{e}$ are equal to $\frac{24\text{C}ss}{e}$.

Q. E. D.

47. Secondly, the next term of the compound ſeries $8e - \Delta$ is $\frac{24\text{E}s^4}{e^3} - \frac{24\text{C}^2s^4}{e^3}$, which is $= 24 \times \frac{10}{243} \times \frac{s^4}{e^3} - 24 \times \frac{1}{9} \times \frac{1}{9} \times \frac{s^4}{e^3} = 8 \times 3 \times \frac{10}{81 \times 3} \times \frac{s^4}{e^3} - 24 \times \frac{1}{9} \times \frac{1}{9} \times \frac{s^4}{e^3} = \frac{80s^4}{81e^3} - \frac{24s^4}{81e^3} = \frac{56s^4}{81}$; and the next term of the compound ſeries $6e - \Pi$ is $\frac{6\text{C}s^4}{e^3} - \frac{6\text{BC}s^4}{e^3} + \frac{6\text{E}s^4}{e^3}$, which is $= 6 \times \frac{1}{9} \times \frac{s^4}{e^3} - 6 \times \frac{1}{3} \times \frac{1}{9} \times \frac{s^4}{e^3} + 6 \times \frac{10}{243} \times \frac{s^4}{e^3} = \frac{6 \times 9}{9 \times 9} \times \frac{s^4}{e^3} - \frac{6 \times 3}{9 \times 9} \times \frac{s^4}{e^3} + \frac{2 \times 10}{81} \times \frac{s^4}{e^3} = \frac{54s^4}{81e^3} - \frac{18s^4}{81e^3} + \frac{20}{81} \times \frac{s^4}{e^3} = \frac{74s^4}{e^3} - \frac{18s^4}{e^3} = \frac{56s^4}{e^3}$. Therefore $\frac{6\text{C}s^4}{e^3} - \frac{6\text{BC}s^4}{e^3} + \frac{6\text{E}s^4}{e^3}$, or the ſecond term of the compound ſeries $6e - \Pi$, is equal to $\frac{24\text{E}s^4}{e^3} - \frac{24\text{C}^2s^4}{e^3}$, or the ſecond term of the compound ſeries $8e - \Delta$.

Q. E. D.

48. The third term of the compound ſeries $8e - \Delta$ is $\frac{24\text{G}s^6}{e^5} - \frac{48\text{CE}s^6}{e^5} + \frac{8\text{C}^3s^6}{e^5}$, which is $= 24 \times \frac{154}{6561} \times \frac{s^6}{e^5} - 48 \times \frac{1}{9} \times \frac{10}{243} \times \frac{s^6}{e^5} + 8 \times \frac{1}{9} \times \frac{1}{9} \times \frac{1}{9} \times \frac{s^6}{e^5} = \frac{8 \times 154}{2187} \times \frac{s^6}{e^5} - \frac{480}{2187} \times \frac{s^6}{e^5} + \frac{24}{2187} \times \frac{s^6}{e^5} = \frac{1232}{2187} \times \frac{s^6}{e^5} - \frac{480}{2187} \times \frac{s^6}{e^5} + \frac{24}{2187} \times \frac{s^6}{e^5} = \frac{1256}{2187} \times \frac{s^6}{e^5} - \frac{480}{2187} \times \frac{s^6}{e^5} = \frac{776s^6}{2187e^5}$; and the third term of the compound ſeries $6e - \Pi$ is $\frac{6\text{D}s^6}{e^5} - \frac{6\text{C}^2s^6}{e^5} - \frac{6\text{BE}s^6}{e^5} + \frac{6\text{G}s^6}{e^5}$, which is $= 6 \times \frac{5}{81} \times \frac{s^6}{e^5} - 6 \times \frac{1}{81} \times \frac{s^6}{e^5} - 6 \times \frac{1}{3} \times \frac{10}{243} \times \frac{s^6}{e^5} + 6 \times \frac{154}{6561} \times \frac{s^6}{e^5} = \frac{6 \times 5 \times 27}{2187} \times \frac{s^6}{e^5} - \frac{6 \times 27}{2187} \times \frac{s^6}{e^5} - \frac{6 \times 10 \times 3}{2187} \times \frac{s^6}{e^5} + \frac{2 \times 154}{2187} \times \frac{s^6}{e^5} = \frac{810}{2187} \times \frac{s^6}{e^5} - \frac{162}{2187} \times \frac{s^6}{e^5} - \frac{180}{2187} \times \frac{s^6}{e^5} + \frac{308}{2187} \times \frac{s^6}{e^5} = \frac{1118}{2187} \times \frac{s^6}{e^5} - \frac{342}{2187} \times \frac{s^6}{e^5} = \frac{776s^6}{e^5}$. Therefore $\frac{6\text{D}s^6}{e^5} - \frac{6\text{C}^2s^6}{e^5} - \frac{6\text{BE}s^6}{e^5} + \frac{6\text{G}s^6}{e^5}$, or the third term of the compound ſeries $6e - \Pi$, is equal to $\frac{24\text{G}s^6}{e^5} - \frac{48\text{CE}s^6}{e^5} + \frac{8\text{C}^3s^6}{e^5}$, or the third term of the compound ſeries $8e - \Delta$.

Q. E. D.

49. The fourth term of the compound ſeries $8e - \Delta$ is $\frac{24\text{I}s^8}{e^7} - \frac{48\text{CG}s^8}{e^7} - \frac{24\text{E}^2s^8}{e^7} + \frac{24\text{C}^2\text{E}s^8}{e^7}$, which is $= 24 \times \frac{935}{59049} \times \frac{s^8}{e^7} - 48 \times \frac{1}{9} \times \frac{154}{6561} \times \frac{s^8}{e^7} - 24 \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^8}{e^7} + 24 \times \frac{1}{81} \times \frac{10}{243} \times \frac{s^8}{e^7} = \frac{8 \times 935}{19683} \times \frac{s^8}{e^7} - \frac{16 \times 154}{3 \times 6561} \times \frac{s^8}{e^7}$

$\frac{s^8}{e^7} - \frac{8 \times 10 \times 10}{81 \times 243} \times \frac{s^8}{e^7} + \frac{24 \times 10}{81 \times 243} \times \frac{s^8}{e^7} = \frac{7480}{19683} \times \frac{s^8}{e^7} - \frac{2464}{19683} \times \frac{s^8}{e^7} - \frac{800}{19683} \times \frac{s^8}{e^7} + \frac{240}{19683} \times \frac{s^8}{e^7} = \frac{7720}{19683} \times \frac{s^8}{e^7} - \frac{3264}{19683} \times \frac{s^8}{e^7} = \frac{4456 s^8}{19683 e^7}$; and the fourth term of the compound series $6e - \Pi$ is $\frac{6 \mathrm{E} s^8}{e^7} - \frac{6 \mathrm{CD} s^8}{e^7} - \frac{6 \mathrm{CE} s^8}{e^7} - \frac{6 \mathrm{BG} s^8}{e^7} + \frac{6 \mathrm{I} s^8}{e^7}$, which is $= 6 \times \frac{10}{243} \times \frac{s^8}{e^7} - 6 \times \frac{1}{9} \times \frac{5}{81} \times \frac{s^8}{e^7} - 6 \times \frac{1}{9} \times \frac{10}{243} \times \frac{s^8}{e^7} - 6 \times \frac{1}{3} \times \frac{154}{6561} \times \frac{s^8}{e^7} + 6 \times \frac{935}{59{,}049} \times \frac{s^8}{e^7} = \frac{6 \times 81 \times 10}{81 \times 243} \times \frac{s^8}{e^7} - \frac{6 \times 27 \times 5}{9 \times 27 \times 81} \times \frac{s^8}{e^7} - \frac{6 \times 9 \times 10}{81 \times 243} \times \frac{s^8}{e^7} - \frac{6 \times 154}{3 \times 6561} \times \frac{s^8}{e^7} + \frac{2 \times 935}{19683} \times \frac{s^8}{e^7} = \frac{4860}{19683} \times \frac{s^8}{e^7} - \frac{810}{19683} \times \frac{s^8}{e^7} - \frac{540}{19683} \times \frac{s^8}{e^7} - \frac{924}{19683} \times \frac{s^8}{e^7} + \frac{1870}{19683} \times \frac{s^8}{e^7} = \frac{6730}{19683} \times \frac{s^8}{e^7} - \frac{2274}{19683} \times \frac{s^8}{e^7} = \frac{4456}{19683} \times \frac{s^8}{e^7}$. Therefore $\frac{6 \mathrm{E} s^8}{e^7} - \frac{6 \mathrm{CD} s^8}{e^7} - \frac{6 \mathrm{CE} s^8}{e^7} - \frac{6 \mathrm{BG} s^8}{e^7} + \frac{6 \mathrm{I} s^8}{e^7}$ is $= \frac{24 \mathrm{I} s^8}{e^7} - \frac{48 \mathrm{CG} s^8}{e^7} - \frac{24 \mathrm{E}^2 s^8}{e^7} + \frac{24 \mathrm{C}^2 \mathrm{E} s^8}{e^7}$.

Q. E. D.

50. The fifth term of the compound series $8e - \Delta$ is $\frac{24 \mathrm{L} s^{10}}{e^9} - \frac{48 \mathrm{CI} s^{10}}{e^9} - \frac{48 \mathrm{BG} s^{10}}{e^9} + \frac{24 \mathrm{C}^2 \mathrm{G} s^{10}}{e^9} + \frac{24 \mathrm{CE}^2 s^{10}}{e^9}$, which is $= 24 \times \frac{55{,}913}{4{,}782{,}969} \times \frac{s^{10}}{e^9} - 48 \times \frac{1}{9} \times \frac{935}{59{,}049} \times \frac{s^{10}}{e^9} - 48 \times \frac{10}{243} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} + 24 \times \frac{1}{81} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} + 24 \times \frac{1}{9} \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{10}}{e^9} = \frac{8 \times 55{,}913}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - 48 \times \frac{3}{27} \times \frac{935}{59{,}049} \times \frac{s^{10}}{e^9} - \frac{48 \times 10 \times 154}{1{,}594{,}323} \times \frac{s^{10}}{e^9} + 24 \times \frac{3}{243} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} + 24 \times \frac{3}{27} \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{10}}{e^9} = \frac{447{,}304}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{134{,}640}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{73{,}920}{1{,}594{,}323} \times \frac{s^{10}}{e^9} + \frac{11{,}088}{1{,}594{,}323} \times \frac{s^{10}}{e^9} + \frac{7200}{1{,}594{,}323} \times \frac{s^{10}}{e^9} = \frac{465{,}592}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{208{,}560}{1{,}594{,}323} \times \frac{s^{10}}{e^9} = \frac{257{,}032}{1{,}594{,}323} \times \frac{s^{10}}{e^9}$; and the fifth term of the compound series $6e - \Pi$ is $\frac{6 \mathrm{F} s^{10}}{e^9} - \frac{6 \mathrm{CE} s^{10}}{e^9} - \frac{6 \mathrm{DE} s^{10}}{e^9} - \frac{6 \mathrm{CG} s^{10}}{e^9} - \frac{6 \mathrm{BI} s^{10}}{e^9} + \frac{6 \mathrm{L} s^{10}}{e^9}$, which is $= 6 \times \frac{22}{729} \times \frac{s^{10}}{e^9} - 6 \times \frac{1}{9} \times \frac{10}{243} \times \frac{s^{10}}{e^9} - 6 \times \frac{5}{81} \times \frac{10}{243} \times \frac{s^{10}}{e^9} - 6 \times \frac{1}{9} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} - 6 \times \frac{1}{3} \times \frac{935}{59{,}049} \times \frac{s^{10}}{e^9} + \frac{6 \times 55{,}913}{4{,}782{,}969} \times \frac{s^{10}}{e^9} = \frac{6 \times 22 \times 2187}{729 \times 2187} \times \frac{s^{10}}{e^9} - \frac{6 \times 10 \times 729}{9 \times 243 \times 729} \times \frac{s^{10}}{e^9} - \frac{6 \times 5 \times 10 \times 81}{81 \times 81 \times 243} \times \frac{s^{10}}{e^9} - \frac{6 \times 27 \times 154}{9 \times 27 \times 6561} \times \frac{s^{10}}{e^9} - \frac{6 \times 9 \times 935}{3 \times 9 \times 59{,}049} \times \frac{s^{10}}{e^9} + \frac{2 \times 55{,}913}{1{,}594{,}323} \times \frac{s^{10}}{e^9} = \frac{288{,}684}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{43{,}740}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{24{,}300}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{24{,}948}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{50{,}490}{1{,}594{,}323} \times \frac{s^{10}}{e^9} + \frac{111{,}826}{1{,}594{,}323} \times \frac{s^{10}}{e^9} = \frac{400{,}510}{1{,}594{,}323} \times \frac{s^{10}}{e^9} - \frac{143{,}478}{1{,}594{,}323} \times \frac{s^{10}}{e^9} = \frac{257{,}032}{1{,}594{,}323} \times \frac{s^{10}}{e^9}$. Therefore $\frac{6 \mathrm{F} s^{10}}{e^9} - \frac{6 \mathrm{CE} s^{10}}{e^9} - \frac{6 \mathrm{DE} s^{10}}{e^9} - \frac{6 \mathrm{CG} s^{10}}{e^9} - \frac{6 \mathrm{BI} s^{10}}{e^9} + \frac{6 \mathrm{L} s^{10}}{e^9}$,

$\frac{6Ls^{10}}{e^{9}}$, or the fifth term of the compound ſeries $6e - \Pi$, is equal to $\frac{24Ls^{10}}{e^{9}} - \frac{48CIs^{10}}{e^{9}} - \frac{48EGs^{10}}{e^{9}} + \frac{24C^{2}Gs^{10}}{e^{9}} + \frac{24CE^{2}s^{10}}{e^{9}}$, or the fifth term of the compound ſeries $8e - \Delta$. Q. E. D.

51. Laſtly, the ſixth term of the compound ſeries $8e - \Delta$ is $\frac{24Ns^{12}}{e^{11}} - \frac{48CLs^{12}}{e^{11}} - \frac{48EIs^{12}}{e^{11}} - \frac{24G^{2}s^{12}}{e^{11}} + \frac{24C^{2}Is^{12}}{e^{11}} + \frac{48CEGs^{12}}{e^{11}} + \frac{8E^{3}s^{12}}{e^{11}}$, which is

$$= 24 \times \frac{1,179,256}{129,140,163} \times \frac{s^{12}}{e^{11}} - 48 \times \frac{1}{9} \times \frac{55,913}{4,782,969} \times \frac{s^{12}}{e^{11}} - 48 \times \frac{10}{243} \times \frac{935}{59,049} \times \frac{s^{12}}{e^{11}} - 24 \times \frac{154}{6561} \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} + 24 \times \frac{1}{81} \times \frac{935}{59,049} \times \frac{s^{12}}{e^{11}} + 48 \times \frac{1}{9} \times \frac{10}{243} \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} + 8 \times \frac{10}{243} \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{12}}{e^{11}}$$

$$= \frac{8 \times 1,179,256}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{48 \times 55,913}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{48 \times 10 \times 3 \times 935}{243 \times 3 \times 59,049} \times \frac{s^{12}}{e^{11}} - \frac{24 \times 154 \times 154}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{24 \times 9 \times 935}{81 \times 9 \times 59.049} \times \frac{s^{12}}{e^{11}} + \frac{48 \times 3 \times 10 \times 154}{9 \times 3 \times 243 \times 6561} \times \frac{s^{12}}{e^{11}} + \frac{8 \times 3 \times 1000}{3 \times 243 \times 243 \times 243} \times \frac{s^{12}}{e^{11}}$$

$$= \frac{9,434,048}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{2,683,824}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{1,346,400}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{569,184}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{201,960}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{221,760}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{24000}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{9,881,768}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{4,599,408}{43,046,721} \times \frac{s^{12}}{e^{11}}$$

$= \frac{5,282,360}{43,046,721} \times \frac{s^{12}}{e^{11}}$; and the ſixth term of the compound ſeries $6e - \Pi$ is $\frac{6Gs^{12}}{e^{11}} - \frac{6CFs^{12}}{e^{11}} - \frac{6E^{2}s^{12}}{e^{11}} - \frac{6DGs^{12}}{e^{11}} - \frac{6CIs^{12}}{e^{11}} - \frac{6BLs^{12}}{e^{11}} + \frac{6Ns^{12}}{e^{11}}$, which is

$$= 6 \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{1}{9} \times \frac{22}{729} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{5}{81} \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{1}{9} \times \frac{935}{59,049} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{1}{3} \times \frac{55,913}{4,782,969} \times \frac{s^{12}}{e^{11}} + \frac{6 \times 1,179,256}{129,140,163} \times \frac{s^{12}}{e^{11}}$$

$$= \frac{6 \times 154 \times 6561}{6561 \times 6561} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 22 \times 6561}{9 \times 729 \times 6561} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 10 \times 10 \times 729}{243 \times 243 \times 729} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 5 \times 154 \times 81}{81 \times 81 \times 6561} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 81 \times 935}{9 \times 81 \times 59,049} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 3 \times 55,913}{3 \times 3 \times 4,782,969} \times \frac{s^{12}}{e^{11}} + \frac{2 \times 1,179,256}{43,046,721} \times \frac{s^{12}}{e^{11}}$$

$$= \frac{6,062,364}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{866,052}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{437,400}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{374,220}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{454,410}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{1,006,434}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{2,358,512}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{8,420,876}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{3,138,516}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{5,282,360}{43,046,721} \times \frac{s^{12}}{e^{11}}.$$

Therefore $\frac{6Gs^{12}}{e^{11}} - \frac{6CFs^{12}}{e^{11}} - \frac{6E^{2}s^{12}}{e^{11}} - \frac{6DGs^{12}}{e^{11}} - \frac{6CIs^{12}}{e^{11}} - \frac{6BLs^{12}}{e^{11}} + \frac{6Ns^{12}}{e^{11}}$, or the ſixth term of the compound ſeries $6e - \Pi$, is equal to $\frac{24Ns^{12}}{e^{11}} - \frac{48CLs^{12}}{e^{11}} - \frac{48EIs^{12}}{e^{11}} - \frac{24G^{2}s^{12}}{e^{11}} + \frac{24C^{2}Is^{12}}{e^{11}} + \frac{48CEGs^{12}}{e^{11}} + \frac{8E^{3}s^{12}}{e^{11}}$, or the ſixth term of the compound ſeries $8e - \Delta$. Q. E. D.

A ge-

A general demonſtration of the equality between the correſponding terms of the two compound ſerieſes $8e - \Delta$ *and* $6e - \Pi$.

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52. We have ſeen in the ſix preceeding articles that each of the firſt ſix compound terms (or vertical columns of ſimple terms involving the ſame powers of s and e) in the compound ſeries $6e - \Pi$ is equal to the correſponding compound term, or vertical column of ſimple terms, in the compound ſeries $8e - \Delta$. We now proceed to ſhew that the ſame equality muſt of neceſſity take place between all the correſpondent terms of the ſaid two ſerieſes, as well as between the firſt ſix terms of them, to whatever number of terms the ſaid ſerieſes may be continued.

53. Now this equality of the correſpondent terms of theſe two ſerieſes will appear from this conſideration, to wit, that the compound ſeries $6e - \Pi$ is conſtantly equal to the compound ſeries $8e - \Delta$ in all the different values of ss and ee that are poſſible, that is, when ee, or $\frac{rr}{4}$, is of any magnitude greater than $\frac{q^3}{27}$ (which is its leaſt poſſible magnitude) and conſequently when ss, or $\frac{rr}{4} - \frac{q^3}{27}$, is of any magnitude greater than $\frac{q^3}{27} - \frac{q^3}{27}$, or o, or, in other words, of any magnitude, how ſmall ſoever. For from hence it may be ſhewn of each of the compound terms of the compound ſeries $6e - \Pi$ ſucceſſively, beginning with the firſt term, that it is equal to the correſponding term of the compound ſeries $8e - \Delta$. This may be done in the manner following.

54. In the firſt place, ſince the whole compound ſeries $6e - \Pi$ is equal to the whole compound ſeries $8e - \Delta$ (as has been ſhewn in art. 44) it follows that, if we divide all the terms of both ſerieſes by the fraction $\frac{ss}{e}$, the quotients of theſe diviſions will be equal to each other, that is, the compound ſeries

$$\begin{array}{rrrrr}
6B & + \frac{6Css}{ee} & + \frac{6Ds^4}{e^4} & + \frac{6Es^6}{e^6} & + \&c \\
+ 6C & - \frac{6BCss}{ee} & - \frac{6C^2s^4}{e^4} & - \frac{6CDs^6}{e^6} & - \&c \\
 & + \frac{6Ess}{ee} & - \frac{6BEs^4}{e^4} & - \frac{6CEs^6}{e^6} & - \&c \\
 & & + \frac{6Gs^4}{e^4} & - \frac{6BGs^6}{e^6} & - \&c \\
 & & & + \frac{6Is^6}{e^6} & - \&c
\end{array}$$

will be equal to the compound ſeries

24 C +

$$\begin{array}{l} 24\,\text{C} + \frac{24\,\text{B}ss}{ee} + \frac{24\,\text{G}s^4}{e^4} + \frac{24\,\text{I}s^6}{e^6} + \&c \\ \quad - \frac{24\,\text{C}^2ss}{ee} - \frac{48\,\text{CE}s^4}{e^4} - \frac{48\,\text{CG}s^6}{e^6} - \&c \\ \qquad\qquad + \frac{8\,\text{C}^3s^4}{e^4} - \frac{24\,\text{E}^2s^6}{e^6} - \&c \\ \qquad\qquad\qquad + \frac{24\,\text{C}^2\text{E}s^6}{e^6} + \&c. \end{array}$$

And this will be true, of how ſmall a magnitude ſoever we may ſuppoſe ss to be taken. And conſequently it will be true likewiſe, when ss is equal to o, or when $\frac{rr}{4} - \frac{q^3}{27}$ is equal to o, or when $\frac{rr}{4}$ is equal to $\frac{q^3}{27}$. But, when ss is $=$ o, all the terms of theſe two compound ſerieſes that involve any power of ss, that is, all the terms, except the firſt terms, of the ſaid ſerieſes, will likewiſe be equal to o, and the ſaid two ſerieſes will become equal to their firſt terms 6 B + 6 C and 24 C reſpectively. Therefore the ſaid firſt terms muſt be equal to each other, that is, 6 B + 6 C, the firſt term of the compound ſeries $\frac{6e - \Pi}{\frac{ss}{e}}$, will be equal to 24 C, the firſt term of the compound ſeries $\frac{8e - \Delta}{\frac{ss}{e}}$. Therefore, if we multiply both theſe firſt terms into $\frac{ss}{e}$, it will follow that $\frac{6\,\text{B}ss}{e} + \frac{6\,\text{C}ss}{e}$, or the firſt term of the compound ſeries $6e - \Pi$, will be equal to $\frac{24\,\text{C}ss}{e}$, or the firſt term of the compound ſeries $8e - \Delta$.

Q. E. D.

55. Secondly, ſince it has been ſhewn in the laſt article that $\frac{6\,\text{B}ss}{e} + \frac{6\,\text{C}ss}{e}$, or the firſt term of the compound ſeries $6e - \Pi$, is equal to $\frac{24ss}{e}$, or the firſt term of the compound ſeries $8e - \Delta$, it follows that, if we ſubtract theſe firſt terms of theſe two ſerieſes from the whole ſerieſes, the remainders will be equal to each other, that is, the compound ſeries

$$\begin{array}{l} \quad\;\; \frac{6\,\text{C}s^4}{e^3} + \frac{6\,\text{D}s^6}{e^5} + \frac{6\,\text{E}s^8}{e^7} + \&c \\ - \frac{6\,\text{BC}s^4}{e^3} - \frac{6\,\text{C}^2s^6}{e^5} - \frac{6\,\text{CD}s^8}{e^7} - \&c \\ + \frac{6\,\text{E}s^4}{e^3} - \frac{6\,\text{BE}s^6}{e^5} - \frac{6\,\text{CE}s^8}{e^7} - \&c \\ \qquad\quad + \frac{6\,\text{G}s^6}{e^5} - \frac{6\,\text{BG}s^8}{e^7} - \&c \\ \qquad\qquad\qquad + \frac{6\,\text{I}s^8}{e^7} - \&c \end{array}$$

will be equal to the compound ſeries

$$\begin{array}{l} \frac{24 E s^4}{e^3} + \frac{24 G s^6}{e^5} + \frac{24 I s^8}{e^7} + \&c \\ - \frac{24 C^2 s^4}{e^3} - \frac{48 CE s^6}{e^5} - \frac{48 CG s^8}{e^7} - \&c \\ \quad + \frac{8 C^3 s^6}{e^5} - \frac{24 E^2 s^8}{e^7} - \&c \\ \quad\quad + \frac{24 C^2 E s^8}{e^7} + \&c. \end{array}$$

Now let all the terms of thefe two laft compound feriefes be divided by the fraction $\frac{s^4}{e^3}$; and it is evident that the quotients thence arifing muft be equal to each other, that is, the compound feries

$$\begin{array}{l} 6C + \frac{6 D ss}{ee} + \frac{6 E s^4}{e^4} + \&c \\ - 6 BC - \frac{6 C^2 ss}{ee} - \frac{6 CD s^4}{e^4} - \&c \\ + 6 E - \frac{6 BE ss}{ee} - \frac{6 CE s^4}{e^4} - \&c \\ \quad + \frac{6 G ss}{ee} - \frac{6 BG s^4}{e^4} - \&c \\ \quad\quad + \frac{6 I s^4}{e^4} - \&c \end{array}$$

will be equal to the compound feries

$$\begin{array}{l} 24 E + \frac{24 G ss}{ee} + \frac{24 I s^4}{e^4} + \&c \\ - 24 C^2 - \frac{48 CE ss}{ee} - \frac{48 CG s^4}{e^4} - \&c \\ \quad + \frac{8 C^3 ss}{ee} - \frac{24 C^2 s^4}{e^4} - \&c \\ \quad\quad + \frac{24 C^2 E s^4}{e^4} + \&c. \end{array}$$

And this will be true, of how fmall a magnitude foever we may fuppofe ss to be taken. And confequently it will likewife be true, when ss is $= 0$, or when $\frac{rr}{4} - \frac{q^3}{27}$ is $= 0$, or when $\frac{rr}{4}$ is $= \frac{q^3}{27}$. But in this cafe all the terms of thefe two compound feriefes that involve any power of ss, that is, all the terms except the firft, muft likewife be equal to o. Confequently thofe firft terms of the faid two feriefes (being equal to the faid whole feriefes refpectively) muft be equal to each other, that is, $6C - 6BC + 6E$, or the firft term of the compound feries derived from the compound feries $6e - \Pi$, will be equal to $24E - 24C^2$, or the firft term of the compound feries derived from the compound feries $8e - \Delta$. Therefore, if we multiply both thefe firft terms into the fraction $\frac{s^4}{e^3}$, we fhall have $\frac{6 C s^4}{e^3} - \frac{6 BC s^4}{e^3} + \frac{6 E s^4}{e^3}$ equal to $\frac{24 E s^4}{e^3} - \frac{24 C^2 s^4}{e^3}$; that is, the fecond term of the compound feries $6e - \Pi$ will be equal to the fecond term of the compound feries $8e - \Delta$. Q. E. D.

56. Thirdly, fince it has been fhewn in art. 54, that $\frac{6 B ss}{e} + \frac{6 C ss}{e}$, or the firft term of the compound feries $6e - \Pi$, is equal to $\frac{24 ss}{e}$, or the firft term of the

the compound ſeries $8e - \Delta$; and it has been ſhewn in art. 55, that $\frac{6cs^4}{e^3} - \frac{6BCs^4}{e^3} + \frac{6Es^4}{e^3}$, or the ſecond term of the compound ſeries $6e - \Pi$, is equal to $\frac{24Es^4}{e^3} - \frac{24C^2s^4}{e^3}$, or the ſecond term of the compound ſeries $8e - \Delta$; it follows that, if we ſubtract theſe two firſt terms of theſe two ſerieſes from the whole ſerieſes, the remainders will be equal to each other, that is, the compound ſeries

$$\begin{array}{rrrl} & \frac{6Ds^6}{e^5} & + \frac{6Es^8}{e^7} & + \&c \\ - & \frac{6C^2s^6}{e^5} & - \frac{6CDs^8}{e^7} & - \&c \\ - & \frac{6BEs^6}{e^5} & - \frac{6CEs^8}{e^7} & - \&c \\ + & \frac{6Gs^6}{e^5} & - \frac{6BGs^8}{e^7} & - \&c \\ & & + \frac{6Is^8}{e^7} & - \&c \end{array}$$

will be equal to the compound ſeries

$$\begin{array}{rrrl} & \frac{24Gs^6}{e^5} & + \frac{24Is^8}{e^7} & + \&c \\ - & \frac{48CEs^6}{e^5} & - \frac{48CGs^8}{e^7} & - \&c \\ + & \frac{8C^3s^6}{e^5} & - \frac{24E^2s^8}{e^7} & - \&c \\ & & + \frac{24C^2Es^8}{e^7} & + \&c. \end{array}$$

Now let all the terms of theſe two laſt compound ſerieſes be divided by the fraction $\frac{s^6}{e^5}$; and it is evident that the quotients thence ariſing muſt be equal to each other; that is, the compound ſeries

$$\begin{array}{rrrl} & 6D & + \frac{6Ess}{ee} & + \&c \\ - & 6C^2 & - \frac{6CDss}{ee} & - \&c \\ - & 6BE & - \frac{6CEss}{ee} & - \&c \\ + & 6G & - \frac{6BGss}{ee} & - \&c \\ & & + \frac{6Iss}{ee} & - \&c \end{array}$$

will be equal to the compound ſeries

$$\begin{array}{rrrl} & 24G & + \frac{24Iss}{ee} & + \&c \\ - & 48CE & - \frac{48CGss}{ee} & - \&c \\ + & 8C^3 & - \frac{24E^2ss}{ee} & - \&c \\ & & + \frac{24C^2Ess}{ee} & + \&c. \end{array}$$

 And

And this will be true, of how ſmall a magnitude ſoever we may ſuppoſe ss to be taken; and conſequently it will likewiſe be true, when ss is $= 0$, or when $\frac{rr}{4} - \frac{q^3}{27}$ is $= 0$, or when $\frac{rr}{4}$ is $= \frac{q^3}{27}$. But in this caſe all the terms of theſe two compound ſerieſes that involve any power of ss, that is, all the terms except the firſt, muſt likewiſe be equal to o. Conſequently thoſe firſt terms of the ſaid two ſerieſes (being equal to the ſaid whole ſerieſes reſpectively) muſt be equal to each other; that is, $6D - 6C^2 - 6BE + 6G$, the firſt term of the compound ſeries derived from the compound ſeries $6e - \Pi$, muſt be equal to $24G - 48CE + 8C^3$, the firſt term of the compound ſeries derived from the compound ſeries $8e - \Delta$. Therefore, if we multiply both theſe firſt terms into the fraction $\frac{s^6}{e^5}$, we ſhall have $\frac{6Ds^6}{e^5} - \frac{6C^2s^6}{e^5} - \frac{6BEs^6}{e^5} + \frac{6Gs^6}{e^5}$ equal to $\frac{24Gs^6}{e^5} - \frac{48CEs^6}{e^5} + \frac{8C^3s^6}{e^5}$; that is, the third term of the compound ſeries $6e - \Pi$ will be equal to the third term of the compound ſeries $8e - \Delta$. Q. E. D.

57. And in the ſame manner we may ſhew that the fourth term of the compound ſeries $6e - \Pi$ is equal to the fourth term of the compound ſeries $8e - \Delta$, and the fifth to the fifth, and the ſixth to the ſixth, and the ſeventh to the ſeventh, and every following term of the one ſeries to the correſponding term of the latter ſeries, to whatever number of terms the ſaid ſerieſes may be continued.

The reduction of the compound ſeries $8e - \Delta$ *to a ſimple ſeries* $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}}$ + *&c* ad infinitum.

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58. Now let the ſeveral numeral co-efficients of the fractions $\frac{ss}{e}$, $\frac{s^4}{e^3}$, $\frac{s^6}{e^5}$, $\frac{s^8}{e^7}$, $\frac{s^{10}}{e^9}$, $\frac{s^{12}}{e^{11}}$, &c in the terms of the compound ſeries $8e - \Delta$ (which is ſet down above in art. 44) be reduced (by performing the neceſſary arithmetical operations of multiplication, addition, and ſubtraction) to ſingle numbers, ſo as to convert the ſaid compound ſeries into a ſimple ſeries, or ſeries of ſimple, or ſingle, terms. This reduction has been already made for the firſt ſix terms of this ſeries in art. 46, 47, 48, 49, 50, and 51; in which it has been ſhewn that the firſt term of the ſaid compound ſeries, to wit, $\frac{24Css}{e}$, is $= \frac{8ss}{3e}$; and that its ſecond term, $\frac{24Es^4}{e^3} - \frac{24C^2s^4}{e^3}$, is $= \frac{56s^4}{81e^3}$; and that its third term, $\frac{24Gs^6}{e^5} - \frac{48CEs^6}{e^5} + \frac{8C^3s^6}{e^5}$, is $= \frac{776s^6}{2187e^5}$; and that its fourth term, $\frac{24Is^8}{e^7} - \frac{48CGs^8}{e^7} - \frac{24E^2s^8}{e^7}$

$\frac{24\,E^2 s^8}{e^7} + \frac{24\,C^2 E\,s^8}{e^7}$, is $= \frac{4456\,s^8}{19{,}683\,e^7}$; and that its fifth term, $\frac{24\,L\,s^{10}}{e^9} - \frac{48\,CI\,s^{10}}{e^9} - \frac{48\,EG\,s^{10}}{e^9} + \frac{24\,C^2 G\,s^{10}}{e^9} + \frac{24\,CE^2 s^{10}}{e^9}$, is $= \frac{257{,}032\,s^{10}}{1{,}594{,}323\,e^9}$; and that its sixth term, $\frac{24\,N\,s^{12}}{e^{11}} - \frac{48\,CL\,s^{12}}{e^{11}} - \frac{48\,EI\,s^{12}}{e^{11}} - \frac{24\,G^2 s^{12}}{e^{11}} + \frac{24\,C^2 I\,s^{12}}{e^{11}} + \frac{48\,CEG\,s^{12}}{e^{11}} + \frac{8\,E^3 s^{12}}{e^{11}}$, is $= \frac{5{,}282{,}360\,s^{12}}{43{,}046{,}721\,e^{11}}$; so that the first six terms of the said compound series $8\,e - \Delta$ are equal to the six following simple quantities, to wit, $\frac{8\,ss}{3e} + \frac{56\,s^4}{81\,e^3} + \frac{776\,s^6}{2187\,e^5} + \frac{4456\,s^8}{19683\,e^7} + \frac{257{,}032\,s^{10}}{1{,}594{,}323\,e^9} + \frac{5{,}282{,}360\,s^{12}}{43{,}046{,}721\,e^{11}}$. And let these co-efficients of the fractions $\frac{ss}{e}, \frac{s^4}{e^3}, \frac{s^6}{e^5}, \frac{s^8}{e^7}, \frac{s^{10}}{e^9}, \frac{s^{12}}{e^{11}}$, &c, in the said simple series be (for the sake of brevity) denoted by the capital letters P, Q, R, S, T, V, &c. Then will the compound series $8\,e - \Delta$ be equal to the simple series $\frac{P\,ss}{e} + \frac{Q\,s^4}{e^3} + \frac{R\,s^6}{e^5} + \frac{S\,s^8}{e^7} + \frac{T\,s^{10}}{e^9} + \frac{V\,s^{12}}{e^{11}}$ + &c *ad infinitum*.

The reduction of the compound series $6\,e - \Pi$ *in like manner to a series of simple terms.*

59. And, in like manner, let the several numeral co-efficients of the same fractions $\frac{ss}{e}, \frac{s^4}{e^3}, \frac{s^6}{e^5}, \frac{s^8}{e^7}, \frac{s^{10}}{e^9}, \frac{s^{12}}{e^{11}}$ &c in the terms of the other compound series $6\,e - \Pi$ (which is also set down above in art. 44) be reduced (by performing the necessary operations of multiplication, addition, and subtraction) to single numbers, so as to convert the said compound series into a simple series, or series of simple, or single, terms. This reduction has been already made for the first six terms of the series in art. 46, 47, 48, 49, 50, and 51; in which it has been shewn that the first term of the said compound series, to wit, $\frac{6\,B\,ss}{e} + \frac{6\,C\,ss}{e}$, is $= \frac{8\,ss}{3\,e}$, as well as the first term of the former compound series $8\,e - \Delta$; and that the second term of this compound series, to wit, $\frac{6\,C\,s^4}{e^3} - \frac{6\,BC\,s^4}{e^3} + \frac{6\,E\,s^4}{e^3}$, is $= \frac{56\,s^4}{81\,e^3}$, as well as the second term of the said former compound series; and that the third term of this compound series, to wit, $\frac{6\,D\,s^6}{e^5} - \frac{6\,C^2 s^6}{e^5} - \frac{6\,BE\,s^6}{e^5} + \frac{6\,G\,s^6}{e^5}$, is $= \frac{776\,s^6}{2187\,e^5}$, as well as the third term of the said former compound series; and that the fourth term of this compound series, to wit, $\frac{6\,E\,s^8}{e^7} - \frac{6\,CD\,s^8}{e^7} - \frac{6\,CE\,s^8}{e^7} - \frac{6\,BG\,s^8}{e^7} + \frac{6\,I\,s^8}{e^7}$, is $= \frac{4456\,s^8}{19683\,e^7}$, as well as the fourth term of the said former

 compound

compound series; and that the fifth term of this compound series, to wit, $\frac{6 F s^{10}}{e^9} - \frac{6 CE s^{10}}{e^9} - \frac{6 DE s^{10}}{e^9} - \frac{6 CG s^{10}}{e^9} - \frac{6 BI s^{10}}{e^9} + \frac{6 L s^{10}}{e^9}$, is $= \frac{257,032 s^{10}}{1,594,323 e^9}$, as well as the fifth term of the said former compound series; and that the sixth term of this compound series, to wit, $\frac{6 G s^{12}}{e^{11}} - \frac{6 CF s^{12}}{e^{11}} - \frac{6 E^2 s^{12}}{e^{11}} - \frac{6 DG s^{12}}{e^{11}} - \frac{6 CI s^{12}}{e^{11}} - \frac{6 BL s^{12}}{e^{11}} + \frac{6 N s^{12}}{e^{11}}$, is $= \frac{5,282,360 s^{12}}{43,046,721 e^{11}}$, as well as the sixth term of the said former compound series; so that the first six terms of the compound series $6e - \Pi$ are equal to the six following simple quantities, to wit, $\frac{8 ss}{3e} + \frac{56 s^4}{81 e^3} + \frac{776 s^6}{2187 e^5} + \frac{4456 s^8}{19683 e^7} + \frac{257,032 s^{10}}{1,594,323 e^9} + \frac{5,282,360 s^{12}}{43,046,721 e^{11}}$, as well as the first six terms of the former compound series $8e - \Delta$. And it is evident, from art. 57, that if the seventh, and eighth, and ninth, and other following terms of the compound series $6e - \Pi$ were in the same manner to be reduced to simple terms, the said simple terms would be equal to the simple terms to which the seventh, eighth, ninth, and other following terms of the said former compound series $8e - \Delta$ would be reduced. And consequently the same simple series $\frac{P ss}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} +$ &c *ad infinitum*, will be equal to both the compound serieses $8e - \Delta$ and $6e - \Pi$.

The values of y^3 and qy expressed by means of the simple series $\frac{P ss}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} +$ &c ad infinitum.

60. We have seen, in art. 41, that the compound series Δ is $= y^3$; and we have seen, in art. 43, that the compound series Π is $= qy$. Therefore $8e - \Delta$ is $= 8e - y^3$, and $6e - \Pi$ is $= 6e - qy$. We shall therefore have $8e - y^3$ ($= 8e - \Delta$) $=$ the simple series $\frac{P ss}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} +$ &c *ad infinitum*, and consequently (adding Δ to both sides) $8e = y^3 +$ the series $\frac{P ss}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} +$ &c *ad infinitum*, and $y^3 = 8e - \frac{P ss}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} -$ &c *ad infinitum*; and we shall also have $6e - qy$ ($= 6e - \Pi$) $=$ the same series $\frac{P ss}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} +$ &c *ad infinitum*, and consequently (adding qy to both sides)

ſides) $6e = qy + \frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} +$ &c *ad infinitum*, and $qy = 6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum*.

Examination of the expreſſion $2\sqrt{^3}g \times$ *the infinite ſeries* $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ *&c* ad infinitum.

61. We muſt now turn our attention to the other tranſcendental expreſſion $2\sqrt{^3}\overline{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum*, in which g is put for $\frac{t}{2}$, or half the abſolute term of the equation $x^3 - qx = t$, as e was before put for $\frac{r}{2}$, or half the abſolute term of the equation $y^3 - qy = r$, and in which zz is put for $\frac{q^3}{27} - \frac{tt}{4}$, or the difference between $\frac{q^3}{27}$ and the ſquare of half the abſolute term t, as ss was before put for $\frac{rr}{4} - \frac{q^3}{27}$, or the difference between $\frac{q^3}{27}$ and the ſquare of half the abſolute term r. This expreſſion we have aſſerted above, in art. 36, to be equal to the root of the equation $x^3 - qx = t$, in which the abſolute term t is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or in which $\frac{tt}{4}$ is leſs than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$. This aſſertion we muſt now endeavour to prove.

62. Now "that this expreſſion $2\sqrt{^3}\overline{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c is equal to the root x of the equation $x^3 - qx = t$, when t is of the magnitude here ſuppoſed," will be evident, if we can ſhew that this expreſſion, being ſubſtituted inſtead of x in the compound quantity $x^3 - qx$, will make that quantity be equal to the abſolute term t of the ſaid equation; or that, if the ſaid expreſſion be cubed, or raiſed to the third power by multiplying it twice into itſelf, and alſo be multiplied into the co-efficient q, the ſaid cube of the ſaid expreſſion will be greater than the ſaid product of its multiplication into q, and that the exceſs, or difference, will be equal to the abſolute term t. This therefore is what I ſhall now endeavour to demonſtrate.

63. The cube of the expression $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum* is $= 8g \times$ the cube of the said series. We must therefore raise the cube of this series by multiplying it twice into itself; which may be done as follows.

The multiplication of the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c ad infinitum *into itself, in order to obtain its square.*

$$
\begin{array}{llllllll}
1 & +\frac{Czz}{gg} & -\frac{Ez^4}{g^4} & +\frac{Gz^6}{g^6} & -\frac{Iz^8}{g^8} & +\frac{Lz^{10}}{g^{10}} & -\frac{Nz^{12}}{g^{12}} & +\&c \\
1 & +\frac{Czz}{gg} & -\frac{Ez^4}{g^4} & +\frac{Gz^6}{g^6} & -\frac{Iz^8}{g^8} & +\frac{Lz^{10}}{g^{10}} & -\frac{Nz^{12}}{g^{12}} & +\&c \\
\hline
1 & +\frac{Czz}{gg} & -\frac{Ez^4}{g^4} & +\frac{Gz^6}{g^6} & -\frac{Iz^8}{g^8} & +\frac{Lz^{10}}{g^{10}} & -\frac{Nz^{12}}{g^{12}} & +\&c \\
 & +\frac{Czz}{gg} & +\frac{C^2z^4}{g^4} & -\frac{CEz^6}{g^6} & +\frac{CGz^8}{g^8} & -\frac{CIz^{10}}{g^{10}} & +\frac{CLz^{12}}{g^{12}} & -\&c \\
 & & -\frac{Ez^4}{g^4} & -\frac{CEz^6}{g^6} & +\frac{E^2z^8}{g^8} & -\frac{EGz^{10}}{g^{10}} & +\frac{EIz^{12}}{g^{12}} & -\&c \\
 & & & +\frac{Gz^6}{g^6} & +\frac{CGz^8}{g^8} & -\frac{EGz^{10}}{g^{10}} & +\frac{G^2z^{12}}{g^{12}} & -\&c \\
 & & & & -\frac{Iz^8}{g^8} & -\frac{CIz^{10}}{g^{10}} & +\frac{EIz^{12}}{g^{12}} & -\&c \\
 & & & & & +\frac{Lz^{10}}{g^{10}} & +\frac{CLz^{12}}{g^{12}} & -\&c \\
 & & & & & & -\frac{Nz^{12}}{g^{12}} & -\&c \\
\hline
1 & +\frac{2Czz}{gg} & -\frac{2Ez^4}{g^4} & +\frac{2Gz^6}{g^6} & -\frac{2Iz^8}{g^8} & +\frac{2Lz^{10}}{g^{10}} & -\frac{2Nz^{12}}{g^{12}} & +\&c \\
 & & +\frac{C^2z^4}{g^4} & -\frac{2CEz^6}{g^6} & +\frac{2CGz^8}{g^8} & -\frac{2CIz^{10}}{g^{10}} & +\frac{2CLz^{12}}{g^{12}} & -\&c \\
 & & & & +\frac{E^2z^8}{g^8} & -\frac{2EGz^{10}}{g^{10}} & +\frac{2EIz^{12}}{g^{12}} & -\&c \\
 & & & & & & +\frac{G^2z^{12}}{g^{12}} & -\&c
\end{array}
$$

This is the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c.

A com-

A comparison between the foregoing compound series, which is equal to the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ ad infinitum, *and the compound series which is equal to the square of the former series* $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ ad infinitum, *and which is set down above in art.* 40.

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64. Now, if we compare this square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$ with the compound series which is equal to the square of the former series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} + \&c$, and which is set down above in art. 40, we shall find that there are the following resemblances and differences between them.

In the first place, the first terms of both these compound serieses are equal to the same quantity, to wit, 1.

Secondly, the second, and third, and fourth, and other following terms of the latter compound series obtained in the foregoing article 63, involve in them the fractions $\frac{zz}{gg}$, $\frac{z^4}{g^4}$, $\frac{z^6}{g^6}$, $\frac{z^8}{g^8}$, $\frac{z^{10}}{g^{10}}$, $\frac{z^{12}}{g^{12}}$, &c, or the several powers of the fraction $\frac{zz}{gg}$, just as the second, and third, and fourth, and other following terms of the former compound series, obtained in art. 40, involve in them the fractions $\frac{ss}{ee}$, $\frac{s^4}{e^4}$, $\frac{s^6}{e^6}$, $\frac{s^8}{e^8}$, $\frac{s^{10}}{e^{10}}$, $\frac{s^{12}}{e^{12}}$, &c, or the several powers of the fraction $\frac{ss}{ee}$. And it is evident that this observation will be true of all the following terms of these two compound serieses, because the simple series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$ contains in its terms the very same powers of the fraction $\frac{zz}{gg}$ as the former simple series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ contains of the fraction $\frac{ss}{ee}$ in its several corresponding terms.

Thirdly, the signs of the several members of the third, fifth, and seventh terms of the latter compound series obtained in the preceeding art. 63, are the same with the signs of the corresponding members of the third, fifth, and seventh terms of the former compound series obtained in art. 40. For the third term of the former compound series is $-\frac{2Es^4}{e^4} + \frac{C^2s^4}{e^4}$, the two members of which are marked with the signs $-$ and $+$; and the third term of the latter compound

compound series is $-\frac{2Ez^4}{g^4}+\frac{C^2z^4}{g^4}$, the two members of which are likewise marked with the signs — and +. And the fifth term of the former compound series is $-\frac{2Is^8}{e^8}+\frac{2CGs^8}{e^8}+\frac{E^2s^8}{e^8}$, the three members of which are marked with the signs —, +, and +; and the fifth term of the latter compound series is $-\frac{2Iz^8}{g^8}+\frac{2CGz^8}{g^8}+\frac{E^2z^8}{g^8}$, the three members of which are likewise marked with the signs —, +, and +. And the seventh term of the former compound series is $-\frac{2Ns^{12}}{e^{12}}+\frac{2CLs^{12}}{e^{12}}+\frac{2EIs^{12}}{e^{12}}+\frac{G^2s^{12}}{e^{12}}$, the four members of which are marked with the signs —, +, +, and +; and the seventh term of the latter compound series is $-\frac{2Nz^{12}}{g^{12}}+\frac{2CLz^{12}}{g^{12}}+\frac{2EIz^{12}}{g^{12}}+\frac{G^2z^{12}}{g^{12}}$, the four members of which are likewise marked with the signs —, +, +, and +. And the same analogy will take place between the signs of the several members of the ninth, eleventh, thirteenth, and other following odd terms of the former compound series obtained in art. 40, and the signs of the corresponding members of the ninth, eleventh, thirteenth, and other following odd terms of the latter compound series obtained in art. 63; as we shall presently endeavour to make appear.

Fourthly, the sign of the second term, $+\frac{2Czz}{gg}$, of the latter compound series, and those of the several members of its fourth term, $+\frac{2Gz^6}{g^6}-\frac{2CEz^6}{g^6}$, and those of the several members of its sixth term, $+\frac{2Lz^{10}}{g^{10}}-\frac{2CIz^{10}}{g^{10}}-\frac{2EGz^{10}}{g^{10}}$, are respectively contrary to the sign of the second term, $-\frac{2Css}{ee}$, of the former compound series, and to those of the several members of its fourth term, $-\frac{2Gs^6}{e^6}+\frac{2CEs^6}{e^6}$, and to those of the several members of its sixth term, $-\frac{2Ls^{10}}{e^{10}}+\frac{2CIs^{10}}{e^{10}}+\frac{2EGs^{10}}{e^{10}}$. And the same contrariety will take place between the signs of the several members of the eighth, tenth, twelfth, and other following even terms of the latter compound series obtained in art. 63, and those of the several corresponding members of the eighth, tenth, twelfth, and other following even terms of the former compound series obtained in art. 40; as we shall now endeavour to make appear.

65. It is evident, from the rules of multiplication in algebra, that, whenever a series of algebraïck quantities is multiplied by either the same, or another, series of algebraïck quantities, all those horizontal lines of terms in the product, which arise from the multiplication of the first series, or multiplicand, into those terms of the multiplicator which are marked with the sign +, will have the same signs + and — prefixed to their several terms as are prefixed to the corresponding terms in the multiplicand; and that all those horizontal lines of terms in the product which arise from the multiplication of the first series, or multiplicand, into those terms of the multiplicator which are marked with the sign

ſign —, will have contrary ſigns prefixed to their ſeveral terms to thoſe which are prefixed to the correſponding terms of the multiplicand. It follows therefore that in the general product of the multiplication of the ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c (of which all the terms after the firſt term are marked with the ſign —) into itſelf, which is ſet down above in art. 40 (I mean by the *general* product the firſt product, before the ſimilar terms in each vertical column of terms are added up together at the bottom, ſo as to make but one term), the firſt horizontal line of terms (which ariſes from the multiplication of the ſaid multiplicand, or ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, into its firſt term 1, or, in other words, which is the ſaid ſeries itſelf) muſt have the ſign + prefixed to its firſt term (or rather, no ſign at all; becauſe, being the firſt term of the whole product, it is that to which all the other terms are to be referred, and to be added to it when they are marked with the ſign +, and to be ſubtracted from it when they are marked with the ſign —) and muſt have the ſign — prefixed to all the following terms; and the ſecond, and third, and fourth, and fifth, and ſixth, and ſeventh, and every following horizontal line of terms in the ſaid product (which ariſe from the multiplication of the ſaid multiplicand, or ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, into the ſeveral terms $\frac{Css}{ee}$, $\frac{Es^4}{e^4}$, $\frac{Gs^6}{e^6}$, $\frac{Is^8}{e^8}$, $\frac{Ls^{10}}{e^{10}}$, $\frac{Ns^{12}}{e^{12}}$, &c, which are all marked with the ſign —) muſt have the ſign — prefixed to their ſeveral firſt terms, and the ſign + prefixed to all their following terms, to whatever number of terms the ſaid horizontal lines may be continued. And it follows likewiſe, that in the general product of the multiplication of the ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c (of which the ſecond, and fourth, and ſixth, and all the following even terms, are marked with the ſign +, and the third, and fifth, and ſeventh, and all the following odd terms, are marked with the ſign —) into itſelf, which is ſet down in art. 63, the firſt horizontal line of terms (which ariſes by the multiplication of the ſaid multiplicand, or ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, into its firſt term 1, or, in other words, which is the ſaid ſeries itſelf) muſt have the ſign + prefixed to its ſecond, fourth, ſixth, and other following even terms, and the ſign — prefixed to its third, fifth, ſeventh, and other following odd terms (to whatever number of terms the ſaid horizontal line may be continued), being the ſame ſigns as thoſe of the ſeveral correſponding terms of the ſaid multiplicand itſelf; and, in like manner, the ſecond horizontal row of terms in the ſaid product, and the fourth, and ſixth, and eighth, and tenth, and every following even horizontal row of terms in it (which ariſe from the multiplication of the ſaid multiplicand, or ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, into the ſecond,

fourth, and fixth terms, $\frac{Czz}{gg}$, $\frac{Gz^6}{g^6}$, $\frac{Lz^{10}}{g^{10}}$, and the other following even terms, of the faid feries, to all which the fign + is prefixed) will have the fame figns + and — prefixed to their feveral terms as are prefixed to the correfponding terms of the faid multiplicand, or feries itfelf, to wit, the fign + prefixed to their firft, and fecond, and fourth, and fixth, and eighth, and tenth, and other following even terms, and the fign — prefixed to their third, and fifth, and feventh, and ninth, and eleventh, and other following odd terms; and the third horizontal row of terms in the faid product, and the fifth, and feventh, and ninth, and eleventh, and every following odd horizontal row of terms in it (which arife from the multiplication of the faid multiplicand, or feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, into the third, and fifth, and feventh, terms $\frac{Ez^4}{g^4}$, $\frac{Iz^8}{g^8}$, $\frac{Nz^{12}}{g^{12}}$, and the other following odd terms of the faid feries, to all which the fign — is prefixed), will have contrary figns prefixed to their feveral terms to thofe which are prefixed to the correfponding terms of the faid multiplicand, or feries itfelf, and therefore will have the fign — prefixed to their firft, and fecond, and fourth, and fixth, and eighth, and tenth, and other following even terms, and the fign + prefixed to their third, and fifth, and feventh, and ninth, and eleventh, and other following odd terms.

66. It has been fhewn, in the laft article, that in the general product of the multiplication of the feries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c. into itfelf, fet down above in art. 40, all the terms in the firft, or higheft, horizontal row of terms, except the firft term 1, will have the fign — prefixed to them, and that in the fecond, and third, and fourth, and other following horizontal rows of terms in the faid product, the firft terms of the faid rows will have the fign — prefixed to them, but all the following terms in them will be marked with the fign +. Now the firft terms of the feveral horizontal rows of terms in the faid general product are the loweft terms of the feveral vertical columns of terms in the faid product, which involve the fame powers of the fraction $\frac{ss}{ee}$; and the fecond, and third, and fourth, and other following terms of the firft, or higheft, horizontal row of terms in the faid product, are the higheft terms of the fecond, and third, and fourth, and other following vertical columns of terms in the faid product. Therefore the higheft term and the loweft term of the fecond, and the third, and the fourth, and every following vertical column of terms in the faid product, will have the fign — prefixed to them, and all the other terms of the faid vertical columns will be marked with the fign +. Now this will likewife be the cafe with the third, and the fifth, and the feventh, and the other following odd vertical columns, of the general product of the multiplication of the other feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c into itfelf, fet down in art. 63. For the firft, or higheft, terms

terms of the ſaid odd vertical columns are the third, fifth, ſeventh, and other following odd terms of the firſt, or higheſt, horizontal row of terms, to wit, the terms $\frac{Ez^4}{g^4}$, $\frac{Iz^8}{g^8}$, $\frac{Nz^{12}}{g^{12}}$, &c, which are all marked with the ſign $-$; and the loweſt terms of the ſame odd vertical columns are equal to their higheſt terms, and are marked alſo with the ſign $-$, becauſe they are the firſt terms of the ſeveral odd horizontal rows of terms, which have been ſhewn to be marked with the ſaid ſign $-$. And the intermediate terms of the ſaid odd vertical columns between the higheſt term and the loweſt muſt all be marked with the ſign $+$, becauſe they are, all of them, the products of the multiplication of factors, which are both marked either with the ſame ſign $+$ or with the ſame ſign $-$. Thus, for example, in the ſeventh vertical column of the ſaid product, of which the higheſt term is $-\frac{Nz^{12}}{g^{12}}$, the next term $\frac{CLz^{12}}{g^{12}}$ is the product of the multiplication of the factors $\frac{Lz^{10}}{g^{10}}$ and $\frac{Czz}{gg}$, which are both marked with the ſign $+$, and therefore it muſt be likewiſe marked with the ſign $+$; and the third term $\frac{EIz^{12}}{g^{12}}$ is the product of the multiplication of the factors $\frac{Iz^8}{g^8}$ and $\frac{Ez^4}{g^4}$, which are both marked with the ſign $-$, and therefore it muſt be marked with the ſign $+$; and the fourth term $\frac{G^2z^{12}}{g^{12}}$ is the product of the multiplication of $\frac{Gz^6}{g^6}$ by $\frac{Gz^6}{g^6}$, which are both marked with the ſign $+$, and therefore it muſt be marked alſo with the ſign $+$; and the fifth term $\frac{EIz^{12}}{g^{12}}$ is the product of the multiplication of the factors $\frac{Ez^4}{g^4}$ and $\frac{Iz^8}{g^8}$, which are both marked with the ſign $-$, and therefore it muſt be marked with the ſign $+$; and the ſixth term, $\frac{CLz^{12}}{g^{12}}$, or the loweſt term but one, or the laſt of the intermediate terms between the higheſt and loweſt terms of the ſaid vertical column, is the product of the factors $\frac{Czz}{gg}$ and $\frac{Lz^{10}}{g^{10}}$, which are both marked with the ſign $+$, and therefore it muſt likewiſe be marked with the ſign $+$; and conſequently all the five intermediate terms of the ſaid ſeventh vertical column between its higheſt and loweſt terms muſt be marked with the ſign $+$. And it is eaſy to perceive that, for the ſame reaſons, all the intermediate terms between the higheſt and loweſt terms of any other vertical column of which the higheſt term was marked with the ſign $-$, that is, of any other odd vertical column, muſt be marked with the ſign $+$. We may therefore conclude that in all the odd vertical columns of terms in the general product ſet down in art. 63, the ſigns $+$ and $-$, which are to be prefixed to the ſeveral terms of the ſaid columns, will be the ſame as thoſe which are to be prefixed to the correſponding terms of the ſame odd vertical columns of terms in the general product ſet down above in art. 40. Q. E. D.

67. Since the ſigns of the terms in the ſeveral odd vertical columns of the general product ſet down in art. 63, are the ſame as thoſe of the correſponding

terms of the like odd vertical columns of the general product set down in art. 40, it is evident that, when the similar terms contained in the said odd vertical columns are added up together at the bottoms of the said general products, so as to make but single terms, the same analogy between the signs of the terms of these vertical columns will continue; that is, in the several odd vertical columns of the reduced compound series, which is equal to the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, the signs of the several terms, or members of the said columns, will be the same with the signs of the several corresponding terms, or members of the like odd vertical columns in the reduced compound series, which is equal to the square of the former series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c, to whatever number of terms the said serieses may be continued. Therefore the third observation made above in art. 64, concerning the signs of the members of the odd vertical columns of these two compound serieses, is universally true. Q. E. D.

68. But in the second, and fourth, and sixth, and other following *even* vertical columns of terms in the general product set down in art. 63, the signs + and −, that are to be prefixed to the several terms or members of the said columns, will be contrary to those which are to be prefixed to the several corresponding terms of the second, and fourth, and sixth, and other following *even* vertical columns in the general product set down in art. 40. For, in the first place, the first, or highest, term in every even vertical column in the general product set down in art. 63, is marked with the sign +; whereas the first, or highest, term of every even vertical column of terms in the general product set down in art. 40 (as well as the first, or highest, term of the third and every following odd vertical column of terms in the same general product), is marked with the sign −. And, in the second place, the last, or lowest, term in every even vertical column of terms in the general product set down in art. 63, is marked with the sign +; because the said lowest term of such even vertical column is the first term of the even horizontal line of terms in which it lies: and it has been shewn in art. 65 that the first terms of the second, and fourth, and sixth, and every following even horizontal row of terms, are marked with the sign +. But the lowest terms of the second, and fourth, and sixth, and every following even vertical column of terms, in the general product set down in art. 40 (as well as of every odd vertical column in the said product), are all marked with the sign −, as has been shewn in art. 66. And, thirdly, the several intermediate terms between the highest and the lowest terms in every even vertical column of terms, in the general product set down in art. 63, must all be marked with the sign −, because they are the products of factors which are marked with different signs. Thus, for example, in the sixth vertical column of the general product, set down in art. 63, of which the highest term is $+ \frac{Lz^{10}}{g^{10}}$, the next term $\frac{CIz^{10}}{g^{10}}$ is the product of the multiplication of the factors $\frac{Iz^8}{g^8}$ and $\frac{Czz}{gg}$, of which the former is marked with the sign −, and the latter is marked with the sign +, in consequence

consequence of which their product must be marked with the sign $-$; and the third term $\frac{EGz^{10}}{g^{10}}$ is the product of the multiplication of the factors $\frac{Gz^6}{g^6}$ and $\frac{Ez^4}{g^4}$, of which the former is marked with the sign $+$, and the latter is marked with the sign $-$, and consequently their product must be marked with the sign $-$; and the fourth term $\frac{EGz^{10}}{g^{10}}$ is the product of the multiplication of the factors $\frac{Ez^4}{g^4}$ and $\frac{Gz^6}{g^6}$, of which the former is marked with the sign $-$, and the latter is marked with the sign $+$, and consequently their product must be marked with the sign $-$; and the fifth term $\frac{CIz^{10}}{g^{10}}$, or the lowest term but one, or the last of the said intermediate terms between the highest and the lowest terms of the said vertical column, is the product of the multiplication of the factors $\frac{Czz}{gg}$ and $\frac{Iz^8}{g^8}$, of which the former is marked with the sign $+$, and the latter is marked with the sign $-$, and consequently their product must be marked with the sign $-$. And it is easy to perceive that, for the same reasons, all the intermediate terms between the highest and the lowest terms of any other vertical column, of which the highest term was marked with the sign $+$, that is, of any other even vertical column, must be marked with the sign $-$. But in the general product set down in art. 40, all the intermediate terms between the highest and lowest terms of all the vertical columns, both odd and even, are marked with the sign $+$. Therefore all the intermediate terms between the highest and the lowest terms of all the even vertical columns, in the general product set down in art. 63, are marked with contrary signs to those of the corresponding intermediate terms of the even vertical columns of the general product set down in art. 40. And the same thing has been shewn concerning the highest and lowest terms of the said even vertical columns of the said two general products. We may therefore conclude that in all the even vertical columns of terms in the general product set down in art. 63, the signs $+$ and $-$, which are to be prefixed to the several terms of the said columns, will be every where contrary to those which are to be prefixed to the corresponding terms of the like even vertical columns of terms in the general product set down above in art. 40. Q. E. D.

69. Since the signs of the terms in the several even vertical columns of the general product set down in art. 63, are respectively contrary to those of the corresponding terms of the like even vertical columns of the general product set down in art. 40, it is evident that, when the similar terms contained in the said even vertical columns are added up together at the bottoms of both the said general products, so as to make but single terms, the same analogy, or rather contrariety, between the signs of the terms of these vertical columns will continue; that is, in the several even vertical columns of the reduced compound series, which is equal to the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, the signs of the several terms, or members of the said columns, will be every where contrary to the signs of the several corresponding terms, or

members of the like even vertical columns, of the reduced compound series, which is equal to the square of the series $1 - \frac{css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, to whatever number of terms the said seriefes may be continued. Therefore the fourth observation made above in art. 64, concerning the signs of the members of the even vertical columns of these two compound seriefes, is universally true. Q. E. D.

70. In the fifth place we may observe concerning these two compound seriefes, which are equal to the squares of the two infinite seriefes $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, that the co-efficients of the several terms, or members of the several vertical columns, of the compound series which is equal to the square of the latter simple series, and which is set down in art. 63, will be equal to the co-efficients of the corresponding terms, or members of the several vertical columns of the former compound series, which is equal to the square of the former simple series, and which is set down above in art. 40; though the signs + and — that are to be prefixed to them will not every where be alike. For, since the two simple seriefes $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, contain the very same co-efficients C, E, G, I, L, N, &c, combined with the same powers of the two fractions $\frac{ss}{ee}$ and $\frac{zz}{gg}$, though with different signs + and — prefixed to the even terms $\frac{Css}{ee}$, $\frac{Gs^6}{e^6}$, $\frac{Ls^{10}}{e^{10}}$, &c, and $\frac{Czz}{gg}$, $\frac{Gz^6}{g^6}$, $\frac{Lz^{10}}{g^{10}}$, &c, of the two seriefes; and since it has been shewn that, in the third, and fifth, and seventh, and other following *odd* vertical columns of terms, in the two compound seriefes which are equal to the squares of the said simple seriefes, the signs + and —, that are to be prefixed to the corresponding terms of the said *odd* vertical columns, are the same in both seriefes; and that in the second, and fourth, and sixth, and other following *even* vertical columns of terms in the said compound seriefes, the signs + and —, that are to be prefixed to the corresponding terms of the said *even* vertical columns, are uniformly contrary to each other in the two seriefes; it follows that the co-efficients of the several terms, or members of the vertical columns, of one series, must arise from the same combinations of the original co-efficients C, E, G, I, L, N, &c, by multiplication and addition, by which the co-efficients of the corresponding terms, or members of the several vertical columns, of the other series are produced: and consequently the said co-efficients must be the same in both seriefes, though the signs + and — that are to be prefixed to them will, in all the terms of the several even vertical columns, be different. Q. E. D.

71. We

71. We now proceed to multiply the compound series obtained above, in art. 63, for the square of the simple series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, into the said simple series itself, in order to obtain the cube of the said simple series. This multiplication is as follows.

The multiplication of the compound series (obtained above in art. 63) which is equal to the square of the simple series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, *into the said simple series, in order to obtain the cube of the said simple series.*

$$
\begin{array}{llllllll}
1 & +\frac{2Czz}{gg} & -\frac{2Ez^4}{g^4} & +\frac{2Gz^6}{g^6} & -\frac{2Iz^8}{g^8} & +\frac{2Lz^{10}}{g^{10}} & -\frac{2Nz^{12}}{g^{12}} & +\&c \\
 & & +\frac{C^2z^4}{g^4} & -\frac{2CEz^6}{g^6} & +\frac{2CGz^8}{g^8} & -\frac{2CIz^{10}}{g^{10}} & +\frac{2CLz^{12}}{g^{12}} & -\&c \\
 & & & & +\frac{E^2z^8}{g^8} & -\frac{2EGz^{10}}{g^{10}} & +\frac{2EIz^{12}}{g^{12}} & -\&c \\
 & & & & & & +\frac{G^2z^{12}}{g^{12}} & -\&c \\
\hline
1 & +\frac{Czz}{gg} & -\frac{Ez^4}{g^4} & +\frac{Gz^6}{g^6} & -\frac{Iz^8}{g^8} & +\frac{Lz^{10}}{g^{10}} & -\frac{Nz^{12}}{g^{12}} & +\&c \\
\hline
1 & +\frac{2Czz}{gg} & -\frac{2Ez^4}{g^4} & +\frac{2Gz^6}{g^6} & -\frac{2Iz^8}{g^8} & +\frac{2Lz^{10}}{g^{10}} & -\frac{2Nz^{12}}{g^{12}} & +\&c \\
 & & +\frac{C^2z^4}{g^4} & -\frac{2CEz^6}{g^6} & +\frac{2CGz^8}{g^8} & -\frac{2CIz^{10}}{g^{10}} & +\frac{2CLz^{12}}{g^{12}} & -\&c \\
 & & & & +\frac{E^2z^8}{g^8} & -\frac{2EGz^{10}}{g^{10}} & +\frac{2EIz^{12}}{g^{12}} & -\&c \\
 & & & & & & +\frac{G^2z^{12}}{g^{12}} & -\&c \\
 & +\frac{Czz}{gg} & +\frac{2C^2z^4}{g^4} & -\frac{2CEz^6}{g^6} & +\frac{2CGz^8}{g^8} & -\frac{2CIz^{10}}{g^{10}} & +\frac{2CLz^{12}}{g^{12}} & -\&c \\
 & & & +\frac{C^3z^6}{g^6} & -\frac{2C^2Ez^8}{g^8} & +\frac{2C^2Gz^{10}}{g^{10}} & -\frac{2C^2Iz^{12}}{g^{12}} & +\&c \\
 & & & & & +\frac{CE^2z^{10}}{g^{10}} & -\frac{2CEGz^{12}}{g^{12}} & +\&c \\
 & & -\frac{Ez^4}{g^4} & -\frac{2CEz^6}{g^6} & +\frac{2E^2z^8}{g^8} & -\frac{2EGz^{10}}{g^{10}} & +\frac{2EIz^{12}}{g^{12}} & -\&c \\
 & & & & -\frac{C^2Ez^8}{g^8} & +\frac{2CE^2z^{10}}{g^{10}} & -\frac{2CEGz^{12}}{g^{12}} & +\&c \\
 & & & & & & -\frac{E^3z^{12}}{g^{12}} & +\&c \\
 & & & +\frac{Gz^6}{g^6} & +\frac{2CGz^8}{g^8} & -\frac{2EGz^{10}}{g^{10}} & +\frac{2G^2z^{12}}{g^{12}} & -\&c \\
 & & & & & +\frac{C^2Gz^{10}}{g^{10}} & -\frac{2CEGz^{12}}{g^{12}} & +\&c
\end{array}
$$

$-\frac{Iz^8}{g^8}$

$$\begin{array}{r}
-\frac{Iz^8}{g^8}-\frac{2CIz^{10}}{g^{10}}+\frac{2EIz^{12}}{g^{12}}-\&c\\
-\frac{C^2Iz^{12}}{g^{12}}+\&c\\
+\frac{Lz^{10}}{g^{10}}+\frac{2CLz^{12}}{g^{12}}-\&c\\
-\frac{Nz^{12}}{g^{12}}-\&c\\
\hline
1+\frac{3Czz}{gg}-\frac{3Ez^4}{g^4}+\frac{3Gz^6}{g^6}-\frac{3Iz^8}{g^8}+\frac{3Lz^{10}}{g^{10}}-\frac{3Nz^{12}}{g^{12}}+\&c\\
+\frac{3C^2z^4}{g^4}-\frac{6CEz^6}{g^6}+\frac{6CGz^8}{g^8}-\frac{6CIz^{10}}{g^{10}}+\frac{6CLz^{12}}{g^{12}}-\&c\\
+\frac{C^3z^6}{g^6}+\frac{3E^2z^8}{g^8}-\frac{6EGz^{10}}{g^{10}}+\frac{6EIz^{12}}{g^{12}}-\&c\\
-\frac{3C^2Ez^8}{g^8}+\frac{3C^2Gz^{10}}{g^{10}}+\frac{3G^2z^{12}}{g^{12}}-\&c\\
+\frac{3CE^2z^{10}}{g^{10}}-\frac{3C^2Iz^{12}}{g^{12}}+\&c\\
-\frac{6CEGz^{12}}{g^{12}}+\&c\\
-\frac{E^3z^{12}}{g^{12}}-\&c.
\end{array}$$

This last compound series (which, for the sake of brevity, we will denote by the small Greek letter γ) is the cube of the series $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+\&c$ *ad infinitum.* Therefore $8g\times$ the cube of the said series $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+\&c$ *ad infinitum* will be $=8g\times$ the compound series γ; and consequently the cube of the expression $2\sqrt[3]{g}\times$ the infinite series $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+\&c$ will be $=8g\times$ the compound series γ.

72. If the foregoing compound series γ be actually multiplied into $8g$, the product will be the following compound series, to wit,

$$\begin{array}{r}
8g+\frac{24Czz}{g}-\frac{24Ez^4}{g^3}+\frac{24Gz^6}{g^5}-\frac{24Iz^8}{g^7}+\frac{24Lz^{10}}{g^9}-\frac{24Nz^{12}}{g^{11}}+\&c\\
+\frac{24C^2z^4}{g^3}-\frac{48CEz^6}{g^5}+\frac{48CGz^8}{g^7}-\frac{48CIz^{10}}{g^9}+\frac{48CLz^{12}}{g^{11}}-\&c\\
+\frac{8C^3z^6}{g^5}+\frac{24E^2z^8}{g^7}-\frac{48EGz^{10}}{g^9}+\frac{48EIz^{12}}{g^{11}}-\&c\\
-\frac{24C^2Ez^8}{g^7}+\frac{24C^2Gz^{10}}{g^9}+\frac{24G^2z^{12}}{g^{11}}-\&c\\
+\frac{24CE^2z^{10}}{g^9}-\frac{24C^2Iz^{12}}{g^{11}}+\&c\\
-\frac{48CEGz^{12}}{g^{11}}+\&c\\
-\frac{8E^3z^{12}}{g^{11}}-\&c;
\end{array}$$

which,

which, for the ſake of brevity, we ſhall denote by the ſmall Greek letter δ. And then the cube of the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum* will be equal to the ſaid compound ſeries δ.

A compariſon between the compound ſeries δ, obtained in the foregoing art. 72, *and the compound ſeries Δ obtained above in art.* 41.

73. If we compare the laſt-mentioned compound ſeries δ with the compound ſeries Δ obtained above in art. 41, we ſhall find that the co-efficients of the correſponding terms are the ſame in both ſerieſes, and likewiſe that the ſigns prefixed to the correſponding terms, or members, of the third, and fifth, and ſeventh, vertical columns of terms in both ſerieſes are the ſame, but that the ſigns prefixed to the ſecond terms of the ſaid two ſerieſes, to wit, the terms $\frac{24Css}{e}$ and $\frac{24Czz}{g}$, and to the correſponding terms, or members, of their fourth and ſixth vertical columns of terms, are contrary to each other. And the ſame agreement of the ſigns of the terms of theſe two ſerieſes will take place in all the following *odd* vertical columns of terms after the ſeventh; and the ſame contrariety between the ſigns of the terms of the ſaid ſerieſes will take place in all the following *even* vertical columns of terms in them after the ſixth terms, to whatever number of terms the ſaid ſerieſes may be continued; as might be ſhewn by reaſonings ſimilar to thoſe contained in art. 65, 66, 67, 68, and 69.

74. Since the co-efficients of the ſeveral terms of the compound ſeries δ are equal to the co-efficients of the correſpondent terms of the compound ſeries Δ; and the ſigns prefixed to the correſponding terms of the ſaid two ſerieſes are exactly the ſame in the third, and fifth, and ſeventh, and other following *odd* vertical columns of terms in the ſaid ſerieſes, but are contrary to each other in the *ſecond* terms of the ſaid ſerieſes, and in the correſponding terms of the fourth, and ſixth, and other following *even* vertical columns of the ſaid ſerieſes; it follows that, if we reduce the ſeveral columns of terms in the compound ſeries Δ into ſingle terms by making the neceſſary multiplications, additions, and ſubtractions (as is done above in art. 46, 47, 48, 49, 50, and 51), and denote the co-efficients of the ſingle terms thereby obtained by the capital letters P, Q, R, S, T, V, &c, and, if we likewiſe reduce the ſeveral columns of terms in the compound ſeries δ into ſingle terms by making the neceſſary multiplications, additions, and ſubtractions, ſo as to convert the ſaid compound ſeries δ into a ſimple ſeries; the co-efficients of the terms of this latter ſimple ſeries, which will be equal to the compound ſeries δ, will be equal to the co-efficients of the correſponding terms of the former ſimple ſeries which is equal to the compound

 ſeries

ſeries Δ; and the ſigns $+$ and $-$ that are to be prefixed to the third, and fifth, and ſeventh, and other following odd terms of the latter ſimple ſeries, will be the ſame as thoſe which are to be prefixed to the third, and fifth, and ſeventh, and other following odd terms of the former ſimple ſeries, which is equal to the compound ſeries Δ; but the ſigns to be prefixed to the ſecond, and fourth, and ſixth, and other following even terms of this ſecond ſimple ſeries will be contrary to thoſe which are to be prefixed to the ſecond, and fourth, and ſixth, and other following even terms of the former ſimple ſeries, which is equal to the compound ſeries Δ.

75. Now it has been ſhewn in art. 60 that y^3 is equal to the ſimple ſeries $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum*. Therefore the compound ſeries Δ (which is equal to y^3) muſt likewiſe be equal to the ſimple ſeries $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum*. It follows therefore, from the foregoing article 74, that the compound ſeries δ will be equal to the ſimple ſeries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*. Therefore the cube of the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c (which is equal to the compound ſeries δ) will alſo be equal to the ſimple ſeries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*.

The value of the co-efficient q of the ſimple power of x in the cubick equation $x^3 - qx = t$ expreſſed by an infinite ſeries involving the powers of z and g.

76. In the next place we will find a tranſcendental expreſſion, involving the powers of z and g, for q, the co-efficient of x in the propoſed cubick equation $x^3 - qx = t$; after which we ſhall multiply the ſaid value of q into the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ $+$ &c, which we have aſſerted to be equal to the value of x in the equation $x^3 - qx = t$. And, if it can be ſhewn that the ſaid product of the multiplication of the expreſſion which is equal to q into the ſaid expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c is leſs than the cube of the ſaid laſt-mentioned expreſſion, or than the ſimple ſeries $8e + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c (which has been ſhewn in the laſt article 75 to be equal to the ſaid cube), and that the difference by which it

3 falls

falls ſhort of the ſaid cube, or of the ſaid ſeries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$, is equal to t (which is the abſolute term of the equation $x^3 - qx = t$) or to $2 \times \frac{t}{2}$, or to $2g$, or that the ſaid product is equal to the ſimple ſeries $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$, which is leſs than the ſeries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$ by the difference $2g$, or t, we ſhall be able juſtly to conclude that the ſaid expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ is equal to the root x of the propoſed equation $x^3 - qx = t$. This therefore is what we muſt now endeavour to prove: and, for this purpoſe, we muſt, in the firſt place, find a tranſcendental expreſſion for the co-efficient q, which ſhall involve in it only the powers of g and z. Now this may be done in the manner following.

Since zz is $= \frac{q^3}{27} - \frac{tt}{4}$, and gg is $= \frac{tt}{4}$, we ſhall have $zz = \frac{q^3}{27} - gg$, and conſequently $gg + zz = \frac{q^3}{27}$, and $q^3 = 27 \times \overline{gg + zz} = 27 \times gg \times \overline{1 + \frac{zz}{gg}}$, and $q = 3 \times \sqrt[3]{gg} \times \sqrt[3]{1 + \frac{zz}{gg}} = 3 \times g^{\frac{2}{3}} \times \overline{1 + \frac{zz}{gg}}|^{\frac{1}{3}}$. But, by the binomial theorem in the caſe of roots, $\overline{1 + \frac{zz}{gg}}|^{\frac{1}{3}}$ is $=$ the infinite ſeries $1 + \frac{1}{3} \times A \times \frac{zz}{gg} - \frac{2}{6} B \times \frac{z^4}{g^4} + \frac{5}{9} C \times \frac{z^6}{g^6} - \frac{8}{12} D \times \frac{z^8}{g^8} + \frac{11}{15} E \times \frac{z^{10}}{g^{10}} - \frac{14}{18} F \times \frac{z^{12}}{g^{12}} + \&c$ *ad infinitum*, or $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ *ad infinitum*. Therefore q will be $= 3 \times g^{\frac{2}{3}} \times$ the infinite ſeries $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ *ad infinitum*. Q. E. I.

77. Therefore the product of the multiplication of q into the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ will be $= 3 \times g^{\frac{2}{3}} \times$ the infinite ſeries $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \times 2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c = 3 \times g^{\frac{2}{3}} \times 2 \times g^{\frac{1}{3}} \times$ the infinite ſeries $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Fz^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c = 6g \times$ the infinite ſeries $1 +$

$\frac{Bzz}{gg}$

$\frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c.$ We muſt therefore now proceed to multiply the ſeries $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ into the ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$; which may be done in the manner following.

The multiplication of the infinite ſeries $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ ad infinitum *into the infinite ſeries* $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ ad infinitum.

$$\begin{array}{llllllll}
1 & + \frac{Bzz}{gg} & - \frac{Cz^4}{g^4} & + \frac{Dz^6}{g^6} & - \frac{Ez^8}{g^8} & + \frac{Fz^{10}}{g^{10}} & - \frac{Gz^{12}}{g^{12}} & + \&c \\
1 & + \frac{Czz}{gg} & - \frac{Ez^4}{g^4} & + \frac{Gz^6}{g^6} & - \frac{Iz^8}{g^8} & + \frac{Lz^{10}}{g^{10}} & - \frac{Nz^{12}}{g^{12}} & + \&c \\
\hline
1 & + \frac{Bzz}{gg} & - \frac{Cz^4}{g^4} & + \frac{Dz^6}{g^6} & - \frac{Ez^8}{g^8} & + \frac{Fz^{10}}{g^{10}} & - \frac{Gz^{12}}{g^{12}} & + \&c \\
 & + \frac{Czz}{gg} & + \frac{BCz^4}{g^4} & - \frac{C^2z^6}{g^6} & + \frac{CDz^8}{g^8} & - \frac{CEz^{10}}{g^{10}} & + \frac{CFz^{12}}{g^{12}} & - \&c \\
 & & - \frac{Ez^4}{g^4} & - \frac{BEz^6}{g^6} & + \frac{CEz^8}{g^8} & - \frac{DEz^{10}}{g^{10}} & + \frac{E^2z^{12}}{g^{12}} & - \&c \\
 & & & + \frac{Gz^6}{g^6} & + \frac{BGz^8}{g^8} & - \frac{CGz^{10}}{g^{10}} & + \frac{DGz^{12}}{g^{12}} & - \&c \\
 & & & & - \frac{Iz^8}{g^8} & - \frac{BIz^{10}}{g^{10}} & + \frac{CIz^{12}}{g^{12}} & - \&c \\
 & & & & & + \frac{Lz^{10}}{g^{10}} & + \frac{BLz^{12}}{g^{12}} & - \&c \\
 & & & & & & - \frac{Nz^{12}}{g^{12}} & - \&c.
\end{array}$$

Therefore the product of the multiplication of the expreſſion $2\sqrt{^3}\, \overline{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ into the coefficient q will be equal to $6g \times$ the compound ſeries juſt now obtained, to wit, the ſeries

1 +

$$\begin{array}{llllllll}
1 & + \frac{Bzz}{gg} & - \frac{Cz^4}{g^4} & + \frac{Dz^6}{g^6} & - \frac{Ez^8}{g^8} & + \frac{Fz^{10}}{g^{10}} & - \frac{Gz^{12}}{g^{12}} & + \&c \\
 & + \frac{Czz}{gg} & + \frac{BCz^4}{g^4} & - \frac{C^2z^6}{g^6} & + \frac{CDz^8}{g^8} & - \frac{CEz^{10}}{g^{10}} & + \frac{CFz^{12}}{g^{12}} & - \&c \\
 & & - \frac{Ez^4}{g^4} & - \frac{BEz^6}{g^6} & + \frac{CEz^8}{g^8} & - \frac{DEz^{10}}{g^{10}} & + \frac{E^2z^{12}}{g^{12}} & - \&c \\
 & & & + \frac{Gz^6}{g^6} & + \frac{BGz^8}{g^8} & - \frac{CGz^{10}}{g^{10}} & + \frac{DGz^{12}}{g^{12}} & - \&c \\
 & & & & - \frac{Iz^8}{g^8} & - \frac{BIz^{10}}{g^{10}} & + \frac{CIz^{12}}{g^{12}} & - \&c \\
 & & & & & + \frac{Lz^{10}}{g^{10}} & + \frac{BLz^{12}}{g^{12}} & - \&c \\
 & & & & & & - \frac{Nz^{12}}{g^{12}} & - \&c.
\end{array}$$

or, if, for the ſake of brevity, we denote this compound ſeries by the ſmall Greek letter λ, the ſaid product will be equal to $6g \times$ the ſaid compound ſeries λ. Q. E. I.

A compariſon between the foregoing compound ſeries λ and the compound ſeries Λ obtained above in art. 42.

78. If we compare this compound ſeries λ with the compound ſeries Λ obtained above in art. 42, we ſhall find that there are the following reſemblances and differences between them.

In the firſt place, the firſt terms of both theſe compound ſerieſes are equal to the ſame quantity, to wit, 1.

Secondly, the ſecond, and third, and fourth, and fifth, and all the following vertical columns of terms of the latter compound ſeries λ, obtained in the foregoing art. 77, involve in them the ſeveral fractions $\frac{zz}{gg}$, $\frac{z^4}{g^4}$, $\frac{z^6}{g^6}$, $\frac{z^8}{g^8}$, $\frac{z^{10}}{g^{10}}$, $\frac{z^{12}}{g^{12}}$, &c, or the ſeveral ſucceſſive powers of the fraction $\frac{zz}{gg}$, juſt as the ſecond, and third, and fourth, and fifth, and other following vertical columns of terms of the former compound ſeries Λ, obtained in art. 42, involve in them the ſeveral fractions $\frac{ss}{ee}$, $\frac{s^4}{e^4}$, $\frac{s^6}{e^6}$, $\frac{s^8}{e^8}$, $\frac{s^{10}}{e^{10}}$, $\frac{s^{12}}{e^{12}}$, or the ſeveral ſucceſſive powers of the fraction $\frac{ss}{ee}$. And it is evident that this obſervation will be true of all the following terms of theſe two compound ſerieſes, to whatever number of terms the ſaid ſerieſes may be continued, as well as of the terms ſet down in art. 42 and 77, becauſe the two ſimple ſerieſes $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c (by the multiplication of which the compound ſeries λ is produced) contain in their terms the very ſame powers of the fraction $\frac{zz}{gg}$ as the two former ſimple ſerieſes $1 - \frac{Bss}{ee}$

$- \frac{Cs^4}{e^4}$

$-\frac{Cs^4}{e^4}-\frac{Ds^6}{e^6}-\frac{Es^8}{e^8}-\frac{Fs^{10}}{e^{10}}-\frac{Gs^{12}}{e^{12}}-$ &c and $1-\frac{Css}{ee}-\frac{Es^4}{e^4}-\frac{Gs^6}{e^6}-\frac{Is^8}{e^8}-\frac{Ls^{10}}{e^{10}}-\frac{Ns^{12}}{e^{12}}-$ &c (by the multiplication of which the former compound series Λ was produced) contain in their terms of the fraction $\frac{ss}{ee}$.

Thirdly, the co-efficients of the several terms in the compound series λ will be equal to the co-efficients of the corresponding terms, or terms placed in the same situations in the several horizontal lines, of the compound series Λ, though they will not every where have the same signs + and — prefixed to them. For these co-efficients are the products of the co-efficients of the terms of the simple serieses, by the multiplication of which into each other the said compound serieses Λ and λ are produced: and therefore, as the co-efficients of the powers of the fraction $\frac{zz}{gg}$ in the terms of the two simple serieses $1+\frac{Bzz}{gg}-\frac{Cz^4}{g^4}+\frac{Dz^6}{g^6}-\frac{Ez^8}{g^8}+\frac{Fz^{10}}{g^{10}}-\frac{Gz^{12}}{g^{12}}+$ &c and $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+$ &c (by the multiplication of which the compound series λ is produced) are the very same with the co-efficients of the powers of the fraction $\frac{ss}{ee}$ in the terms of the two simple serieses $1-\frac{Bss}{ee}-\frac{Cs^4}{e^4}-\frac{Ds^6}{e^6}-\frac{Es^8}{e^8}-\frac{Fs^{10}}{e^{10}}-\frac{Gs^{12}}{e^{12}}-$ &c and $1-\frac{Css}{ee}-\frac{Es^4}{e^4}-\frac{Gs^6}{e^6}-\frac{Is^8}{e^8}-\frac{Ls^{10}}{e^{10}}-\frac{Ns^{12}}{e^{12}}-$ &c (by the multiplication of which the compound series Λ is produced) it follows that the several co-efficients of the terms of the compound series λ must be the same combinations of the several original co-efficients B, C, D, E, F, G, I, L, N, &c, as the co-efficients of the corresponding terms of the compound series Λ are of the same original co-efficients, and consequently that they must be equal to the said co-efficients of the corresponding terms of the compound series Λ. Q. E. D.

Fourthly, the signs + and — that are prefixed to the terms contained in the third, and fifth, and seventh, vertical columns of the compound series λ, are the same as those which are prefixed to the corresponding terms contained in the third, and fifth, and seventh vertical columns of the compound series Λ, as will be evident upon the inspection of the said two compound serieses. And the same similitude will take place between the signs prefixed to the terms contained in the following odd vertical columns, after the seventh, of the compound series λ, and those prefixed to the corresponding terms of the following odd vertical columns, after the seventh, of the compound series Λ; as we shall presently endeavour to make appear.

And, fifthly, the signs + and — that are prefixed to the terms contained in the second, and fourth, and sixth vertical columns of the compound series λ are every where contrary to those which are prefixed to the corresponding terms of the second, and fourth, and sixth vertical columns of the compound series Λ; as will be evident upon the inspection of the said two compound serieses. And the same contrariety will take place between the signs to be prefixed to the terms contained in the following even vertical columns, after the sixth, of the compound

compound series λ and the signs to be prefixed to the corresponding terms of the following even vertical columns, after the sixth, of the compound series Λ; as we shall now endeavour to make appear.

79. In the compound series Λ, or the product of the multiplication of the series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c (of which all the terms, after the first term 1, are marked with the sign —) into the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c (of which all the terms, after the first term 1, are likewise marked with the sign —) set down above in art. 42, it is evident that all the terms, after the first term 1, of the first, or highest, horizontal row of terms must be marked with the sign —, to whatever number of terms the said horizontal row of terms may be continued; and it is evident likewise, that in all the following horizontal rows of terms in the said product the first terms of the said horizontal rows (being the products of the multiplication of 1 by $-\frac{Css}{ee}, -\frac{Es^4}{e^4}, -\frac{Gs^6}{e^6}, -\frac{Is^8}{e^8}, -\frac{Ls^{10}}{e^{10}}, -\frac{Ns^{12}}{e^{12}}$, &c) must be marked with the sign —, but that all the following terms of the said horizontal rows, to whatever number of terms the said rows may be continued, will be marked with the sign +, because they are the products of the multiplication of the several terms $-\frac{Bss}{ee}, -\frac{Cs^4}{e^4}, -\frac{Ds^6}{e^6}, -\frac{Es^8}{e^8}, -\frac{Fs^{10}}{e^{10}}, -\frac{Gs^{12}}{e^{12}}$, &c, as multiplicands, into the several terms $-\frac{Css}{ee}, -\frac{Es^4}{e^4}, -\frac{Gs^6}{e^6}, -\frac{Is^8}{e^8}, -\frac{Ls^{10}}{e^{10}}, -\frac{Ns^{12}}{e^{12}}$, &c, as multiplicators, and, wherever the sign — is prefixed to both the factors of any multiplication, the sign + is to be prefixed to the product. But the second, and third, and other following terms of the first, or highest, horizontal row of terms in the said product, are the highest terms of the second, and third, and other following vertical columns of terms in it; and the first terms of the second, and third, and other following horizontal rows of terms of the said product are the lowest terms of the second, and third, and other following vertical columns of terms in the same. Therefore, the highest and the lowest terms in the second, and the third, and every following vertical column of the said product will be marked with the sign —, and all the intermediate terms in the said vertical columns between the highest and the lowest terms will be marked with the sign +.

These will be the signs to be prefixed to the several terms of the compound series Λ, or the product, set down above in art. 42, of the multiplication of the series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c *ad infinitum* into the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum*. We must now enquire what signs are to be prefixed to the several corresponding terms of the compound series λ, or the product, set down in art. 77, of the multiplication of the series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c. *ad infinitum* into

into the ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum.*

80. Now the firſt, or higheſt, horizontal row of terms in this product, to wit, $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c, conſiſts of the very ſame terms as the multiplicand, being the product of the multiplication of the ſaid multiplicand into 1, which will make no change either in the terms of the multiplicand themſelves, or in the ſigns to be prefixed to them. Therefore in this firſt, or higheſt, horizontal row of terms, the ſecond, and fourth, and ſixth, and eighth, and tenth, terms, and all the following even terms (to whatever number of terms the ſaid horizontal row of terms may be continued) will be marked with the ſign +, and the third, and fifth, and ſeventh, and ninth, and eleventh, terms, and all the following odd terms (whatever be their number) will be marked with the ſign —. But the ſecond, and fourth, and ſixth, and other following even terms of the firſt, or higheſt, horizontal row of terms in the ſaid product, are the firſt, or higheſt, terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the ſaid product; and the third, and fifth, and ſeventh, and other following odd terms of the ſaid firſt, or higheſt, horizontal row of terms in the ſaid product are the firſt, or higheſt, terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the ſaid product. Therefore the firſt, or higheſt, terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of the ſaid product will be marked with the ſign +; and the firſt, or higheſt, terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the ſaid product will be marked with the ſign —.

81. We will next enquire what ſigns muſt be prefixed to the laſt, or loweſt, terms of the ſeveral vertical columns of the ſaid product.

Now, becauſe the firſt term of the ſeries which is the multiplicand in the multiplication ſet down above in art. 77, is 1, and the ſecond, and fourth, and ſixth, and other following even terms of the ſeries which is the multiplicator in that multiplication, are marked with the ſign +, and the third, and fifth, and ſeventh, and other following odd terms of the ſaid multiplicator are marked with the ſign —, it follows that in the ſecond, and fourth, and ſixth, and other following even horizontal rows of the ſaid product, the ſigns + and — that are to be prefixed to the ſeveral terms in the ſaid rows will be the ſame as thoſe of the correſponding terms of the multiplicand, and conſequently the two firſt terms of each of the ſaid even horizontal rows will be marked with the ſign +, and the third, and fourth, and fifth, and ſixth, and other following terms of the ſaid even horizontal rows will be marked with the ſign — and the ſign + alternately; and it follows alſo, that in the third, and fifth, and ſeventh, and other following odd horizontal rows of terms, the ſigns of the terms will be contrary to thoſe of the correſponding terms of the multiplicand, and conſequently the two firſt terms of each of the ſaid odd horizontal rows of terms will be marked with the

the ſign —, and the third, and fourth, and fifth, and ſixth, and other following terms of the ſaid odd horizontal rows of terms will be marked with the ſign + and the ſign — alternately. But the firſt terms of the ſecond, and third, and fourth, and other following horizontal rows of terms are the loweſt terms of the ſecond, and third, and fourth, and other following vertical columns of terms. Therefore the loweſt terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the ſaid product will be marked with the ſign + ; and the loweſt terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the ſaid product will be marked with the ſign —.

82. It appears therefore that in the third, and fifth, and ſeventh, and other following odd vertical columns of the ſaid compound ſeries λ, or product, ſet down in art. 77, both the higheſt and the loweſt terms will have the ſign — prefixed to them, as is the caſe in the compound ſeries Λ, or the product ſet down above in art. 42 ; and that in the ſecond, and fourth, and ſixth, and other following even vertical columns of the ſaid compound ſeries λ, or product ſet down in art. 77, both the higheſt and loweſt terms will be marked with the ſign +, which is contrary to the ſign which is prefixed to the higheſt and loweſt terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the compound ſeries Λ, or product ſet down above in art. 42. It remains that we enquire into the ſigns that are to be prefixed to the ſeveral intermediate terms between the higheſt and the loweſt terms of the ſeveral vertical columns of terms in the ſaid compound ſeries λ, or product ſet down in art. 77.

83. Now, wherever the higheſt term of one of the vertical columns of the product ſet down in art. 77 is marked with the ſign — (which has been ſhewn to be the caſe in the third, and fifth, and ſeventh, and all the following odd vertical columns), all the intermediate terms of the ſaid column between the higheſt term and the loweſt will be marked with the ſign +, becauſe they will be the product of two factors which are both marked with the ſame ſign, to wit, firſt with the ſign +, then with the ſign —, then with the ſign +, and then with the ſign — again, and ſo on alternately. Thus, for example, in the ſeventh vertical column of the ſaid product, the higheſt term is $-\frac{Gz^{12}}{g^{12}}$, which is the product of the multiplication of $-\frac{Gz^{12}}{g^{12}}$, the ſeventh term of the multiplicand, by 1, the firſt term of the multiplicator; and the ſecond term of the ſaid vertical column is $+\frac{CFz^{12}}{g^{12}}$, which is marked with the ſign +, becauſe it is the product of the multiplication of $\frac{Fz^{10}}{g^{10}}$, the ſixth term of the multiplicand, by $\frac{Czz}{gg}$, the ſecond term of the multiplicator, which terms are both marked with the ſign + ; and the third term of the ſaid vertical column is $+\frac{E^2z^{12}}{g^{12}}$, which is marked

marked with the ſign +, becauſe it is the product of the multiplication of $\frac{Ez^8}{g^8}$, the fifth term of the multiplicand, by $\frac{Ez^4}{g^4}$, the third term of the multiplicator, which terms are both marked with the ſign —; and the fourth term of the ſaid vertical column is $+\frac{DGz^{12}}{g^{12}}$, which is marked with the ſign +, becauſe it is the product of the multiplication of $\frac{Dz^6}{g^6}$, the fourth term of the multiplicand, by $\frac{Gz^6}{g^6}$, the fourth term of the multiplicator, which terms are both marked with the ſign +; and the fifth term of the ſaid vertical column is $+\frac{CIz^{12}}{g^{12}}$, which is in like manner marked with the ſign +, becauſe it is the product of the multiplication of $\frac{Cz^4}{g^4}$, the third term of the multiplicand, by $\frac{Iz^8}{g^8}$, the fifth term of the multiplicator, which terms are both marked with the ſign —; and the ſixth term of the ſaid vertical column, or the loweſt term but one, of the ſaid column, or the laſt of the intermediate terms of the ſaid column between its higheſt and its loweſt terms, is $+\frac{BLz^{12}}{g^{12}}$, which is alſo marked with the ſign +, becauſe it is the product of the multiplication of $\frac{Bzz}{gg}$, the ſecond term of the multiplicand, by $\frac{Lz^{10}}{g^{10}}$, the ſixth term of the multiplicator, which terms are both marked with the ſame ſign +. And thus it appears, that all the intermediate terms between the higheſt and loweſt terms of the ſaid ſeventh vertical column muſt be marked with the ſign +, becauſe the two factors, by the multiplication of which they are produced, are always marked with the ſame ſign, whether + or —. And it is eaſy to ſee, from the alternate ſucceſſion of the ſigns + and — to each other, in both the multiplicand and multiplicator, that the ſame thing muſt take place with reſpect to the intermediate terms of any other vertical column of terms of which the higheſt term is marked with the ſign —, that is, of any other odd vertical column. We may therefore conclude that in all the odd vertical columns of the compound ſeries λ, or the product ſet down in art. 77 (to whatever number of terms the ſaid product may be continued), the higheſt and loweſt terms of the ſaid vertical columns will be marked with the ſign —, and all the intermediate terms will be marked with the ſign +, and conſequently (by what is ſhewn in art. 79) that the ſigns of the ſeveral terms of all the ſaid odd vertical columns of this product, or of the compound ſeries λ, will be the ſame with thoſe of the correſponding terms of the like odd vertical columns of terms in the product ſet down above in art 42, or in the compound ſeries Λ; or that the fourth obſervation made above in art. 78 is univerſally true of the terms contained in all the odd vertical columns of the ſaid two products, or compound ſerieſes Λ and λ, ſet down in art. 42 and 77 after the ſeventh vertical columns (to whatever number of columns the ſaid ſerieſes may be continued), as well as of the terms contained in their third, and fifth, and ſeventh vertical columns.

Q. E. D.

6 84. In

84. In the ſeveral even vertical columns of the compound ſeries λ, ſet down in art. 77, the ſigns + and —, that are to be prefixed to the terms of the ſaid columns, will be every where contrary to thoſe which are to be prefixed to the correſponding terms of the like even vertical columns of the compound ſeries Λ, ſet down above in art. 42. For it has been already ſhewn in art. 80 and 81, that the higheſt and loweſt terms of the ſecond, and fourth, and ſixth, and all the following even vertical columns of terms in the ſaid compound ſeries λ (to whatever number they may be continued) will be marked with the ſign +, which is contrary to that with which the higheſt and loweſt terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of terms, in the compound ſeries Λ, are marked. And all the intermediate terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of the compound ſeries λ, between the higheſt and the loweſt terms of the ſaid columns, muſt be marked with the ſign —, becauſe they will be the products of two factors which are marked with different ſigns + and —. Thus, for example, in the ſixth vertical column of the ſaid product, the higheſt term is $+ \frac{\text{F}z^{10}}{g^{10}}$, which is the product of the multiplication of $+ \frac{\text{F}z^{10}}{g^{10}}$, the ſixth term of the multiplicand, by 1, the firſt term of the multiplicator; and the ſecond term of the ſaid vertical column is $- \frac{\text{CE}z^{10}}{g^{10}}$, which is marked with the ſign —, becauſe it is the product of the multiplication of $\frac{\text{E}z^{8}}{g^{8}}$, the fifth term of the multiplicand, which is marked with the ſign —, by $\frac{\text{C}zz}{gg}$, the ſecond term of the multiplicator, which is marked with the ſign +; and the third term of the ſaid vertical column is $- \frac{\text{DE}z^{10}}{g^{10}}$, which is alſo marked with the ſign —, becauſe it is the product of the multiplication of $\frac{\text{D}z^{6}}{g^{6}}$, the fourth term of the multiplicand, which is marked with the ſign +, by $\frac{\text{E}z^{4}}{g^{4}}$, the third term of the multiplicator, which is marked with the ſign —; and the fourth term of the ſaid vertical column is $- \frac{\text{CG}z^{10}}{g^{10}}$, which is alſo marked with the ſign —, becauſe it is the product of the multiplication of $\frac{\text{C}z^{4}}{g^{4}}$, the third term of the multiplicand, which is marked with the ſign —, by $\frac{\text{G}z^{6}}{g^{6}}$, the fourth term of the multiplicator, which is marked with the ſign +; and the fifth term of the ſaid vertical column, or the loweſt term but one of the ſaid column, or the laſt of the intermediate terms of the ſaid column between the higheſt term and the loweſt, is $- \frac{\text{BI}z^{12}}{g^{12}}$, which is alſo marked with the ſign —, becauſe it is the product of the multiplication of $\frac{\text{B}zz}{gg}$, the ſecond term of the multiplicand, which is marked with the ſign +, by $\frac{\text{I}z^{8}}{g^{8}}$, the fifth term of the multiplicator, which is marked with the ſign —. And thus it appears that all the intermediate terms between the higheſt and the loweſt terms of

of the fixth vertical column muft be marked with the fign —, becaufe the two factors, by the multiplication of which they are produced, are always marked with contrary figns. And it is eafy to fee, from the alternate fucceffion of the figns + and — to each other in both the multiplicand and the multiplicator, that the fame thing muft take place with refpect to the intermediate terms of every other vertical column of which the higheft term is marked with the fign +, that is, of every other even vertical column. We may therefore conclude, that in all the even vertical columns of the compound feries λ, or the product fet down in art. 77 (to whatever number of terms the faid product may be continued) the higheft and loweft terms will be marked with the fign +, and all the intermediate terms will be marked with the fign —, and confequently (by what is fhewn in art. 79) that the figns of the feveral terms of all the faid even vertical columns of terms in this product, or compound feries λ, will be every where contrary to the figns of the correfponding terms of the like even vertical columns of terms in the compound feries Λ, or the product fet down above in art. 42; or that the fifth obfervation made above in art. 78 is univerfally true of the terms contained in all the even vertical columns of terms of the faid two products, or compound feriefes Λ and λ, fet down above in art. 42 and 77 (to whatever number of columns of terms the faid feriefes may be continued), as well as of the terms contained in their fecond, and fourth, and fixth vertical columns. Q. E. D.

85. If the foregoing compound feries λ, fet down in art. 77, be actually multiplied into $6g$, the product thence arifing will be the following compound feries, to wit,

$$\begin{array}{llllllll}
6g + \frac{6Bzz}{g} & - \frac{6Cz^4}{g^3} & + \frac{6Dz^6}{g^5} & - \frac{6Ez^8}{g^7} & + \frac{6Fz^{10}}{g^9} & - \frac{6Gz^{12}}{g^{11}} & + \&c \\
\quad + \frac{6Czz}{g} & + \frac{6BCz^4}{g^3} & - \frac{6C^2z^6}{g^5} & + \frac{6CDz^8}{g^7} & - \frac{6CEz^{10}}{g^9} & + \frac{6CFz^{12}}{g^{11}} & - \&c \\
 & - \frac{6Ez^4}{g^3} & - \frac{6BEz^6}{g^5} & + \frac{6CEz^8}{g^7} & - \frac{6DEz^{10}}{g^9} & + \frac{6E^2z^{12}}{g^{11}} & - \&c \\
 & & + \frac{6Gz^6}{g^5} & + \frac{6BGz^8}{g^7} & - \frac{6CGz^{10}}{g^9} & + \frac{6DGz^{12}}{g^{12}} & - \&c \\
 & & & - \frac{6Iz^8}{g^7} & - \frac{6BIz^{10}}{g^9} & + \frac{6CIz^{12}}{g^{11}} & - \&c \\
 & & & & + \frac{6Lz^{10}}{g^9} & + \frac{6BLz^{12}}{g^{11}} & - \&c \\
 & & & & & - \frac{6Nz^{12}}{g^{11}} & - \&c;
\end{array}$$

which, for the fake of brevity, we will denote by the fmall Greek letter π. Then will the product of the multiplication of the expreffion $2\sqrt{^3}\,\overline{g}$ × the feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum* into the coefficient q be equal to the compound feries π.

We muft now compare this compound feries π with the compound feries Π obtained above in art. 43.

A Com-

A Comparison between the foregoing compound series π and the compound series Π *obtained above in art.* 43.

86. The compound series Π is equal to $6e \times$ the compound series Λ; and the compound series π is equal to $6g \times$ the compound series λ. Therefore the signs $+$ and $-$ which are to be prefixed to the terms of the compound series Π will be the same with the signs which are prefixed to the corresponding terms of the compound series Λ; and the signs $+$ and $-$, which are to be prefixed to the terms of the compound series π will be the same with the signs which are to be prefixed to the correspondent terms of the compound series λ; because the multiplications of the terms of the two compound serieses Λ and λ into $6e$ and $6g$ cannot make any change in the signs of the terms which are multiplied. It follows therefore that there will be the same similitude and the same differences between the signs $+$ and $-$ that are to be prefixed to the several terms of the two compound serieses Π and π (which are equal to $6e \times \Lambda$ and $6g \times \lambda$) as there are between the signs which are to be prefixed to the terms of the two former compound serieses Λ and λ. But it has been shewn in art. 78, 79, 80, 81, 82, 83, 84, that the signs $+$ and $-$ that are to be prefixed to the several terms of the third, fifth, seventh, and other following odd vertical columns of terms in the compound series λ are the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Λ; and that the signs $+$ and $-$ which are to be prefixed to the several terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series λ are uniformly contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series Λ. It therefore follows that the signs $+$ and $-$ which are to be prefixed to the several terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series π will be the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Π; and that the signs $+$ and $-$, which are to be prefixed to the several terms of the second, and fourth, and sixth, and other following even vertical columns of the compound series π will be uniformly contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of the compound series Π.

87. It has been shewn in the third observation of art. 78, that the co-efficients of the several terms of the compound series λ are equal to the co-efficients of the corresponding terms of the compound series Λ. Therefore, if the co-efficients of all the terms of both these serieses Λ and λ be multiplied by the same number, the new numbers, or co-efficients, thereby produced in the one series will be equal to the corresponding new numbers, or co-efficients, thereby produced in the other series. But the co-efficients of the terms of the compound series Π are produced by the multiplication of the co-efficients of the terms

terms of the compound ſeries Λ by 6, becauſe the ſeries Π is $= 6e \times$ the ſeries Λ; and the co-efficients of the terms of the compound ſeries π are produced by the multiplication of the co-efficients of the terms of the compound ſeries λ by the ſame number 6, becauſe the ſeries π is $= 6g \times$ the ſeries λ. Therefore the co-efficients of the terms of the compound ſeries π will be equal to the co-efficients of the correſponding terms of the compound ſeries Π.

88. It appears therefore from the two laſt articles, that the co-efficients of the ſeveral terms of the compound ſeries π are equal to the co-efficients of the correſponding terms of the compound ſeries Π; and that in the third and fifth, and ſeventh, and other following odd vertical columns of terms in the compound ſeries π, the ſigns + and — that are to be prefixed to the ſeveral terms of the ſaid columns will be the ſame with thoſe which are to be prefixed to the correſponding terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the compound ſeries Π; and, laſtly, that in the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the ſaid compound ſeries π, the ſigns + and —, that are to be prefixed to the ſeveral terms of the ſaid columns, will be reſpectively contrary to thoſe which are to be prefixed to the correſponding terms of the ſecond and fourth, and ſixth, and other following even vertical columns of terms in the compound ſeries Π. Now from hence it follows that, if all the vertical columns of terms in the compound ſeries Π be reduced to ſingle terms, by the addition, or ſubtraction, of their ſeveral members according to the marks + or — which are prefixed to them, and the compound ſeries Π be thereby converted into a ſimple ſeries, or ſeries conſiſting of ſingle terms; and, if all the vertical columns of terms in the compound ſeries π be in like manner reduced to ſingle terms, by the addition and ſubtraction of their ſeveral members according to the ſigns + and — that are prefixed to them, and the ſaid compound ſeries π be thereby converted into a ſimple ſeries, or ſeries conſiſting of ſingle terms;—I ſay, it follows that, if theſe two compound ſerieſes Π and π be ſo converted into ſimple ſerieſes, the co-efficients of the ſecond, and third, and fourth, and all the following terms of the ſimple ſeries which is equal to the compound ſeries π, will be reſpectively equal to the co-efficients of the ſecond, and third, and fourth, and other following correſponding terms of the ſimple ſeries which is equal to the compound ſeries Π; and the ſigns + and — that are to be prefixed to the third, and fifth, and ſeventh, and other following odd terms of the ſimple ſeries which is equal to the compound ſeries π will be the ſame with thoſe which are to be prefixed to the third, and fifth, and ſeventh, and other following odd terms of the ſimple ſeries which is equal to the compound ſeries Π; but the ſigns + and — that are to be prefixed to the ſecond, and fourth, and ſixth, and other following even terms of the ſimple ſeries which is equal to the compound ſeries π will be reſpectively contrary to thoſe which are to be prefixed to the correſponding terms of the ſimple ſeries which is equal to the compound ſeries Π.

89. It has been ſhewn above in art. 60, that the compound ſeries Π (which is equal to y^3 in the equation $y^3 - qy = r$) is equal to the ſimple ſeries $6e -$

$\frac{ru}{e}$

$\frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}}$ — &c *ad infinitum*. It follows therefore from the foregoing art. 88, that the compound ſeries π will be equal to the ſimple ſeries $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}}$ + &c *ad infinitum*.

90. But it has been ſhewn in art. 85, that the product of the multiplication of the tranſcendental expreſſion $2\sqrt[3]{}\,(g \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c *ad infinitum* into q (the co-efficient of x in the equation $x^3 - qx = t$) is equal to the compound ſeries π. Therefore the product ariſing from the ſaid multiplication of the expreſſion $2\sqrt[3]{}\,(g \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c *ad infinitum* into the co-efficient q will be equal to the ſaid ſimple ſeries $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}}$ + &c *ad infinitum*.

91. We have juſt now ſeen that, if the tranſcendental expreſſion $2\sqrt[3]{}\,(g \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c *ad infinitum* be multiplied into q (the co-efficient of x in the equation $x^3 - qy = t$) the product ariſing from the ſaid multiplication will be equal to the ſimple ſeries $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}}$ + &c *ad infinitum*. And it was ſhewn above in art. 75, that the cube of the ſame tranſcendental expreſſion $2\sqrt[3]{}\,(g \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c *ad infinitum* is equal to the ſimple ſeries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}}$ + &c *ad infinitum*. But the ſimple ſeries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}}$ + &c *ad infinitum* is greater than the ſimple ſeries $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}}$ + &c, and the difference is $8g - 6g$, or $2g$, or $2 \times \frac{t}{2}$, or t. Therefore the cube of the ſaid tranſcendental expreſſion will be greater than the product of its multiplication into the co-efficient q, and the exceſs or difference will be equal to t, or the abſolute term of the equation $x^3 - qx = t$. Therefore the ſaid tranſcendental expreſſion $2\sqrt[3]{}\,(g \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c *ad infinitum* muſt be equal to the root x of the ſaid equation, agreeably to the aſſertion made above in art. 36. Q. E. D.

End of the Demonſtration of the Propoſition, or Theorem, laid down in art. 36.

A Re-

A Recapitulation of the Contents of the foregoing Demonſtration of the Propoſition laid down above in art. 36.

92. The foregoing demonſtration of the propoſition laid down above in art. 36 is in itſelf very ſubtle and intricate, and therefore has been ſet forth in the foregoing articles at very great length. But now, after it has, I hope, been rendered intelligible and ſatisfactory to the attentive reader, by the full expoſition that has been given of it, he will probably be glad to ſee the matter of it compreſſed into a ſmaller compaſs, by which the connection of the ſeveral ſteps of which it is compoſed will be more clearly and eaſily perceived. This therefore I ſhall now proceed to do in the following articles.

93. It has been ſhewn in art. 25 that y, the root of the cubick equation $y^3 - qy = r$ (in which the abſolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) is equal to the tranſcendental expreſſion $2\sqrt{^3}\overline{|e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum*, in which all the terms, after the firſt term 1, are marked with the ſign $-$. Let this infinite ſeries be denoted by v; and then we ſhall have $y = 2\sqrt{^3}\overline{|e} \times$ the ſeries v.

94. In this expreſſion $2\sqrt{^3}\overline{|e} \times$ the ſeries v the letter e is put for $\frac{r}{2}$, or half the abſolute term r of the equation $y^3 - qy = r$, and ss is $= \frac{rr}{4} - \frac{q^3}{27}$, or $ee - \frac{q^3}{27}$; and conſequently $ss + \frac{q^3}{27}$ is $= ee$, and $\frac{q^3}{27}$ is $= ee - ss = ee \times \overline{|1 - \frac{ss}{ee}}$, and q^3 is $= 27 \times ee \times \overline{|1 - \frac{ss}{ee}}$, and q is $= 3 \times e^{\frac{2}{3}} \times \overline{|1 - \frac{ss}{ee}|}^{\frac{1}{3}} =$ (by the reſidual theorem in the caſe of roots) $3 \times e^{\frac{2}{3}} \times$ the infinite ſeries $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c *ad infinitum*.

95. Since y is $= 2\sqrt{^3}\overline{|e} \times$ the ſeries v, it follows that y^3 will be $= 8e \times$ the ſeries v^3, which will be a compound ſeries, or ſeries conſiſting of ſeveral different horizontal lines, or rows, of terms placed one under another. And the firſt term of this ſeries v^3 muſt evidently be 1, becauſe the firſt term of the ſeries v is 1. Call this compound ſeries Γ. And we ſhall have y^3, or $8e \times v^3$, $= 8e \times$ the compound ſeries Γ.

96. Let all the terms of the compound ſeries Γ be actually multiplied into $8e$; and we ſhall thereby obtain another compound ſeries, of which the firſt term will be $8e$ (becauſe the firſt term of the compound ſeries Γ is 1), and which will be equal to y^3. Let this ſecond compound ſeries be called Δ. And we ſhall then have $y^3 =$ the compound ſeries Δ.

97. Now

97. Now let the expreſſion $2\sqrt[3]{e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c (which is equal to y) be multiplied into the expreſſion $3 \times e^{\frac{2}{3}} \times$ the infinite ſeries $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c *ad infinitum*, which is equal to the co-efficient q. And we ſhall have $qy = 2\sqrt[3]{e} \times 3 \times e^{\frac{2}{3}} \times$ the product of the ſaid two infinite ſerieſes, which product will be a certain compound ſeries, of which the firſt term will be 1. Let this compound ſeries be called Λ. Then will qy be $(= 2\sqrt[3]{e} \times 3 \times e^{\frac{2}{3}} \times$ the compound ſeries $\Lambda = 2 \times e^{\frac{1}{3}} \times 3 \times e^{\frac{2}{3}} \times$ the compound ſeries $\Lambda) =$ $6e \times$ the compound ſeries Λ.

98. Let all the terms of the compound ſeries Λ be actually multiplied into $6e$; and we ſhall thereby obtain another compound ſeries, of which the firſt term will be $6e$ (becauſe the firſt term of the compound ſeries Λ is 1), and which will be equal to qy. Let this new compound ſeries be called Π. And we ſhall then have $qy =$ the compound ſeries Π.

99. Since $y^3 - qy$ is $= r$, and y^3 is $=$ the compound ſeries Δ, and qy is $=$ the compound ſeries Π, it follows that the exceſs of the compound ſeries Δ above the compound ſeries Π will alſo be equal to r, that is, $\Delta - \Pi$ will be $= r = 2 \times \frac{r}{2} = 2e$. But the exceſs of the firſt term of the compound ſeries Δ (which is $8e$) above the firſt term of the compound ſeries Π (which is $6e$) is alſo $=$ $2e$. Therefore the exceſs of the whole compound ſeries Δ above the whole compound ſeries Π is equal to the exceſs of the firſt term of the former compound ſeries above the firſt term of the latter compound ſeries; that is, $\Delta - \Pi$ is $= 8e - 6e$. Conſequently (adding $6e$ to both ſides) we ſhall have $\Delta + 6e - \Pi = 8e$. But the compound ſeries Δ is leſs than its firſt term $8e$, becauſe the ſecond term of it is marked with the ſign —. Therefore in the laſt equation $\Delta + 6e - \Pi = 8e$ it is poſſible to ſubtract Δ from both ſides. Let it be ſo ſubtracted; and we ſhall then have $6e - \Pi = 8e - \Delta$.

100. Now let the compound ſeries $8e - \Delta$ be reduced to a ſimple ſeries, or ſeries conſiſting of ſingle terms, by making the ſeveral multiplications, additions, and ſubtractions that are neceſſary for that purpoſe; and let the ſimple ſeries, to which it is thereby reduced, be $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} +$ &c *ad infinitum*. Then will the ſaid ſimple ſeries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}}$ + &c *ad infinitum* be equal to the compound ſeries $6e - \Pi$. And, becauſe this

this equality between the ſimple ſeries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}}$ + &c *ad infinitum*, and the compound ſeries $6e - \Pi$ takes place in all the different relative magnitudes of ss and ee, or of $\frac{rr}{4} - \frac{q^3}{27}$ and $\frac{rr}{4}$, and conſequently when $\frac{rr}{4}$ approaches as near as we pleaſe to an equality with $\frac{q^3}{27}$, and $\frac{rr}{4} - \frac{q^3}{27}$ approaches as near as we pleaſe to $\frac{q^3}{27} - \frac{q^3}{27}$, or to o, and $\frac{ss}{ee}$ approaches as near as we pleaſe to $\frac{0}{ee}$, or to o, it follows that every term in the ſimple ſeries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}}$ + &c *ad infinitum* will be equal to the correſponding vertical column of terms, or column of terms involving the ſame powers of ss and e in their numerators and denominators, in the compound ſeries $6e - \Pi$. And therefore, if the compound terms, or vertical columns of terms of the ſaid compound ſeries $6e - \Pi$ be reduced to ſingle terms by making the ſeveral multiplications, additions, and ſubtractions, that are neceſſary for that purpoſe; the ſimple ſeries thence ariſing will be the foregoing ſimple ſeries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}}$ + &c *ad infinitum*, which aroſe from a ſimilar reduction of the terms of the compound ſeries $8e - \Delta$.

101. Since $8e - \Delta$ is equal to the ſimple ſeries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}}$ + &c, we ſhall have $8e = \Delta +$ the ſaid ſeries, and $\Delta = 8e -$ the ſaid ſeries = the ſeries $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}}$ — &c. Therefore y^3 (which is equal to the compound ſeries Δ) will alſo be equal to the ſeries $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}}$ — &c.

102. And, ſince the compound ſeries $6e - \Pi$ is equal to the ſaid ſimple ſeries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}}$ + &c *ad infinitum*, we ſhall have $6e = \Pi +$ the ſaid ſimple ſeries, and $\Pi = 6e -$ the ſaid ſimple ſeries, or $\Pi =$ the ſeries $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}}$ — &c *ad infinitum*. Therefore the product qy (which is $= \Pi$) will alſo be equal to the ſeries $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}}$ — &c *ad infinitum*.

103. It appears therefore that y^3 is equal to the ſeries $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}}$ — &c *ad infinitum*, and that qy is equal to the ſeries $6e$

$- \frac{Pss}{e}$

$-\frac{Pss}{e}-\frac{Qs^4}{e^3}-\frac{Rs^6}{e^5}-\frac{Ss^8}{e^7}-\frac{Ts^{10}}{e^9}-\frac{Vs^{12}}{e^{11}}-$ &c *ad infinitum*, which is leſs than the former ſeries $8e-\frac{Pss}{e}-\frac{Qs^4}{e^3}-\frac{Rs^6}{e^5}-\frac{Ss^8}{e^7}-\frac{Ts^{10}}{e^9}-\frac{Vs^{12}}{e^{11}}-$ &c *ad infinitum*, by the quantity, or difference, $8e-6e$, or $2e$, or $2\times\frac{r}{2}$, or r, as it ought to be, agreeably to the original equation $y^3-qy=r$.

104. We then endeavoured to prove that the tranſcendental expreſſion $2\sqrt[3]{g}$ $\times$ the infinite ſeries $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+$ &c *ad infinitum* (in which g is equal to $\frac{t}{2}$, or half the abſolute term of the equation $x^3-qx=t$, and zz is equal to $\frac{q^3}{27}-\frac{tt}{4}$, or $\frac{q^3}{27}-gg$) was equal to the root x of the ſame equation. And the method we took to prove the ſaid equality, was to ſhew that, if the ſaid tranſcendental expreſſion was ſubſtituted inſtead of x in the compound quantity x^3-qx (which forms the left-hand ſide of the ſaid equation) it would make the ſaid compound quantity equal to the abſolute term t, or that, if the ſaid expreſſion were to be cubed by multiplying it twice into itſelf, and were likewiſe to be multiplied into the co-efficient q, the ſaid cube would be greater than the ſaid product, and the difference would be equal to the abſolute term t. Theſe ſubſtitutions, and the reaſoning grounded on them, are as follows.

105. Let us, for the ſake of brevity, denote the infinite ſeries $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+$ &c *ad infinitum*, by the letter w, and the product of the multiplication of $2\sqrt[3]{g}$ into the ſaid ſeries by the letter m. Then it will be neceſſary for us to prove that the tranſcendental quantity m will be equal to the root x of the equation $x^3-qx=t$, or that, if we raiſe the tranſcendental quantity m to its cube, ſo as to obtain the quantity m^3, and alſo multiply m into the co-efficient q, or its value expreſſed in powers of g and z, ſo as to obtain the value of the product $q\times m$, the ſaid cube of m will be greater than the ſaid product qm by the quantity, or difference, $2g$, or $2\times\frac{t}{2}$, or t, the abſolute term of the equation $x^3-qx=t$; from which it will follow that the ſaid tranſcendental quantity m is equal to x.

106. Now, ſince m is $=2\sqrt[3]{g}\times$ the ſeries w, it follows that m^3 will be $=8g\times$ the ſeries w^3, which will be a compound ſeries, or ſeries conſiſting of ſeveral different horizontal lines, or rows, of terms placed one under another. Call this compound ſeries γ. And we ſhall have $m^3=8g\times$ the compound ſeries γ.

107. Since the terms of the ſeries w, or $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+$ &c *ad infinitum*, have the ſame co-efficients C, E, G, I, L, N, &c, as

as the correſponding terms of the ſeries v, or $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum* (though differently connected together by the ſigns + and —), it follows that the co-efficients of the ſeveral terms of the compound ſeries w^3, or γ, will be equal to the co-efficients of the correſponding terms of the compound ſeries v^3, or Γ, both ſets of terms being ſimilar combinations, or products, of the ſame original factors, or co-efficients, C, E, G, I, L, N, &c. And from the continued repetition of the ſign — in the ſecond and other following terms of the ſeries v, or $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c, and the alternate ſucceſſion of the ſign + and the ſign — in the ſecond and other following terms of the ſeries w, or $1 + \frac{Cxx}{gg} - \frac{Ex^4}{g^4} + \frac{Gx^6}{g^6} - \frac{Ix^8}{g^8} + \frac{Lx^{10}}{g^{10}} - \frac{Nx^{12}}{g^{12}} +$ &c, it follows that, in the third, and fifth, and ſeventh, and other following odd vertical columns of terms of the compound ſeries w^3, or γ, the ſigns + and — that are to be prefixed to the terms of the ſaid columns will be exactly the ſame with thoſe which are to be prefixed to the correſponding terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the compound ſeries v^3, or Γ; and that in the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the compound ſeries w^3, or γ, the ſigns + and —, that are to be prefixed to the terms of the ſaid columns, will be every where contrary to thoſe which are to be prefixed to the correſponding terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the compound ſeries v^3, or Γ.

And, as to the firſt terms of theſe two compound ſerieſes v^3 and w^3, or Γ and γ, they will be equal to each other, becauſe each of them will be equal to 1.

108. Let all the terms of the compound ſeries w^3, or γ, be multiplied by $8g$. And we ſhall thereby obtain another compound ſeries of which the firſt term will be $8g$ (becauſe the firſt term of the compound ſeries w^3, or γ, is 1) and which will be equal to m^3. Let this new compound ſeries be called δ. And we ſhall then have $m^3 =$ the compound ſeries δ.

109. Now there will evidently be the ſame analogies and the ſame differences between the compound ſeries Δ and δ, or $8e \times \Gamma$ and $8g \times \gamma$, with reſpect to the co-efficients of their ſeveral correſponding terms and the ſigns + and —, which are to be prefixed to them, as between the compound ſerieſes Γ and γ themſelves; becauſe the ſaid multiplications by $8e$ and $8g$ can make no difference at all in the ſigns of the terms multiplied, and will increaſe the co-efficients of all the ſaid terms in the ſame proportion of 8 to 1, which will not affect their antecedent equality. For as to the quantities e and g (which are involved in the multiplicators $8e$ and $8g$) the multiplications of the terms by them will not affect the numeral co-efficients of the terms ariſing from the combinations

tions of the original numeral co-efficients C, E, G, I, L, N, &c, but will only affect the literal parts of the said terms by removing one power of e and g from their several denominators. It therefore follows from art. 107, and 108, in the first place, that the first term of the compound series δ, or $8g \times \gamma$, will be ($8g \times 1$, or) $8g$; and, secondly, that the co-efficients of the second term, and all the following terms of the compound series δ, or $8g \times \gamma$, will be respectively equal to the co-efficients of the second term, and all the following corresponding terms of the compound series Δ, or $8e \times \Gamma$; and, thirdly, that the signs + and —, which are to be prefixed to the several terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series δ, or $8g \times \gamma$, will be the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Δ, or $8e \times \Gamma$; and, fourthly, that the signs + and —, which are to be prefixed to the several terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series δ, or $8g \times \gamma$, will be every where contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series Δ, or $8e \times \Gamma$.

110. It follows from the last article, that, if the compound series δ, or $8g \times \gamma$, be reduced into a simple series, or series of single terms, by performing all the multiplications, additions, and subtractions that are necessary for that purpose, the co-efficients of the second and other following terms of the simple series thereby obtained will be respectively equal to the co-efficients of the second and other following terms of the simple series which is equal to the compound series Δ, or $8e \times \Gamma$, that is (by art. 101) to the co-efficients P, Q, R, S, T, V, &c; and the signs to be prefixed to the third, and fifth, and seventh, and other following odd terms of the said simple series, which is equal to the compound series δ, or $8g \times \gamma$, will be the same with those which are prefixed to the third, and fifth, and seventh, and other following odd terms of the simple series which is equal to the compound series Δ, or $8e \times \Gamma$, or to the odd terms of the simple series $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c; and the signs to be prefixed to the second, and fourth, and sixth, and other following even terms of the simple series which is equal to the said compound series δ, or $8g \times \gamma$, will be respectively contrary to those which are to be prefixed to the second, and fourth, and sixth, and other following even terms of the simple series which is equal to the compound series Δ, or $8e \times \Gamma$, or to the even terms of the simple series $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c. And consequently the simple series that will be equal to the compound series δ, or $8g \times \gamma$, will be $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*. Therefore m^3, or the cube of the transcendental expression m, or $2\sqrt{^3}g \times$ the series w (which, by art. 108, is equal to the compound series δ) will be equal to the simple series $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*.

111. In

111. In the expression $2\sqrt[3]{g} \times$ the series w, or $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum*, the letter g is put for $\frac{t}{2}$, or half the absolute term t of the equation $x^3 - qx = t$, and zz is $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$. Therefore $\frac{q^3}{27}$ is $= gg + zz = gg \times \overline{1 + \frac{zz}{gg}}$, and q^3 is $= 27 \times gg \times \overline{1 + \frac{zz}{gg}}$, and q is $= 3 \times g^{\frac{2}{3}} \times \overline{1 + \frac{zz}{gg}}\big|^{\frac{1}{3}} =$ (by the binomial theorem in the case of roots) $3 \times g^{\frac{2}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c *ad infinitum*.

112. Now let the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c be multiplied into the expression $3 \times g^{\frac{2}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c, which is equal to q; and the product will be ($= 2\sqrt[3]{g} \times 3 \times g^{\frac{2}{3}} \times$ the product of the said two infinite serieses $= 2 \times g^{\frac{1}{3}} \times 3 \times g^{\frac{2}{3}} \times$ the product of the said two infinite serieses) $= 6g \times$ the product of the said two infinite serieses. Therefore $q \times$ the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, or $q \times 2\sqrt[3]{g} \times w$, or $q \times m$, will be $= 6g \times$ the product of the said two infinite serieses, which product will be a certain compound series, or series consisting of several different horizontal rows of terms, placed one under another. Let this compound series be denoted by the small Greek letter λ. And then $q \times 2\sqrt[3]{g} \times$ the simple series w, or $q \times m$, will be $= 6g \times$ the compound series λ.

113. Now let the several terms of the compound series λ be multiplied into $6g$, and let the new compound series thence arising be denoted by the small Greek letter π. Then will $q \times 2\sqrt[3]{g} \times$ the simple series w, or $q \times m$, be $=$ the new compound series π.

And in this compound series π it is evident that the first term must be $6g$, because the first term of the compound series λ (which is the product of the multiplication of the series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} +$ &c into the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} +$ &c) must be 1. And the following terms of the said compound series π will involve the successive even powers of z, to wit, zz,

zz, z^4, z^6, z^8, z^{10}, z^{12}, &c, in their numerators, and the ſucceſſive odd powers of g, to wit, g, g^3, g^5, g^7, g^9, g^{11}, &c, in their denominators.

114. Since the co-efficients of the terms of the two infinite ſerieſes $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c (by the multiplication of which the compound ſeries λ is produced) are the ſame with thoſe of the correſponding terms of the two infinite ſerieſes $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c, and $1 - \frac{Css}{e} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c (by the multiplication of which the compound ſeries Λ is produced in art. 97), though the terms of the two former ſerieſes are not connected with each other by the ſigns + and — in the ſame manner as the terms of the two latter ſerieſes; it follows that the co-efficients of the ſeveral terms contained in the compound ſeries λ will be equal to the co-efficients of the correſponding terms contained in the compound ſeries Λ; becauſe the co-efficients of the terms of both the ſaid compound ſerieſes are ſimilar combinations, or products, of the ſame original factors or co-efficients, B, C, D, E, F, G, H, I, K, L, M, N, &c, of the terms of the ſimple ſerieſes by the multiplication of which into each other the ſaid compound ſerieſes are produced. And from the continued repetition of the ſign — in the ſecond and other following terms of the two ſerieſes $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c, and $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c (by the multiplication of which the compound ſeries Λ is produced), and the alternate ſucceſſion of the ſign + and the ſign — in the ſecond and other following terms of the two ſerieſes $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c, and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c (by the multiplication of which into each other the compound ſeries λ is produced), it follows that in the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the compound ſeries λ the ſigns + and — that are to be prefixed to the ſeveral terms of the ſaid columns, will be the ſame with thoſe which are to be prefixed to the correſponding terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the compound ſeries Λ; and that, in the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the ſaid compound ſeries λ the ſigns + and —, that are to be prefixed to the ſeveral terms of the ſaid columns, will be reſpectively contrary to thoſe which are to be prefixed to the correſponding terms of the ſecond and fourth, and ſixth, and other following even vertical columns of terms in the ſaid compound ſeries Λ.

115. Since the compound ſeries Π is $= 6e \times$ the compound ſeries Λ, and the compound ſeries π is $= 6g \times$ the compound ſeries λ, it is evident that there will

will be the ſame analogies and the ſame differences between the two compound ſerieſes Π and π, with reſpect to the co-efficients of their ſeveral correſponding terms, and the ſigns + and − which are to be prefixed to them, as between the two compound ſerieſes Λ and λ themſelves; becauſe the ſaid multiplications by $6e$ and $6g$ can make no difference at all in the ſigns of the terms multiplied, and will increaſe the co-efficients of all the ſaid terms in the ſame proportion of 6 to 1, which will not change their antecedent equality. For, as to the quantities e and g (which are involved in the multiplicators $6e$ and $6g$) the multiplications of the terms by them will not affect the numeral co-efficients of the terms ariſing from the combinations of the original numeral co-efficients B, C, D, E, F, G, H, I, K, L, M, N, &c, but will only affect the literal parts of the ſaid terms by removing one power of e and g out of their ſeveral denominators. It therefore follows from the two laſt articles 113 and 114, in the firſt place, that the firſt term of the compound ſeries π, or $6g \times \lambda$, will be ($6g \times 1$, or) $6g$; and, ſecondly, that the co-efficients of all the terms of the compound ſeries π, or $6g \times \lambda$, will be, reſpectively, equal to the co-efficients of the correſponding terms of the compound ſeries Π, or $6e \times \Lambda$; and, thirdly, that the ſigns + and −, which are to be prefixed to the ſeveral terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the compound ſeries π, or $6g \times \lambda$, will be the ſame with thoſe which are to be prefixed to the correſponding terms of the third, and fifth, and ſeventh, and other following odd vertical columns of terms in the compound ſeries Π, or $6e \times \Lambda$; and, 4thly, that the ſigns + and −, which are to be prefixed to the ſeveral terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the compound ſeries π, or $6g \times \lambda$, will be, reſpectively, contrary to thoſe which are to be prefixed to the correſponding terms of the ſecond, and fourth, and ſixth, and other following even vertical columns of terms in the compound ſeries Π, or $6e \times \Lambda$.

116. It follows, from the laſt article, that, if the compound ſeries π, or $6g \times \lambda$, be reduced into a ſimple ſeries, or ſeries conſiſting of ſingle terms, by making all the multiplications, additions, and ſubtractions, that are neceſſary for that purpoſe, the co-efficients of the ſecond and other following terms of the ſimple ſeries thereby obtained will be reſpectively equal to the co-efficients of the ſecond and other following terms of the ſimple ſeries which is equal to the compound ſeries Π, or $6e \times \Lambda$; and it follows in the ſecond place, that the ſigns + or −, that are to be prefixed to the third, and fifth, and ſeventh, and other following odd terms of the ſimple ſeries which is equal to the compound ſeries π, or $6g \times \lambda$, will be the ſame with thoſe which are to be prefixed to the third, and fifth, and ſeventh, and other following odd terms of the ſimple ſeries which is equal to the compound ſeries Π, or $6e \times \Lambda$; and it follows, in the third place, that the ſigns + or −, that are to be prefixed to the ſecond, and fourth, and ſixth, and other following even terms of the ſimple ſeries which is equal to the compound ſeries π, or $6g \times \lambda$, will be contrary to thoſe which are to be prefixed to the ſecond, and fourth, and ſixth, and other following even terms of the ſimple ſeries that is equal to the compound ſeries Π, or $6e \times \Lambda$. But it has been ſhewn in art. 100, 101, 102, that the ſimple ſeries that is equal to the compound

pound series Π, or $6e \times \Lambda$, is $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum*. Therefore the simple series that is equal to the compound series π, or $6g \times \lambda$, will be $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vs^{12}}{g^{11}} +$ &c *ad infinitum*. But by art. 112, $q \times m$, or $q \times$ the transcendental expression $2\sqrt[3]{g} \times$ the series w, or $q \times$ the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, is equal to the compound series π, or $6g \times \lambda$. Therefore $q \times m$ is equal to the simple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c, *ad infinitum*.

117. But it was shewn in art. 110, that m^3, or the cube of the transcendental expression m, or $2\sqrt[3]{g} \times$ the series w, or $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{11}} +$ &c is equal to the simple series $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*, which is greater than the simple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*, by the difference ($8g - 6g$, or) $2g$. Therefore m^3 is greater than $q \times m$ by the same difference $2g$, or the residual quantity $m^3 - q \times m$ is $= 2g$, or $2 \times \frac{t}{2}$, or t, the absolute term of the equation $x^3 - qx = t$; and consequently m, or the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum*, must be equal to the root x of the said equation. Q. E. D.

End of the Recapitulation of the foregoing Demonstration of the Proposition laid down above in art. 36.

Another Recapitulation of the Substance of the foregoing Demonstration, somewhat shorter than the former.

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118. The substance of the foregoing demonstration may be expressed in a more concise manner as follows.

Having shewn in art. 25, that, if $\frac{rr}{4}$ be greater than $\frac{q^3}{27}$, or r be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the root y of the cubick equation $y^3 - qy = r$ will be equal to the transcendental expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8}$

$- \frac{Ls^{10}}{e^{10}}$

$-\frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}}$ — &c, *ad infinitum*, in which the letter e denotes $\frac{r}{2}$, or half the abſolute term r of the equation $y^3 - qy = r$, and ss is $= \frac{rr}{4} - \frac{q^3}{27}$, we raiſed the ſaid value of y to its cube, or third power, by multiplying it twice into itſelf, whereby we obtained a certain compound ſeries, of which the firſt term was $8e$, and the following terms involved the ſeveral fractions $\frac{ss}{e}$, $\frac{s^4}{e^3}$, $\frac{s^6}{e^5}$, $\frac{s^8}{e^7}$, $\frac{s^{10}}{e^9}$, $\frac{s^{12}}{e^{11}}$, &c, or the ſeveral ſucceſſive powers of ss divided by the ſucceſſive odd powers of e. This compound ſeries we called Δ, and ſo we had $y^3 =$ the compound ſeries Δ.

119. We then found an expreſſion of the value of q (the co-efficient of y in the equation $y^3 - qy = r$) in terms involving the quantities e and s, to the end that we might multiply the ſaid value of q into the foregoing tranſcendental expreſſion which is equal to y, and thereby obtain a quantity expreſſed in powers of e and s, that ſhould be equal to the product qy, which is ſubtracted from y^3 in the equation $y^3 - qy = r$. This value of q we found to be $3 \times g^{\frac{2}{3}} \times$ the infinite ſeries $1 - \frac{B ss}{ee} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}}$ — &c, which, being multiplied into the foregoing tranſcendental expreſſion of the value of y, produced a certain compound ſeries, of which the firſt term was $6e$, and the following terms involved the ſeveral fractions $\frac{ss}{e}$, $\frac{s^4}{e^3}$, $\frac{s^6}{e^5}$, $\frac{s^8}{e^7}$, $\frac{s^{10}}{e^9}$, $\frac{s^{12}}{e^{11}}$, &c, or the ſeveral ſucceſſive even powers of s divided by the ſucceſſive odd powers of e. This compound ſeries we called Π, and ſo had the product qy equal to the compound ſeries Π.

120. We then obſerved that, ſince the compound ſeries Δ was equal to y^3, and the compound ſeries Π was equal to qy, the exceſs of Δ above Π would be equal to $y^3 - qy$, and conſequently to the abſolute term r, or $2 \times \frac{r}{2}$, or $2e$. But this, we obſerved, was the exceſs of $8e$, the firſt term of the compound ſeries Δ above $6e$, the firſt term of the compound ſeries Π. And hence it followed that $8e - \Delta$, or the exceſs of the firſt term of the ſeries Δ above the whole of the ſaid ſeries, would be equal to $6e - \Pi$, or the exceſs of the firſt term of the ſeries Π above the whole of the ſaid ſeries. And thus we obtained two compound ſerieſes, $8e - \Delta$ and $6e - \Pi$, which were equal to each other.

121. We then obſerved that, ſince this equality between the compound ſerieſes $8e - \Delta$ and $6e - \Pi$ took place in all the poſſible magnitudes of ss and ee, it followed that each ſeparate compound term, or vertical column of ſingle terms, involving any powers of ss and e, in the compound ſeries Π, muſt be equal to the correſponding compound term, or vertical column of ſingle terms, involving the ſame powers of ss and e, in the compound ſeries Δ; and conſequently that, if theſe two compound ſerieſes Δ and Π were to be reduced to ſimple ſerieſes, or

or feriefes confifting of fingle terms, by making all the multiplications, additions, and fubtractions, that were neceffary for that purpofe, the terms of the two fimple feriefes that would be thereby obtained, would be exactly equal to eac other, and marked with the fame figns + or —; and confequently that, if the fimple feries that was equal to the compound feries $8e - \Delta$ was $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} +$ &c *ad infinitum,* the compound feries $6e - \Pi$ would likewife be equal to the fame fimple feries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} +$ &c *ad infinitum.*

122. And from the equality of the compound feries $8e - \Delta$ to the fimple feries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} +$ &c *ad infinitum,* we inferred that the compound feries Δ would be equal to the fimple feries $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum;* and, in like manner, from the equality of the compound feries $6e - \Pi$ to the fame fimple feries $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} +$ &c *ad infinitum,* we inferred that the compound feries Π would be equal to the fimple feries $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum.*

123. And then we concluded that y^3 (which is equal to the compound feries Δ) muft be equal to the fimple feries $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum,* and that qy (which is equal to the compound feries Π) muft be equal to the fimple feries $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c *ad infinitum,* which is lefs than the former fimple feries (which is equal to y^3) by the difference $8e - 6e$, or $2e$, or $2 \times \frac{r}{2}$, or r, as it ought to be.

124. We then turned our attention to the other tranfcendental expreffion, which had been declared above in art. 36 to be equal to the root x of the cubick equation $x^3 - qx = t$, in which t is lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is lefs than $\frac{q^3}{27}$, to wit, the expreffion $2\sqrt[3]{g} \times$ the infinite feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, in which expreffion g is $= \frac{t}{2}$, or half the abfolute term t of the equation $x^3 - qx = t$, and zz is $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$.

 125. This

125. This expreffion we multiplied twice into itfelf, fo as to obtain its cube, which was a compound feries, or feries confifting of feveral horizontal lines, or rows, of terms, placed one under another. This compound feries we called δ, and made the four following obfervations concerning it; to wit, 1ft, That its firft term would be $8g$, and all the following terms of it would involve the feveral quantities zz, z^4, z^6, z^8, z^{10}, z^{12}, &c, or the feveral fucceffive even powers of z, in their numerators, and the feveral quantities g, g^3, g^5, g^7, g^9, g^{11}, &c, or the feveral fucceffive odd powers of g, in their denominators; and, fecondly, That, from the equality of the co-efficients of the feveral terms of the infinite feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c to the co-efficients of the correfponding terms of the infinite feries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c, it followed that the co-efficients of the fecond and other following terms of the compound feries δ would be equal to the co-efficients of the fecond and other following terms of the compound feries Δ; and, thirdly, that from the alternate fucceffion of the fign + and the fign − in the terms of the feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, it followed that, in the third, and fifth, and feventh, and other following odd vertical columns of terms in the compound feries δ, the figns + and − that were to be prefixed to the feveral terms of the faid columns, would be the fame with thofe which were to be prefixed to the correfponding terms of the third, and fifth, and feventh, and other following odd vertical columns in the compound feries Δ; and fourthly, that in the fecond, and fourth, and fixth, and other following even vertical columns of terms in the faid compound feries δ, the figns + and −, that are to be prefixed to the feveral terms of the faid columns, would be, refpectively, contrary to thofe of the correfponding terms of the fecond, and fourth, and fixth, and other following even vertical columns of terms in the compound feries Δ.

126. And, from the foregoing obfervations we concluded, that, fince the fimple feries to which the compound feries Δ was reduced by making all the multiplications, additions, and fubtractions that were neceffary for that purpofe, was $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$ &c, the fimple feries to which the compound feries δ would be reduced in like manner by making all the multiplications, additions, and fubtractions neceffary for that purpofe, would be $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*. And hence it followed that the cube of the tranfcendental expreffion $2\sqrt[3]{g} \times$ the infinite feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum* was equal to the fimple feries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}}$ + &c *ad infinitum*.

6 127. We

127. We then ſought the value of q in terms that involved the powers of g and z, and found that, ſince zz was $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$, we ſhould have $\frac{q^3}{27} = gg + zz = gg \times \overline{1 + \frac{zz}{gg}}$, and $q^3 = 27 \times gg \times \overline{1 + \frac{zz}{gg}}$, and $q = 3 \times g^{\frac{2}{3}} \times \overline{1 + \frac{zz}{gg}}\Big|^{\frac{1}{3}}$ = (by the binomial theorem in the caſe of roots) $3 \times g^{\frac{2}{3}} \times$ the infinite ſeries $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c *ad infinitum.*

128. Having thus found this value of q, we multiplied it into the expreſſion $2\sqrt{}^3\overline{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum*, and found the product to be a certain compound ſeries which we called π, of which the firſt term was $6g$, and the ſecond and other following terms involved the ſeveral quantities zz, z^4, z^6, z^8, z^{10}, z^{12}, &c, or the ſeveral ſucceſſive even powers of z, in their numerators, and the ſeveral quantities g, g^3, g^5, g^7, g^9, g^{11}, &c, or the ſeveral ſucceſſive odd powers of g, in their denominators.

129. We then compared this compound ſeries π with the compound ſeries Π, which was produced by the multiplication of the tranſcendental expreſſion $3 \times e^{\frac{2}{3}} \times$ the infinite ſeries $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c (which was the value of q in the former equation $y^3 - qy = r$) into the tranſcendental expreſſion $2\sqrt{}^3\overline{e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c *ad infinitum*, and we obſerved that, from the equality of the co-efficients of the terms of the two ſimple ſerieſes, by the multiplication of which into each other and into $3 \times g^{\frac{2}{3}} \times 2g^{\frac{1}{3}}$ the compound ſeries π is produced, to the co-efficients of the correſponding terms of the two ſimple ſerieſes, by the multiplication of which into each other and into $3 \times e^{\frac{2}{3}} \times 2 \times e^{\frac{1}{3}}$ the compound ſeries Π was produced, it followed that the numeral co-efficients of the ſeveral terms of the compound ſeries π would be equal to the numeral co-efficients of the correſponding terms of the compound ſeries Π; and we likewiſe obſerved that, from the continued repetition of the ſign — before the ſecond and all the following terms of the ſeries $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}}$ — &c, and likewiſe before the ſecond and all the following terms of the ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, and the alternate ſucceſſion of the ſign + and the ſign — before the ſecond and all the following terms of the ſeries

1 +

$1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c, and likewise before the second and all the following terms of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, it followed that in the compound series π (which was produced by the multiplication of the two latter serieses into each other, and into $3g^{\frac{2}{3}} \times 2g^{\frac{1}{3}}$) the signs + and —, that would be to be prefixed to the several terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the said compound series would be the same with those which were to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Π, which was produced by the multiplication of the two former simple serieses into each other and into $3e^{\frac{2}{3}} \times 2e^{\frac{1}{3}}$; and that in the second, and fourth, and sixth, and other following even vertical columns of the said two compound serieses Π and π the signs prefixed to the terms of the one series would be every where contrary to the signs prefixed to the corresponding terms of the other series.

130. And from the foregoing observations we concluded, that, if the said two compound serieses Π and π were to be reduced to simple serieses, by making the several multiplications, additions, and subtractions that were necessary for that purpose, the co-efficients of the second, and other following terms of the simple series that was equal to the compound series π, would be equal to the co-efficients of the second and other following terms of the simple series that was equal to the compound series Π, and the signs that would be prefixed to the third, and fifth, and seventh, and other following odd terms of both these simple serieses would be the same, but those which would be prefixed to the second, and fourth, and sixth, and other following even terms of these simple serieses would be different in the two serieses.

131. And hence it followed that, since the compound series Π was found before to be equal to the simple series $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} +$ &c *ad infinitum*, the compound series π must be equal to the simple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*.

132. But the compound series π was equal to the product of the multiplication of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum* into the expression $3 \times g^{\frac{2}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c *ad infinitum*, which is equal to q. Therefore the product of the multiplication of q into

into the tranſcendental expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum*, is equal to the ſimple ſeries $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum.*

133. It appears, therefore, that the cube of the tranſcendental expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum* is equal to the ſimple ſeries $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*, and that the product of the multiplication of q into the ſaid tranſcendental expreſſion is equal to the ſimple ſeries $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*, which is leſs than the former ſimple ſeries by $8g - 6g$, or by $2g$, or $2 \times \frac{t}{2}$, or t. Therefore the cube of the tranſcendental expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum* is greater than the product of the multiplication of the ſaid tranſcendental expreſſion into q, and their difference is equal to t, the abſolute term of the equation $x^3 - qx = t$. Therefore the ſaid tranſcendental expreſſion muſt be equal to the root x of the ſaid equation. Q. E. D.

End of the ſecond and ſhorter Recapitulation of the Subſtance of the foregoing Demonſtration of the Propoſition laid down above in art. 36.

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134. I have now compleated the demonſtration of the propoſition, or theorem, laid down above in art. 36, and which is the principal ſubject of this diſcourſe, to wit, "That, if the abſolute term, t, of the cubick equation $x^3 - qx = t$ be leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$ (or tt be leſs than $\frac{4q^3}{27}$, but greater than $\frac{2q^3}{27}$), or $\frac{tt}{4}$ be leſs than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$, and g be put $= \frac{t}{2}$, or half the abſolute term t of the ſaid equation, and zz be $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$, the root x of the ſaid equation will be equal to the following tranſcendental quantity, to wit, $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum*, in which ſeries the numeral co-efficients C, E, G, I, L, N, &c of the ſecond and other following terms are the co-efficients

co-efficients of the third, and fifth, and ſeventh, and ninth, and eleventh, and thirteenth, and other following odd terms of the infinite ſeries which is equal to the cube-root of a binomial quantity, as $1 + x$, and the ſecond and third, and fourth, and fifth, and ſixth, and ſeventh, and other following terms are marked with the ſign + and with the ſign — alternately." This demonſtration has been, I confeſs, exceedingly long and intricate; but I did not know how to make it ſhorter or eaſier, without leſſening the evidence of the reaſonings contained in it. And I hope that the reader will have found it convincing and ſatisfactory, ſo as to have no doubt remaining on his mind of the truth of the propoſition it is intended to demonſtrate. But it may poſſibly be aſked, "How could ſuch an expreſſion be diſcovered to be equal to the root of the cubick equation $x^3 - qx = r$, if we did not already know that it is ſo? ſeeing that, when we have found out that it is equal to the ſaid root, it requires ſuch a long train of ſubtle and abſtruſe reaſoning to prove that it is ſo." In anſwer to this queſtion I will now proceed to ſtate a method of inveſtigating the value of x in this equation $x^3 - qx = t$ by which we might have been led to this diſcovery.

A method of inveſtigating the value of the root x of the cubick equation $x^3 - qx = t$, in which the abſolute term t is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is leſs than $\frac{q^3}{27}$, by means of the tranſcendental expreſſion $2\sqrt[3]{e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c ad infinitum, *which has been ſhewn above in art. 25 to be equal to the root y of the cubick equation $y^3 - qy = r$, in which the abſolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.*

135. Since the tranſcendental expreſſion $2\sqrt[3]{e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c is equal to the root y of the cubick equation $y^3 - qy = r$, it follows that, if the ſaid expreſſion were to be raiſed to its cube, or third power, by multiplying it twice into itſelf, and if it were alſo to be multiplied into the co-efficient q, or into the value of the ſaid co-efficient expreſſed in powers of e and s, the ſaid cube would be greater than the ſaid product, and their difference would be equal to r. Now, becauſe e is $= \frac{r}{2}$, and ss is $= \frac{rr}{4} - \frac{q^3}{27}$, or $ee - \frac{q^3}{27}$, we ſhall have $ss + \frac{q^3}{27} = ee$, and $\frac{q^3}{27} = ee - ss = ee \times \overline{1 - \frac{ss}{ee}}$, and $q^3 = 27 \times ee \times \overline{1 - \frac{ss}{ee}}$, and conſequently $q = 3 \times e^{\frac{2}{3}}$

×

$\times \overline{1 - \frac{ss}{ee}}|^{\frac{1}{3}} =$ (by the residual theorem in the case of roots) $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}}$ — &c. It follows therefore that the product of the multiplication of this last expression into the expression $2\sqrt{}^3|e \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c will be less than the cube of the said expression $2\sqrt{}^3|e \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, and that their difference will be equal to r, or to $2 \times \frac{r}{2}$, or $2 \times e$, or $2e$.

136. Now from hence it seems natural to conjecture that the root x of the second equation $x^3 - qx = t$, in which the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, will be equal to a transcendental expression that shall bear a great resemblance to the expression $2\sqrt{}^3|e \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c (which is equal to the root y of the former equation $y^3 - qy = r$), and shall differ from it only in such circumstances as shall be the consequences of the absolute term's being less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, instead of being greater than that quantity, as it was in the foregoing equation $y^3 - qy = r$. Therefore, if in the equation $x^3 - qx = t$ we were to put g for $\frac{t}{2}$, or half the absolute term t (as we before put e for $\frac{r}{2}$, or half the absolute term r) and were to put zz for the excess of $\frac{q^3}{27}$ above $\frac{tt}{4}$, or gg (as we before put ss for the excess of $\frac{rr}{4}$, or ee, above $\frac{q^3}{27}$) it seems reasonable to conjecture that the root x of the equation $x^3 - qx = t$ will be equal to a transcendental expression of this kind, to wit, $2\sqrt{}^3|g \times$ the infinite series $1, \frac{Czz}{gg}, \frac{Ez^4}{g^4}, \frac{Gz^6}{g^6}, \frac{Iz^8}{g^8}, \frac{Lz^{10}}{g^{10}}, \frac{Nz^{12}}{g^{12}}$, &c, in which I have not prefixed any signs + or — to the second and other following terms, because it does not hitherto appear whether they ought all to have the sign — prefixed to them (as is the case with the second and other following terms of the former series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c), or whether some of them are to be marked with the sign —, and others of them to be marked with the sign +. And we shall have reason to suppose that these terms are not, all of them, to be marked with the sign — (as the corresponding terms of the former series were) because the signs prefixed to the terms of the infinite series which enters into the expression of the value of q in this case, will not be the same as those which are prefixed to the terms of the infinite series

which enters into the expreſſion of the value of q in the former caſe. For we have ſeen in art. 135, that the value of q in the former caſe is $3 \times e^{\frac{2}{3}} \times$ the infinite ſeries $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c, in which all the terms after the firſt term 1 are marked with the ſign —. But in the preſent caſe we have $zz = \frac{q^3}{27} - \frac{tt}{4} = \frac{q^3}{27} - gg$, and conſequently $\frac{q^3}{27} = gg + zz = gg \times \overline{1 + \frac{zz}{gg}}$, and $q^3 = 27 \times gg \times \overline{1 + \frac{zz}{gg}}$, and $q = 3 \times g^{\frac{2}{3}} \times \overline{1 + \frac{zz}{gg}}\Big|^{\frac{1}{3}}$ = (by the binomial theorem in the caſe of roots) $3 \times g^{\frac{2}{3}} \times$ the infinite ſeries $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} +$ &c, in which the ſecond and other following terms are marked with the ſign + and the ſign — alternately. Therefore, if this value of q were to be multiplied into the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} -$ &c, the ſigns + and —, that would be prefixed to the ſeveral terms of the product, or compound ſeries, thence ariſing, would in many inſtances be different from thoſe which would be prefixed to the correſponding terms of the compound ſeries or product, ariſing from the multiplication of the former value of q, to wit, $3 \times e^{\frac{2}{3}} \times$ the infinite ſeries $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} -$ &c, into the expreſſion $2\sqrt[3]{e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c. Therefore, ſince the expreſſion $2\sqrt[3]{e} \times$ the infinite ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c, being ſubſtituted inſtead of y in compound quantity $y^3 - qy$ makes the ſaid quantity equal to r, or $2e$, it follows that the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} -$ &c, being ſubſtituted inſtead of x in the compound quantity $x^3 - qx$, cannot (on account of the different ſigns prefixed to the terms of the value of the product qx from thoſe which are prefixed to the correſponding terms of the value of the product qy) make the ſaid quantity equal to $2g$, or t, and conſequently the ſaid expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} -$ &c cannot be equal to the root x of the equation $x^3 - qx = t$.

137. Having thus found that the expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} -$ &c is not equal to the root x of the equation $x^3 - qx = t$;—and having obſerved that the ſigns of the ſecond and other

other following terms of the infinite ſeries that enters into the expreſſion of the value of q in this caſe are alternately $+$ and $-$, whereas in the former caſe the ſecond and all the following terms of the infinite ſeries which entered into the expreſſion of the value of q, were marked with the ſign $-$;—it ſeems reaſonable to conjecture, in the next place, that the root x of the equation $x^3 - qx = t$ may be equal to the expreſſion $2\sqrt{3}\,\overline{)g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, in which the ſecond and all the following terms are marked with the ſign $+$ and the ſign $-$ alternately. And to this expreſſion we ſhould find that the ſaid root was really equal, if, when we had made this conjecture, we had tried it in a few numerical examples, by computing the values of $2\sqrt{3}\,\overline{)g}$ and of a few of the firſt terms of the ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, and multiplying $2\sqrt{3}\,\overline{)g}$ into the reſult of the ſaid terms, and then ſubſtituting the product thence ariſing inſtead of x in the compound quantity $x^3 - qx$. For it would always appear, upon ſuch ſubſtitutions, that the values of $x^3 - qx$ thence ariſing would be equal to the abſolute terms, t, of the ſaid equations. And thus we might have diſcovered that the ſaid expreſſion $2\sqrt{3}\,\overline{)g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c is equal to the root x of the equation $x^3 - qx = t$; after which it would have become neceſſary to our further ſatisfaction on the ſubject, to endeavour to demonſtrate the truth of this propoſition ſynthetically; which we have done at great length, and with as much exactneſs as poſſible, in the foregoing articles of this diſcourſe.

138. But there is alſo another method of diſcovering that the ſaid tranſcendental expreſſion $2\sqrt{3}\,\overline{)g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c is equal to the root x of the equation $x^3 - qx = t$. And this method is more direct than the former, and carries with it a proof that the ſaid expreſſion muſt be equal to the ſaid root, inſtead of affording us only a probable conjecture that it is ſo, which is afterwards to be confirmed by trials, in particular numerical examples, and by a ſynthetical demonſtration. This method I ſhall now endeavour to explain.

Another method of discovering that the transcendental expression $2\sqrt{^3}\overline{g} \times$ *the infinite series* $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + *&c is equal to the root* x *of the cubick equation* $x^3 - qx = t$, *in which* t *is less than* $\frac{2q\sqrt{q}}{3\sqrt{3}}$, *but greater than* $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$.

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139. We have seen, that in the equation $y^3 - qy = r$ (in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) the transcendental expression $2\sqrt{^3}\overline{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ − &c is equal to the root y. Now there are two different ways of computing the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ − &c contained in this expression, which (though not equally short and convenient in practice) are, nevertheless, equally just and true: and therefore they must both produce the same result for the value of the said series. These different methods of computing the said series are as follows.

140. The first way of computing it is the common one, which consists of the following processes; to wit, first, to compute the quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$; as was done in the foregoing example in art. 30, 31, where $\frac{rr}{4}$ was found to be = 1,110,916, and $\frac{q^3}{27}$ was found to be = 1000,000; and, secondly, to subtract $\frac{q^3}{27}$ from $\frac{rr}{4}$, in order to obtain the quantity ss, which is equal to their difference $\frac{rr}{4} - \frac{q^3}{27}$, and which in the foregoing example was 110,916; and, thirdly, to divide ss by ee, so as to obtain the value of the fraction $\frac{ss}{ee}$; as in the foregoing example we found the fraction $\frac{110,916}{1,110,916}$ to be = 0.099,841,932,2; and, fourthly, to compute the powers of the value found for the fraction $\frac{ss}{ee}$; as in the foregoing example we computed the powers of the decimal fraction 0.099,841,932,2, and found its square to be = 0.009,968,411,4, and its cube to be = 0.000,995,265,4, and its fourth power to be = 0.000,099,369,2, and its fifth, sixth, seventh, and eighth powers to be equal to 0.000,009,921,2, and 0.000,000,990,5, and 0.000,000,098,8, and 0.000,000,009,8, respectively; and, fifthly, to multiply $\frac{ss}{ee}$, and its powers $\frac{s^4}{e^4}, \frac{s^6}{e^6}, \frac{s^8}{e^8}, \frac{s^{10}}{e^{10}}, \frac{s^{12}}{e^{12}}, \frac{s^{14}}{e^{14}}, \frac{s^{16}}{e^{16}}, \frac{s^{18}}{e^{18}}$, &c, into the co-efficients C, E, G, I, L, N, P, R, T, &c, respectively; as in the foregoing example we multiplied 0.099,841,932,2 into the fraction $\frac{1}{9}$ (which is = C), and

0.009,

0.009,968,411,4 into the fraction $\frac{10}{243}$ (which is $=$ E), and 0.000,995,265,4 into the fraction $\frac{154}{6561}$ (which is $=$ G), and 0.000,099,369,2 into the fraction $\frac{935}{59,049}$ (which is $=$ I), and 0.000,009,921,2 into the fraction $\frac{55,913}{4,782,969}$ (which is $=$ L), and 0.000,000,990,5 into the fraction $\frac{1,179,256}{129,140,163}$ (which is $=$ N), and 0.000,000,098,8 into the fraction $\frac{8,617,640}{1,162,261,467}$ (which is $=$ P), and 0.000,000,009,8 into the fraction $\frac{194,327,782}{31,381,059,609}$ (which is $=$ R), and found the products to be 0.011,093,548,0, 0.000,410,222,6, 0.000,023,360,9, 0.000,001,573,4, 0.000,000,115,9, 0.000,000,009,0, 0.000,000,000,7, and 0.000,000,000,0; and, sixthly, to subtract the sum of all the products so obtained from 1, the first term of the series. This is the common, and the proper, way of computing the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} -$ &c, when we want to make use of it in practice. But it may also be computed in another manner, which may be described as follows.

141. Instead of ss let us insert the compound quantity $\frac{rr}{4} - \frac{q^3}{27}$ itself, to which ss is equal, in all the terms of the said series. And it will thereby be converted into the following series, to wit, $1 - \frac{C}{ee} \times \overline{\frac{rr}{4} - \frac{q^3}{27}} - \frac{E}{e^4} \times \overline{\frac{rr}{4} - \frac{q^3}{27}}|^2 - \frac{G}{e^6} \times \overline{\frac{rr}{4} - \frac{q^3}{27}}|^3 - \frac{I}{e^8} \times \overline{\frac{rr}{4} - \frac{q^3}{27}}|^4 - \frac{L}{e^{10}} \times \overline{\frac{rr}{4} - \frac{q^3}{27}}|^5 - \frac{N}{e^{12}} \times \overline{\frac{rr}{4} - \frac{q^3}{27}}|^6 -$ &c, or (if, for the sake of brevity, we substitute ee for $\frac{rr}{4}$, and ff for $\frac{q^3}{27}$) into the series $1 - \frac{C}{ee} \times \overline{ee - ff} - \frac{E}{e^4} \times \overline{ee - ff}|^2 - \frac{G}{e^6} \times \overline{ee - ff}|^3 - \frac{I}{e^8} \times \overline{ee - ff}|^4 - \frac{L}{e^{10}} \times \overline{ee - ff}|^5 - \frac{N}{e^{12}} \times \overline{ee - ff}|^6 -$ &c *ad infinitum*, or (substituting instead of $\overline{ee - ff}|^2$, $\overline{ee - ff}|^3$, $\overline{ee - ff}|^4$, $\overline{ee - ff}|^5$, $\overline{ee - ff}|^6$, &c their expanded values) into the series $1 - \frac{C}{ee} \times \overline{ee - ff} - \frac{E}{e^4} \times \overline{e^4 - 2e^2f^2 + f^4}$

$$- \frac{G}{e^6} \times \overline{e^6 - 3e^4f^2 + 3e^2f^4 - f^6}$$
$$- \frac{I}{e^8} \times \overline{e^8 - 4e^6f^2 + 6e^4f^4 - 4e^2f^6 + f^8}$$
$$- \frac{L}{e^{10}} \times \overline{e^{10} - 5e^8f^2 + 10e^6f^4 - 10e^4f^6 + 5e^2f^8 - f^{10}}$$
$$- \frac{N}{e^{12}} \times \overline{e^{12} - 6e^{10}f^2 + 15e^8f^4 - 20e^6f^6 + 15e^4f^8 - 6e^2f^{10} + f^{12}}$$

— &c *ad infinitum*, or (multiplying the fractions $\frac{C}{e^2}$, $\frac{E}{e^4}$, $\frac{G}{e^6}$, $\frac{I}{e^8}$, $\frac{L}{e^{10}}$, $\frac{N}{e^{12}}$, &c into the separate terms of the compound quantities, or powers of $ee - ff$, with which they are connected, respectively) into the series

1 — C

$$\begin{aligned}
1 & - C + \frac{Cff}{ee} \\
& - E + \frac{2Eff}{ee} - \frac{Ef^4}{e^4} \\
& - G + \frac{3Gff}{ee} - \frac{3Gf^4}{e^4} + \frac{Gf^6}{e^6} \\
& - I + \frac{4Iff}{ee} - \frac{6If^4}{e^4} + \frac{4If^6}{e^6} - \frac{If^8}{e^8} \\
& - L + \frac{5Lff}{ee} - \frac{10Lf^4}{e^4} + \frac{10Lf^6}{e^6} - \frac{5Lf^8}{e^8} + \frac{Lf^{10}}{e^{10}} \\
& - N + \frac{6Nff}{ee} - \frac{15Nf^4}{e^4} + \frac{20Nf^6}{e^6} - \frac{15Nf^8}{e^8} + \frac{6Nf^{10}}{e^{10}} - \frac{Nf^{12}}{e^{12}}
\end{aligned}$$

— &c *ad infinitum*; which ſeries conſiſts of a much greater number of terms than the correſponding part of the ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, from which it is derived, and many of its terms are more complicated, and more difficult to compute, than the terms of the ſaid former ſeries. Nevertheleſs, ſince the compound quantity $\frac{rr}{4} - \frac{q^3}{27}$, or $ee - ff$, is equal to ss, the inſertion of it inſtead of ss in the terms of the ſaid former ſeries cannot alter its value, or magnitude, though it will make it much more difficult to compute. It muſt therefore be true of the new and complicated ſeries juſt now obtained, as well as of the former ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c (from which it was derived), that, if it be multiplied into $2\sqrt[3]{e}$, or $2\sqrt[3]{\frac{r}{2}}$, the quantity thereby produced will be equal to the value of y in the equation $y^3 - qy = r$, or that, if the ſaid laſt-mentioned quantity be raiſed to its cube, or third power, by multiplying it twice into itſelf, and alſo be multiplied into the co-efficient q, the product of the ſaid multiplication will be leſs than the ſaid cube, and their difference will be equal to the abſolute term r.

142. Having ſhewn that the product of the multiplication of $2\sqrt[3]{e}$ or $2\sqrt[3]{\frac{r}{2}}$, into the new and complicated ſeries laſt obtained in the foregoing article (which ſeries we will, for the ſake of brevity, denote by the Greek capital letter Γ) is equal to the root y of the cubick equation $y^3 - qy = r$ in the firſt caſe of that equation, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, or ee is greater than ff, as well as the product of the multiplication of $2\sqrt[3]{e}$ into the ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c, from which the ſeries Γ was derived;—we may, by means of this equality between theſe two products and the root of the equation $y^3 - qy = r$ in the ſaid firſt caſe of that equation, deduce from the ſeries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}}$ — &c another ſeries (reſembling it in the powers of the literal quantity involved in its ſecond and other following

lowing terms, and likewise in the numeral co-efficients C, E, G, I, L, N, &c, by which the said powers are to be multiplied, but differing from it in the signs which are to be prefixed to some of its terms) which will be of such a magnitude that, if it be multiplied into $2\sqrt[3]{e}$, or $2\sqrt[3]{\frac{r}{2}}$, the product of the said multiplication shall be equal to the root y of the said cubick equation $y^3 - qy = r$ in the second case of the said equation, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which cannot be resolved by Cardan's second rule above explained. This may be done in the manner following.

143. If in this second case of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or (if we still denote $\frac{rr}{4}$ by ee, and $\frac{q^3}{27}$ by ff) ee is less than ff, we subtract $\frac{rr}{4}$ from $\frac{q^3}{27}$, or ee from ff, and call the difference, or remainder, zz, and then raise the several powers of zz, to wit, $\overline{zz}|^2$, $\overline{zz}|^3$, $\overline{zz}|^4$, $\overline{zz}|^5$, $\overline{zz}|^6$, $\overline{zz}|^7$, $\overline{zz}|^8$, $\overline{zz}|^9$, &c, or z^4, z^6, z^8, z^{10}, z^{12}, z^{14}, z^{16}, z^{18}, &c, and also the corresponding powers of its value, the residual quantity $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$, to wit, $\overline{ff-ee}|^2$, $\overline{ff-ee}|^3$, $\overline{ff-ee}|^4$, $\overline{ff-ee}|^5$, $\overline{ff-ee}|^6$, $\overline{ff-ee}|^7$, $\overline{ff-ee}|^8$, $\overline{ff-ee}|^9$, &c, the even powers of the difference $ff - ee$, to wit, $\overline{ff-ee}|^2$, $\overline{ff-ee}|^4$, $\overline{ff-ee}|^6$, $\overline{ff-ee}|^8$, &c, will consist of the very same terms, or the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or ee and ff, and with the same signs + and − prefixed to them respectively, as were before contained in the same even powers of the opposite difference $\frac{rr}{4} - \frac{q^3}{27}$, or $ee - ff$, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$, or ee was greater than ff.

Thus, for example, the square of $ee - ff$ in the former case was $e^4 - 2e^2f^2 + f^4$; and in the present case, the square of $ff - ee$ is $f^4 - 2f^2e^2 + e^4$, which consists of the same terms (or the same powers and products of ee and ff) as were contained in the square of the difference $ee - ff$, to wit, $e^4 - 2e^2f^2 + f^4$, and agrees with the said former square in the signs which are prefixed to its several terms, and differs from it only in the order in which the extreme terms e^4 and f^4, and the letters e and f in the middle terms $2e^2f^2$ and $2f^2e^2$, are placed. And the same observation is true of the fourth power of the difference $ee - ff$, to wit, $e^8 - 4e^6f^2 + 6e^4f^4 - 4e^2f^6 + f^8$, and the fourth power of the opposite difference $ff - ee$, to wit, $f^8 - 4f^6e^2 + 6f^4e^4 - 4f^2e^6 - e^8$, and of all the following even powers of the said opposite differences $ee - ff$ and $ff - ee$.

144. Also the odd powers of the difference $ff - ee$, to wit, $ff - ee$ itself, and $\overline{ff-ee}|^3$, $\overline{ff-ee}|^5$, $\overline{ff-ee}|^7$, $\overline{ff-ee}|^9$, &c, will consist of the same terms (or of

of the fame powers, products, and multiples of the two original quantities *ee* and *ff*) as were contained in the fame odd powers of the oppofite difference $ee - ff$, when *ee*, or $\frac{rr}{4}$, was greater than *ff*, or $\frac{q^3}{27}$. But the figns prefixed to the faid terms will be contrary to thofe which were prefixed to them in the former cafe.

Thus, for example, the cube of the difference $ee - ff$ in the former cafe was $e^6 - 3e^4f^2 + 3e^2f^4 - f^6$; and the cube of the difference $ff - ee$ in the prefent cafe is $f^6 - 3f^4e^2 + 3f^2e^4 - e^6$, which confifts of the fame terms (or the fame powers, and products of *ee* and *ff*) as are contained in the cube of $ee - ff$, to wit, $e^6 - 3e^4f^2 + 3e^2f^4 - f^6$; but they are placed in a contrary order to that in which they ftood in the former cafe, and the figns, which are prefixed to them, are contrary in every term to what they were before. And the fame thing is true of the fifth power of the difference $ee - ff$, to wit, $e^{10} - 5e^8f^2 + 10e^6f^4 - 10e^4f^6 + 5e^2f^8 - f^{10}$, and the fifth power of the oppofite difference $ff - ee$, to wit, $f^{10} - 5f^8e^2 + 10f^6e^4 - 10f^4e^6 + 5f^2e^8 - e^{10}$, and of all the following odd powers of the faid oppofite differences $ee - ff$ and $ff - ee$.

145. It follows, therefore, that, if *zz* be put for $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$, in the fecond cafe of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, or *ee* is lefs than *ff*, the even powers of *zz*, to wit, $\overline{zz}|^2$, $\overline{zz}|^4$, $\overline{zz}|^6$, $\overline{zz}|^8$, &c, or z^4, z^8, z^{12}, z^{16}, &c, will reprefent, or be equal to, the fame terms (or the fame powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or *ee* and *ff*) in the prefent cafe, as were reprefented by the fame even powers of *ss*, to wit, $\overline{ss}|^2$, $\overline{ss}|^4$, $\overline{ss}|^6$, $\overline{ss}|^8$, &c, or s^4, s^8, s^{12}, s^{16}, &c, in the former cafe, when $\frac{rr}{4}$, or *ee*, was greater than $\frac{q^3}{27}$, or *ff*, and *ss* was put for the difference $\frac{rr}{4} - \frac{q^3}{27}$, or $ee - ff$; and the feveral terms (or powers, products, and multiples, of the original quantities *ee* and *ff*) reprefented by the faid even powers of *zz* in the fecond cafe of the equation $y^3 - qy = r$, when $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, or *ee* is lefs than *ff*, will have the fame figns + and − prefixed to them as they had in the former cafe of the faid equation, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$, or *ee* was greater than *ff*, and the faid terms were reprefented by the even powers of *ss*; and the only difference between the terms reprefented by the even powers of *zz* in the latter cafe of the faid equation and the terms reprefented by the even powers of *ss* in the former cafe of the faid equation will be in the order in which the faid terms will be placed in the two cafes, and in which the letters *e* and *f* in the feveral middle terms of the values of the faid even powers of *zz* and *ss* (which will involve both the quantities *ee* and *ff*) will follow each other.

146. And

146. And it likewise follows, in the second place, that the odd powers of zz, to wit, zz itself and $\overline{zz}|^3$, $\overline{zz}|^5$, $\overline{zz}|^7$, $\overline{zz}|^9$, &c, or z^6, z^{10}, z^{14}, z^{18}, &c, will also represent, or be equal to, the same terms (or the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or ee and ff) in the present case as were represented by the same odd powers of ss, to wit, ss itself, and $\overline{ss}|^3$, $\overline{ss}|^5$, $\overline{ss}|^7$, $\overline{ss}|^9$, &c, or s^6, s^{10}, s^{14}, s^{18}, &c, in the former case, or when $\frac{rr}{4}$, or ee, was greater than $\frac{q^3}{27}$, or ff, and ss was put for the difference $\frac{rr}{4} - \frac{q^3}{27}$, or $ee - ff$. But the signs + and —, that will be prefixed to the terms that are represented by the said odd powers of zz, to wit, zz, z^6, z^{10}, z^{14}, z^{18}, &c, will be respectively contrary to those which were prefixed to the same terms in the former case, when they were represented by the same odd powers of ss, to wit, ss, s^6, s^{10}, s^{14}, s^{18}, &c: and the terms represented by the said odd powers of zz will likewise differ from the terms represented by the same odd powers of ss in the order in which they will be placed, and in which the letters e and f in the several middle terms of each of the said powers of $ee - ff$ and $ff - ee$ (which middle terms will involve both the quantities ee and ff) will follow each other.

147. If therefore, in the second case of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or ee is less than ff, we put zz for $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$, the series $1 - \frac{Czz}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} -$ &c, *ad infinitum*, will represent, or be equal to, a system of terms derived from the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or ee and ff, that will be the very same in point of composition, that is, will be the very same powers, products, and multiples of ee and ff, as the terms of the complicated series Γ, which was derived above, in art. 141, from the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c, *ad infinitum*, by substituting $ee - ff$ in its terms instead of ss in the former case of the equation $y^3 - qy = r$, or when $\frac{rr}{4}$, or ee, was greater than $\frac{q^3}{27}$, or ff. But the terms of the said two complicated serieses, or systems of terms, so represented by the two serieses $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c, *ad infinitum*, and $1 - \frac{Czz}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} -$ &c, *ad infinitum*, will not *all* have the same signs + and — prefixed to them; but those terms only of the latter complicated series, or system of terms, which is represented by the series $1 - \frac{Czz}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} -$ &c (which latter system of terms we will, for the sake of brevity, denote by the Greek capital letter Δ), which correspond to, or are represented by, the terms which involve the even powers of zz, to wit, the terms

terms $\frac{Ez^4}{e^4}$, $\frac{Iz^8}{e^8}$, $\frac{Nz^{12}}{e^{12}}$, &c, will have the fame figns + and − prefixed to them as were prefixed to them in the former fyftem of terms, or complicated feries, called Γ (which was derived above in art. 141 from the feries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c), when they correfponded to, or were reprefented by, the terms which involved the even powers of ss, to wit, the terms $\frac{Es^4}{e^4}$, $\frac{Is^8}{e^8}$, $\frac{Ns^{12}}{e^{12}}$, &c; and thofe terms of the complicated feries Δ which correfpond to, or are reprefented by, the terms which involve the odd powers of zz, to wit, the terms $\frac{Czz}{ee}$, $\frac{Gz^6}{e^6}$, $\frac{Lz^{10}}{e^{10}}$, &c, will have contrary figns prefixed to them to thofe which were prefixed to them in the former complicated feries Γ, when they correfponded to, or were reprefented by, thofe terms of the feries $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} -$ &c, which involved the odd powers of ss, to wit, the terms $\frac{Css}{ee}$, $\frac{Gs^6}{e^6}$, $\frac{Ls^{10}}{e^{10}}$, &c. And, laftly, the order in which the feveral terms, or powers, products and multiples of the two original quantities ee and ff are placed in the complicated feries Δ, will be contrary to that in which the fame terms are placed in the complicated feries Γ. But this laft circumftance will make no change in the magnitude of the terms, nor in that of the whole feries that is compofed of them, and therefore need not be attended to any further.

148. It follows from the laft article that, if the terms of the complicated feries Δ be placed in the fame order as the terms of the complicated feries Γ, the faid complicated feries Δ will be as follows; to wit,

$$\begin{aligned}
1 &+ C - \frac{Cff}{ee} \\
&- E + \frac{2Eff}{ee} - \frac{Ef^4}{e^4} \\
&+ G - \frac{3Gff}{ee} + \frac{3Gf^4}{e^4} - \frac{Gf^6}{e^6} \\
&- I + \frac{4Iff}{ee} - \frac{6If^4}{e^4} + \frac{4If^6}{e^6} - \frac{If^8}{e^8} \\
&+ L - \frac{5Lff}{ee} + \frac{10Lf^4}{e^4} - \frac{10Lf^6}{e^6} + \frac{5Lf^8}{e^8} - \frac{Lf^{10}}{e^{10}} \\
&- N + \frac{6Nff}{ee} - \frac{15Nf^4}{e^4} + \frac{20Nf^6}{e^6} - \frac{15Nf^8}{e^8} + \frac{6Nf^{10}}{e^{10}} - \frac{Nf^{12}}{e^{12}} \\
&+ \text{\&c, } \textit{ad infinitum}.
\end{aligned}$$

149. Therefore, if in the feries $1 - \frac{Czz}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} -$ &c, we change the figns of thofe terms which involve the odd powers of zz, to wit, the terms $\frac{Czz}{ee}$, $\frac{Gz^6}{e^6}$, $\frac{Lz^{10}}{e^{10}}$, &c, and in the terms of the new feries thereby produced,

produced, to wit, the series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c, we substitute $ff - ee$ instead of zz, and thereby produce a third complicated series (which, for the sake of brevity, we will denote by the Greek capital letter Λ), this third complicated series Λ will consist of the same terms in point of composition (or the same powers, products, and multiples, of the two original quantities ee and ff) as each of the two former complicatedserieses denoted by the Greek capital letters Γ and Δ, and its terms will have the same signs + and — prefixed to them as are prefixed to the same terms in the complicated series Γ, which is set down above, in art. 141; and the only difference between the said two complicated serieses Γ and Λ will be in the order in which their terms are ranged, those of the series Γ proceeding according to the powers of the letter e, and those of the series Λ proceeding according to the powers of the letter f.

150. Now, since this last complicated series Λ consists of the same terms in point of composition (or of the same powers, products, and multiples of the original quantities ee and ff) as the first complicated series Γ, which is set down in art. 141, and has every where the same signs + and — prefixed to its terms; and it has been shewn above, in the same art. 141, that if the said series Γ be multiplied into $2\sqrt[3]{e}$, or $2\sqrt[3]{\frac{r}{2}}$, and the series thereby produced, to wit, the complicated series $2\sqrt[3]{e} \times \Gamma$, be cubed, or raised to its third power, by multiplying it twice into itself, and also be multiplied into the co-efficient q, the said product, to wit, $q \times 2\sqrt[3]{e} \times \Gamma$, will be less than the said cube, to wit, $8e \times \Gamma^3$, and their difference, to wit, $8e \times \Gamma^3 - q \times 2\sqrt[3]{e} \times \Gamma$, will be equal to the absolute term r;—it will follow that, if the third complicated series Λ (which agrees so entirely in all its terms, and the signs which are prefixed to them, with the complicated series Γ) be multiplied into $2\sqrt[3]{e}$, or $2\sqrt[3]{\frac{r}{2}}$, and the complicated series thereby produced, to wit, the complicated series $2\sqrt[3]{e} \times \Lambda$, be cubed, or raised to its third power by multiplying it twice into itself, and also be multiplied into the co-efficient q, the said product, to wit, $q \times 2\sqrt[3]{e} \times \Lambda$, will be less than the said cube, or $8e \times \Lambda^3$, and their difference, to wit, $8e \times \Lambda^3 - q \times 2\sqrt[3]{e} \times \Lambda$ will be equal to the absolute term r.

151. But the complicated series Λ is equal to the series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c, *ad infinitum*, from which it was derived by substituting $ff - ee$ in its terms, instead of zz. Therefore $2\sqrt[3]{e} \times$ the said series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c will be equal to $2\sqrt[3]{e} \times$ the complicated series Λ, or to the complicated series $2\sqrt[3]{e} \times \Lambda$. And consequently, if the quantity $2\sqrt[3]{e} \times$ the infinite series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} -$

 $\frac{Iz^8}{e^8}$

$\frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c (which is equal to the complicated series $2\sqrt{^3}\overline{e} \times$ Λ) be raised to its cube, or third power, by multiplying it twice into itself, and likewise be multiplied into the co-efficient q, the said product will be less than the said cube, and their difference will be equal to the absolute term r. Therefore the said quantity $2\sqrt{^3}\overline{e} \times$ the infinite series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c, *ad infinitum*, is equal to the root y of the cubick equation $y^3 - qy = r$ in the second case of it, or when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or ee is less than ff, and zz is put for the difference $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$; or, if in this second case of the equation $y^3 - qy = r$, we substitute t instead of r for the absolute term of the equation; and put g, instead of e, for half the absolute term t; and denote the root of the equation by the letter x instead of the letter y, the transcendental quantity $2\sqrt{^3}\overline{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, *ad infinitum*, will be equal to the root x of the cubick equation $x^3 - qx = t$. Q. E. I.

End of the Investigation of the Transcendental Expression $2\sqrt{^3}\overline{g} \times$ *the infinite series* $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ *&c, which is equal to the root* x *of the cubick equation* $x^3 - qx = t$, *in which* t *is less than* $\frac{2q\sqrt{q}}{3\sqrt{3}}$, *but greater than* $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$.

A Remark on the foregoing Investigation.

152. The foregoing investigation is very abstruse and difficult, and therefore has been set forth at great length, in order to make the reasonings used in it as clear as possible: and I hope the attentive reader will have found them, in general, intelligible and satisfactory. There is, however, one part of the deduction which is more subtle than the rest, and may therefore require some further elucidation; I mean that part of it which is contained in art. 150. In that article the reasoning is as follows: "Since the complicated series Λ consists of the very same terms in point of composition (or the very same powers, products, and multiples of the two original quantities ee and ff) as the complicated series Γ, and with the same signs + and — prefixed to them; and since it has been shewn in art. 141, that, if the expression $2\sqrt{^3}\overline{e} \times$ the complicated series Γ be cubed, or raised to its third power by multiplying it twice into itself, and also be multiplied into the co-efficient q, the said product $q \times 2\sqrt{^3}\overline{e} \times \Gamma$ will be less than the said cube, or $8e \times \Gamma^3$, and their difference will be equal to the absolute

ſute term r, or $2e$;—it will follow that, if the expreſſion $2\sqrt[3]{e} \times$ the complicated ſeries Λ be cubed, or raiſed to its third power by multiplying it twice into itſelf, and alſo be multiplied into the co-efficient q, the ſaid product $q \times 2\sqrt[3]{e} \times \Lambda$ muſt in like manner be leſs than the ſaid cube, or $8e \times \Lambda^3$, and their difference muſt be equal to the abſolute term r, or $2e$."

An Objection to the Concluſion ſtated in the laſt Article.

153. Now to this concluſion it may be objected, "That the letters r and e do not denote the ſame quantities in the two expreſſions $2\sqrt[3]{e} \times \Gamma$ and $2\sqrt[3]{e} \times \Lambda$, and in the two caſes of the equation $y^3 - qy = r$ which correſpond to thoſe expreſſions; but that in the firſt caſe of the equation $y^3 - qy = r$ the letter r ſignifies a quantity greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the correſponding expreſſion $2\sqrt[3]{e} \times \Gamma$ the letter e ſignifies a quantity greater than f, and that in the ſecond caſe of the equation $y^3 - qy = r$ the letter r ſignifies a quantity leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the correſponding expreſſion $2\sqrt[3]{e} \times \Lambda$ the letter e ſignifies a quantity leſs than f; and that conſequently the ſuppoſed reſemblance between the expreſſion $2\sqrt[3]{e} \times \Gamma$ and $2\sqrt[3]{e} \times \Lambda$ is only apparent, and not real, and therefore will not warrant the concluſion drawn from it."

An Anſwer to the ſaid Objection.

154. In anſwer to this objection we muſt obſerve that it never has been aſſerted, that the expreſſion $2\sqrt[3]{e} \times$ the ſeries Λ was equal to the expreſſion $2\sqrt[3]{e} \times$ the ſeries Γ. For that would not be true; becauſe $2\sqrt[3]{e} \times \Lambda$ is always leſs than $2\sqrt[3]{e} \times \Gamma$. But it was only ſaid that theſe two expreſſions conſiſted of terms compoſed in the ſame manner of the two original quantities ee and ff, and that therefore, ſince the cube of $2\sqrt[3]{e} \times \Gamma$ was greater than the product of the multiplication of $2\sqrt[3]{e} \times \Gamma$ into the co-efficient q, and their difference was equal to r, or $2e$, that is, to the greater value of r or $2e$, which belongs to the firſt caſe of the equation $y^3 - qy = r$, it followed that the cube of $2\sqrt[3]{e} \times \Lambda$ muſt be greater than the product of the multiplication of $2\sqrt[3]{e} \times \Lambda$ into the co-efficient q, and that their difference muſt be equal to the correſponding, or leſſer, value of r, or $2e$, which belongs to the ſecond caſe of the ſaid equation $y^3 - qy = r$. And this concluſion is moſt certainly juſt and true, notwithſtanding the inequality of the two values of r in the two different caſes of the equation $y^3 - qy = r$, and of the two values of e in the two expreſſions $2\sqrt[3]{e} \times \Gamma$ and $2\sqrt[3]{e} \times \Lambda$.

A fur-

A further Proof of the Conclusion stated in Art. 152.

155. But that the truth of this conclusion may be made as plain as possible, let us substitute the letter g instead of e in the expression $2\sqrt[3]{e} \times$ the series Λ. Then will this expression become $2\sqrt[3]{g} \times$ the series Λ, or $2\sqrt[3]{g} \times$ the following complicated series, to wit,

$$\begin{array}{l} 1 - C + \frac{cff}{gg} \\ - E + \frac{2Eff}{gg} - \frac{Ef^4}{g^4} \\ - G + \frac{3Gff}{gg} - \frac{3Gf^4}{g^4} + \frac{Gf^6}{g^6} \\ - I + \frac{4Iff}{gg} - \frac{6If^4}{g^4} + \frac{4If^6}{g^6} - \frac{If^8}{g^8} \\ - L + \frac{5Lff}{gg} - \frac{10Lf^4}{g^4} + \frac{10Lf^6}{g^6} - \frac{5Lf^8}{g^8} + \frac{Lf^{10}}{g^{10}} \\ - N + \frac{6Nff}{gg} - \frac{15Nf^4}{g^4} + \frac{20Nf^6}{g^6} - \frac{15Nf^8}{g^8} + \frac{6Nf^{10}}{g^{10}} - \frac{Nf^{12}}{g^{12}} \\ - \text{\&c, } \textit{ad infinitum}. \end{array}$$

We must therefore endeavour to prove that, since the cube of the expression $2\sqrt[3]{e} \times$ the complicated series Γ (which is set down above in art. 141) is greater than the product of the multiplication of the said expression $2\sqrt[3]{e} \times \Gamma$ into the co-efficient q, and their difference is equal to r, or $2e$, it must follow that the cube of the other expression $2\sqrt[3]{g} \times$ the complicated series Λ (which has been just now set down) will be greater than the product of the multiplication of the said expression $2\sqrt[3]{g} \times \Lambda$ into the co-efficient q, and that their difference will be equal to t, or $2g$.

156. Since ff is $= \frac{q^3}{27}$, we shall have $q^3 = 27 \times ff$, and $q = 3 \times f^{\frac{2}{3}}$. Therefore the product of the multiplication of the expression $2\sqrt[3]{e} \times \Gamma$ into the co-efficient q will be equal to $2\sqrt[3]{e} \times \Gamma \times 3f^{\frac{2}{3}}$, or to $6e^{\frac{1}{3}}f^{\frac{2}{3}} \times$ the complicated series Γ; and the product of the multiplication of the expression $2\sqrt[3]{g} \times \Lambda$ into the co-efficient q will be equal to $2\sqrt[3]{g} \times \Lambda \times 3f^{\frac{2}{3}}$, or to $6g^{\frac{1}{3}}f^{\frac{2}{3}} \times$ the complicated series Λ. And consequently we must now endeavour to prove that, since the cube of the expression $2\sqrt[3]{e} \times$ the complicated series Γ, or of $2e^{\frac{1}{3}} \times$ the complicated series Γ, is greater than the product of the multiplication of $6e^{\frac{1}{3}}f^{\frac{2}{3}}$ into the complicated series Λ, and their difference is equal to $2e$, or r, it must follow that the cube of the other expression $2\sqrt[3]{g} \times$ the complicated series Λ, or of $2g^{\frac{1}{3}} \times$ the complicated series Λ, will be greater than the product of the multiplication of $6g^{\frac{1}{3}}f^{\frac{2}{3}}$ into the

 com-

complicated ſeries Λ, and that their difference will be equal to $2g$, or t. Now this may be ſhewn in the manner following.

157. The complicated ſeries Γ (which is ſet down above in art. 141) conſiſts of 1 together with the ſeveral numeral co-efficients C, E, G, I, L, N, &c, *ad infinitum* (which are all ſubtracted from 1), and of ſeveral following vertical columns of terms involving the fraction $\frac{ff}{ee}$ and its powers $\frac{f^4}{e^4}$, $\frac{f^6}{e^6}$, $\frac{f^8}{e^8}$, $\frac{f^{10}}{e^{10}}$, $\frac{f^{12}}{e^{12}}$, &c, *ad infinitum*. Therefore, if we multiply this complicated ſeries Γ into $2\sqrt[3]{e}$, or $2e^{\frac{1}{3}}$, the product of the ſaid multiplication will be another complicated ſeries which will involve either the quantity e or the quantity f, or ſome of their powers, products, or multiples, in all its terms, and conſequently $8e \times \Gamma^3$, or the cube of the ſaid product will be another and ſtill more complicated ſeries, which will alſo involve the quantities e and f, or ſome of their powers, products, or multiples, in all its terms. And, for the ſame reaſon, the quantity $6e^{\frac{1}{3}}f^{\frac{2}{3}} \times \Gamma$, or the product of the multiplication of $6e^{\frac{1}{3}}f^{\frac{2}{3}}$ into the complicated ſeries Γ, will be a complicated ſeries which will involve the quantities e and f, or ſome of their powers, products, and multiples, in all its terms.

And in like manner the complicated ſeries Λ (which is ſet down in art. 155) conſiſts of 1, together with the ſeveral numeral co-efficients C, E, G, I, L, N, &c, *ad infinitum* (which are all ſubtracted from 1, as in the ſeries Γ), and of ſeveral following vertical columns of terms involving the fraction $\frac{ff}{gg}$, and its powers $\frac{f^4}{g^4}$, $\frac{f^6}{g^6}$, $\frac{f^8}{g^8}$, $\frac{f^{10}}{g^{10}}$, $\frac{f^{12}}{g^{12}}$, &c, *ad infinitum*. Therefore, if we multiply this complicated ſeries Λ into $2\sqrt[3]{g}$, or $2g^{\frac{1}{3}}$, the product of the ſaid multiplication will be another complicated ſeries which will involve either the quantity g or the quantity f, or ſome of their powers, products, or multiples, in all its terms; and conſequently $8g \times \Lambda^3$, or the cube of the ſaid product, will be another and ſtill more complicated ſeries, which will alſo involve the quantities g and f, or ſome of their powers, products, or multiples, in all its terms. And, for the ſame reaſon, the quantity $6g^{\frac{1}{3}}f^{\frac{2}{3}} \times \Lambda$, or the product of the multiplication of $6g^{\frac{1}{3}}f^{\frac{2}{3}}$ into the complicated ſeries Λ, will be another complicated ſeries which will involve the quantities g and f, or ſome of their powers, products, or multiples, in all its terms.

Further, ſince the terms of the complicated ſeries Λ agree exactly with thoſe of the complicated ſeries Γ in their co-efficients, and in the ſigns + and — that are prefixed to them, and in every thing except the quantities which form the denominators of the literal parts of them, which in the ſeries Γ conſiſt of the powers of ee, and in the ſeries Λ conſiſt of the powers of gg, it is evident that the ſame analogy, or reſemblance, muſt take place between all the complicated ſerieſes derived in any manner from the ſeries Γ and the complicated ſerieſes derived in the ſame manner from the ſeries Λ; and conſequently the complicated ſeries

ſeries $8g \times \Lambda^3$ will conſiſt of terms that will be exactly ſimilar to thoſe of the complicated ſeries $8e \times \Gamma^3$, in their co-efficients, and in the ſigns $+$ and $-$ that are prefixed to them, and in every particular whatſoever, except that, wherever the letter e occurs in the terms of the ſeries $8e \times \Gamma^3$, the letter g will occur in the correſponding terms of the ſeries $8g \times \Lambda^3$; and, in like manner, the complicated ſeries $6g^{\frac{1}{3}}f^{\frac{2}{3}} \times \Lambda$ will conſiſt of terms that will be exactly ſimilar to thoſe of the complicated ſeries $6e^{\frac{1}{3}}f^{\frac{2}{3}} \times \Gamma$, in their co-efficients, and in the ſigns $+$ and $-$ which are prefixed to them, and in every particular whatſoever, except that wherever the letter e occurs in the terms of the ſeries $6e^{\frac{1}{3}}f^{\frac{2}{3}} \times \Gamma$, the letter g will occur in the correſponding terms of the ſeries $6g^{\frac{1}{3}}f^{\frac{2}{3}} \times \Lambda$.

And hence it follows, that, ſince, when we ſubtract the complicated ſeries $6e^{\frac{1}{3}}f^{\frac{2}{3}} \times \Gamma$ from the complicated ſeries $8e \times \Gamma^3$, the ſeveral terms of the one ſeries ſo balance and deſtroy the correſponding terms of the other ſeries, as to leave a remainder that is equal to $2e$ or r, the like effect muſt reſult from the ſubtraction of the complicated ſeries $6g^{\frac{1}{3}}f^{\frac{2}{3}} \times \Lambda$ from the complicated ſeries $8g \times \Lambda^3$, or that the ſeveral terms of the one ſeries will ſo balance and deſtroy the correſponding terms of the other ſeries as to leave a remainder that ſhall be equal to $2g$, or t. Q. E. D.

158. This demonſtration will, I hope, be thought ſufficient to refute the objection ſtated in art. 153 to the reaſoning uſed in art. 150, and conſequently to render the whole of the inveſtigation given above in art. 139, 140, 141 ..... 151, of the tranſcendental expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, *ad infinitum*, for the value of x in the equation $x^3 - qx = t$, perfectly ſatisfactory.

Of the Method of reſolving the Cubick Equation $x^3 - qx = t$, when t is leſs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is leſs than $\frac{q^3}{54}$.

159. The foregoing infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c will converge only when zz is leſs than gg, that is, when $\frac{q^3}{27} - \frac{tt}{4}$ is leſs than $\frac{tt}{4}$, or $\frac{q^3}{27}$ is leſs than $\frac{2tt}{4}$, or $\frac{q^3}{2 \times 27}$ is leſs than $\frac{tt}{4}$, or $\frac{q^3}{54}$ is leſs than $\frac{tt}{4}$. Therefore it is only when $\frac{tt}{4}$ (though leſs than $\frac{q^3}{27}$) is greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than

than $\frac{q^3}{54}$, that the foregoing expression $2\sqrt{^3}\overline{|g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c will be equal to the root of the equation $x^3 - qx = t$. But, when $\frac{tt}{4}$ is less than $\frac{q^3}{54}$, we may find the lesser root of the opposite equation $qx - x^3 = t$ by the method set forth in the preceeding tract contained in pages 379, 380, 381, 382 . . . . . 440 of the present volume of tracts; and, calling this lesser root v, we shall have $qv - v^3 (= t) = x^3 - qx$, and $qv = x^3 + v^3 - qx$, and $qx + qv = x^3 + v^3$, and $q = \frac{x^3+v^3}{x+v} = xx - xv + vv$, and $xx - vx = q - vv$, and $xx - vx + \frac{vv}{4} (= q - vv + \frac{vv}{4} = q - \frac{3vv}{4}) = \frac{4q - 3vv}{4}$, and consequently $x - \frac{v}{2} = \frac{\sqrt{4q-3vv}}{2}$, and $x = \frac{v + \sqrt{4q - 3vv}}{2}$. So that, of whatever magnitude less than $\frac{q^3}{27}$ the quantity $\frac{tt}{4}$ may happen to be, or of whatever magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ the absolute term t of the equation $x^3 - qx = t$ may be supposed to be taken, the value of the root x may always be found either by the method explained in the present discourse, or by the method explained in the preceeding tract.

160. The last investigation of this transcendental expression $2\sqrt{^3}\overline{|g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, *ad infinitum* (which has been explained in art. 139, 140, 141, &c . . . . . 157), was that by which I discovered this expression to be equal to the root x of the equation $x^3 - qx = t$, after having seen it asserted to be so by Monsieur *Clairaut* in his *Elémens d'Algébre*. But, as the proposition appeared to me a very curious one, and worthy to be established by more than one method of proof, I afterwards sought for, and discovered, the long synthetical demonstration of it which has taken up so great a part of this discourse, and which, I apprehend, will have confirmed the truth of it beyond any possibility of doubt.

161. I will now proceed to give a few examples of the resolution of cubick equations of the aforesaid form $x^3 - qx = t$, in which t is supposed to be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or tt is supposed to be less than $\frac{4q^3}{27}$, but greater than $\frac{2q^3}{27}$, or $\frac{tt}{4}$ is supposed to be less than $\frac{q^3}{27}$, but greater than $\frac{2q^3}{4 \times 27}$, or $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$; by means of the foregoing transcendental expression $2\sqrt{^3}\overline{|g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, *ad infinitum*, in order to confirm the truth of the reasonings by which the said series has been obtained.

Examples of the resolution of cubick equations of the foregoing form, $x^3 - qx = t$, when the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, by means of the transcendental expression $2\sqrt{}^3 (g \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, ad infinitum.

EXAMPLE I.

162. Let it be required to resolve the equation $x^3 - 50x = 120$ by means of the said transcendental expression.

Here q is $= 50$; t is $= 120$; $\frac{t}{2}$, or g, is $= 60$; $\frac{tt}{4}$, or gg, is $= 3600$; q^3 is $= 125,000$; and $\frac{q^3}{27}$ is $= \frac{125,000}{27} = 4629.629,629,629,629$, &c, which is greater than 3600, or $\frac{tt}{4}$. Therefore this equation cannot be resolved by Cardan's rule, but may by the expression $2\sqrt{}^3 (g \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, provided that the said series converges.

Now, since $\frac{q^3}{27}$ is $= 4629.629,629,629,629$, &c, and $\frac{tt}{4}$ is 3600, we shall have zz ($= \frac{q^3}{27} - \frac{tt}{4} = 4629.629,629,629,629$, &c $- 3600$) $= 1029.629,629,629,629$, &c, which is much less than 3600, or gg; and consequently the series will converge.

163. We shall therefore have $\frac{zz}{gg} = \frac{1029.629,629,629,6}{3600} = 0.286,008,230,4$,

and consequently $\frac{z^4}{g^4}$ ($= \overline{0.286,008,230,4}|^2$) $= 0.081,800,707,8$;

and $\frac{z^6}{g^6}$ ($= \frac{z^4}{g^4} \times \frac{zz}{gg} = 0.081,800,707,8 \times 0.286,008,230,4$) $= 0.023,395,675,6$;

and $\frac{z^8}{g^8}$ ($= \frac{z^6}{g^6} \times \frac{zz}{gg} = 0.023,395,675,6 \times 0.286,008,230,4$) $= 0.006,691,355,7$;

and $\frac{z^{10}}{g^{10}}$ ($= \frac{z^8}{g^8} \times \frac{zz}{gg} = 0.006,691,355,7 \times 0.286,008,230,4$) $= 0.001,913,782,8$;

 and

and $\frac{z^{12}}{g^{12}}$ $(= \frac{z^{10}}{g^{10}} \times \frac{zz}{gg} = 0.001,913,782,8 \times 0.286,008,230,4) = 0.000,547,357,6$;

and $\frac{z^{14}}{g^{14}}$ $(= \frac{z^{12}}{g^{12}} \times \frac{zz}{gg} = 0.000,547,357,6 \times 0.286,008,230,4) = 0.000,156,548,7$;

and $\frac{z^{16}}{g^{16}}$ $(= \frac{z^{14}}{g^{14}} \times \frac{zz}{gg} = 0.000,156,548,7 \times 0.286,008,230,4) = 0.000,044,774,2$;

and $\frac{z^{18}}{g^{18}}$ $(= \frac{z^{16}}{g^{16}} \times \frac{zz}{gg} = 0.000,044,774,2 \times 0.286,008,230,4) = 0.000,012,805,7$.

And confequently $\frac{Czz}{gg}$ will be $(= C \times 0.286,008,230,4 = \frac{1}{9} \times 0.286,008,230,4 = \frac{0.286,008,230,4}{9}) = 0.031,778,692,2$;

and $\frac{Ez^4}{g^4}$ will be $(= E \times 0.081,800,707,8 = \frac{10}{243} \times 0.081,800,707,8 = \frac{10 \times 0.081,800,707,8}{243} = \frac{0.818,007,078,0}{243}) = 0.003,366,284,2$;

and $\frac{Gz^6}{g^6}$ will be $(= G \times 0.023,395,675,6 = \frac{154}{6561} \times 0.023,395,675,6 = \frac{154 \times 0.023,395,675,6}{6561} = \frac{3.602,934,042,4}{6561}) = 0.000,549,144,0$;

and $\frac{Iz^8}{g^8}$ will be $(= I \times 0.006,691,355,7 = \frac{935}{59,049} \times 0.006,691,355,7 = \frac{935 \times 0.006,691,355,7}{59,049} = \frac{6.256,417,579,5}{59,049}) = 0.000,105,952,9$;

and $\frac{Lz^{10}}{g^{10}}$ will be $(= L \times 0.001,913,782,8 = \frac{55,913}{4,782,969} \times 0.001,913,782,8 = \frac{55,913 \times 0.001,913,782,8}{4,782,969} = \frac{107,005,337,696,4}{4,782,969}) = 0.000,022,372,1$;

and $\frac{Nz^{12}}{g^{12}}$ will be $(= N \times 0.000,547,357,6 = \frac{1,179,256}{129,140,163} \times 0.000,547,357,6 = \frac{1,179,256 \times 0.000,547,357,6}{129,140,163} = \frac{645.474,733,945,6}{129,140,163}) = 0.000,004,998,2$;

and $\frac{Pz^{14}}{g^{14}}$ will be $(= P \times 0.000,156,548,7 = \frac{8,617,640}{1,162,261,467} \times 0.000,156,548,7 = \frac{8,617,640 \times 0.000,156,548,7}{1,162,261,467} = \frac{1,349.080,339,068,0}{1,162,261,467}) = 0.000,001,160,7$;

and $\frac{Rz^{16}}{g^{16}}$ will be $(= R \times 0.000,044,774,2 = \frac{194,327,782}{31,381,059,609} \times 0.000,044,774,2 = \frac{194,327,782 \times 0.000,044,774,2}{31,381,059,609} = \frac{8,700.870,976,824,4}{31,381,059,609}) = 0.000,000,277,2$;

and $\frac{Tz^{18}}{g^{18}}$ will be $(= T \times 0.000,012,805,7 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,012,805,7 = \frac{13,431,479,050 \times 0.000,012,805,7}{2,541,865,828,329} = \frac{171,999.491,270,585,0}{2,541,865,828,329}) = 0.000,000,067,6$;

 And

And consequently the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c will be

$$
\begin{aligned}
&= 1.000,000,000,0 \\
&+ 0.031,778,692,2 - 0.003,366,284,2 \\
&+ 0.000,549,144,0 - 0.000,105,952,9 \\
&+ 0.000,022,372,1 - 0.000,004,998,2 \\
&+ 0.000,001,160,7 - 0.000,000,277,2 \\
&+ 0.000,000,067,6 - \&c .
\end{aligned}
$$

$$
\begin{aligned}
&= 1.032,351,436,6 - 0.003,477,512,5 \\
&= 1.028,873,924,1.
\end{aligned}
$$

Further, since g is $= 60$, we shall have $\sqrt{^3}\overline{)g} = \sqrt{^3}\overline{)60} = 3.914,867,641,1$, and consequently $2\sqrt{^3}\overline{)g}$ $(= 2 \times 3.914,867,641,1) = 7.829,735,282,2$. Therefore the transcendental expression $2\sqrt{^3}\overline{)g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c is $= 7.829,735,282,2 \times 1.028,873,924,1 = 8.055,810,464,4$. Therefore the root x of the proposed equation $x^3 - 50x = 120$ is $= 8.055,810,464,4$. Q. E. I.

164. This value of the root x in the equation $x^3 - 50x = 120$ is exact in the first seven figures 8.055,810, its more accurate value being 8.055,810,345, 702, as may easily be found by Mr. Raphson's method of approximation.

N. B. This equation $x^3 - 50x = 120$ expresses the relation between the diameter of a circle and three chords in it that lie contiguous to each other, and together take up the arch of a semicircle, and form a trapezium of which the diameter of the circle is the fourth side. For, if the three chords are called b, k, and t, and the diameter of the circle is called x, the relation between them will be expressed by the cubick equation $x^3 \left.\begin{matrix} -bb \\ -kk \\ -tt \end{matrix}\right\} \times x = 2bkt$; which, if the numbers 3, 4, and 5 are substituted instead of the letters b, k, and t, will become $x^3 - 50x = 120$. See Sir Isaac Newton's *Arithmetica Universalis*, edit. 2d, A. D. 1722, page 101.

EXAMPLE II.

165. Let it be required to find by means of the same transcendental expression $2\sqrt{^3}\overline{)g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, the root of the equation $x^3 - x = \frac{1}{3}$.

In

In this equation q is $= 1$, and t is $= \frac{1}{3}$; and consequently $\frac{t}{2}$, or g, is $= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, and $\frac{tt}{4}$, or gg, is $= \frac{1}{36}$; and $\frac{q}{3}$ is $= \frac{1}{3}$, and $\frac{q^3}{27}$ is $= \frac{1}{27}$, which is greater than $\frac{1}{36}$, or $\frac{tt}{4}$. Therefore this equation cannot be resolved by Cardan's rule, but may by the transcendental expression $2\sqrt{}^3 \lceil g \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} +$ &c, in case that series is a converging one.

Now, since $\frac{q^3}{27}$ is $= \frac{1}{27}$, and $\frac{tt}{4}$ is $= \frac{1}{36}$, we shall have zz, or $\frac{q^3}{27} - \frac{tt}{4} = \frac{1}{27} - \frac{1}{36} = \frac{36}{27 \times 36} - \frac{27}{27 \times 36} = \frac{9}{27 \times 36} = \frac{1}{3 \times 36} = \frac{1}{108}$, which is less than $\frac{1}{36}$, or $\frac{tt}{4}$, or gg, in the proportion of 1 to 3. Consequently the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} +$ &c will be a converging series, and the expression $2\sqrt{}^3 \lceil g \times$ the said series will be equal to the root of the equation $x^3 - x = \frac{1}{3}$.

166. Since zz is $= \frac{1}{3 \times 36}$, and $\frac{tt}{4}$, or gg, is $= \frac{1}{36}$, we shall have $\frac{zz}{gg} = \frac{1}{3} =$ 0.333,333,333,3;

and consequently $\frac{z^4}{g^4}$ ($= \frac{zz}{gg} \times \frac{zz}{gg} = 0.333,333,333,3 \times \frac{1}{3} = \frac{0.333,333,333,3}{3}$ $= 0.111,111,111,1$;

and $\frac{z^6}{g^6}$ ($= \frac{z^4}{g^4} \times \frac{zz}{gg} = 0.111,111,111,1 \times \frac{1}{3} = \frac{0.111,111,111,1}{3} = 0.037,037,037,0$;

and $\frac{z^8}{g^8}$ ($= \frac{z^6}{g^6} \times \frac{zz}{gg} = 0.037,037,037,0 \times \frac{1}{3} = \frac{0.037,037,037,0}{3}$) $= 0.012,345,679,0$;

and $\frac{z^{10}}{g^{10}}$ ($= \frac{z^8}{g^8} \times \frac{zz}{gg} = 0.012,345,679,0 \times \frac{1}{3} = \frac{0.012,345,679,0}{3}$) $= 0.004,115,226,3$;

and $\frac{z^{12}}{g^{12}}$ ($= \frac{z^{10}}{g^{10}} \times \frac{zz}{gg} = 0.004,115,226,3 \times \frac{1}{3} = \frac{0.004,115,226,3}{3}$) $= 0.001,371,742,1$;

and $\frac{z^{14}}{g^{14}}$ ($= \frac{z^{12}}{g^{12}} \times \frac{zz}{gg} = 0.001,371,742,1 \times \frac{1}{3} = \frac{0.001,371,742,1}{3}$) $= 0.000,457,247,3$;

and $\frac{z^{16}}{g^{16}}$ ($= \frac{z^{14}}{g^{14}} \times \frac{zz}{gg} = 0.000,457,247,3 \times \frac{1}{3} = \frac{0.000,457,247,3}{3}$) $= 0.000,152,415,7$;

and $\frac{z^{18}}{g^{18}}$ ($= \frac{z^{16}}{g^{16}} \times \frac{zz}{gg} = 0.000,152,415,7 \times \frac{1}{3} = \frac{0.000,152,415,7}{3}$) $= 0.000,050,805,2$.

Therefore $\frac{Czz}{gg}$ will be ($= C \times 0.333,333,333,3 = \frac{1}{9} \times 0.333,333,333,3$ $\frac{0.333,333,333,3}{9}$) $= 0.037,037,037,0$;

and

and $\frac{Ez^4}{g^4}$ will be ($= E \times 0.111,111,111,1 = \frac{10}{243} \times 0.111,111,111,1 = \frac{10 \times 0.111,111,111,1}{243} = \frac{1.111,111,111,1}{243}$) $= 0.004,572,473,7$;

and $\frac{Gz^6}{g^6}$ will be ($= G \times 0.037,037,037,0 = \frac{154}{6561} \times 0.037,037,037,0 = \frac{154 \times 0.037,037,037,0}{6561} = \frac{5.703,703,698,0}{6561}$) $= 0.000,869,344,5$;

and $\frac{Iz^8}{g^8}$ will be ($= I \times 0.012,345,679,0 = \frac{935}{59,049} \times 0.012,345,679,0 = \frac{935 \times 0.012,345,679,0}{59,049} = \frac{11.543,209,865,0}{59,049}$) $= 0.000,195,485,2$;

and $\frac{Lz^{10}}{g^{10}}$ will be ($= L \times 0.004,115,226,3 = \frac{55,913}{4,782,969} \times 0.004,115,226,3 = \frac{55,913 \times 0.004,115,226,3}{4,782,969} = \frac{230.094,648,111,9}{4,782,969}$) $= 0.000,048,107,0$;

and $\frac{Nz^{12}}{g^{12}}$ will be ($= N \times 0.001,371,742,1 = \frac{1,179,256}{129,140,163} \times 0.001,371,742,1 = \frac{1,179,256 \times 0.001,371,742,1}{129,140,163} = \frac{1617.635,101,877,6}{129,140,163}$) $= 0.000,012,526,1$;

and $\frac{Pz^{14}}{g^{14}}$ will be ($= P \times 0.000,457,247,3 = \frac{8,617,640}{1,162,261,467} \times 0.000,457,247,3 = \frac{8,617,640 \times 0.000,457,247,3}{1,162,261,467} = \frac{3940.392,622,372,0}{1,162,261,467}$) $= 0.000,003,390,2$;

and $\frac{Rz^{16}}{g^{16}}$ will be ($= R \times 0.000,152,415,7 = \frac{194,327,782}{31,381,059,609} \times 0.000,152,415,7 = \frac{194,327,782 \times 0.000,152,415,7}{31,381,059,609} = \frac{29618.604,922,977,4}{31,381,059,609}$) $= 0.000,000,943,8$;

and $\frac{Tz^{18}}{g^{18}}$ will be ($= T \times 0.000,050,805,2 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,050,805,2 = \frac{13,431,479,050 \times 0.000,050,805,2}{2,541,865,828,329} = \frac{682,388.979,431,060,0}{2,541,865,828,329}$) $= 0.000,000,268,4$;

and consequently the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c will be

$$\begin{aligned}
&= 1.000,000,000,0 \\
&+ 0.037,037,037,0 - 0.004,572,473,7 \\
&+ 0.000,869,334,5 - 0.000,195,485,2 \\
&+ 0.000,048,107,0 - 0.000,012,526,1 \\
&+ 0.000,003,390,2 - 0.000,000,943,8 \\
&+ 0.000,000,268,4 - \&c
\end{aligned}$$

$$\begin{aligned}
&= 1.037,958,137,1 - 0.004,781,428,8 \\
&= 1.033,176,708,3.
\end{aligned}$$

Further, since g is $= \frac{1}{6}$, we shall have $\sqrt[3]{g} = \sqrt[3]{\frac{1}{6}} = \frac{1}{\sqrt[3]{6}} = \frac{1}{1.817,121}$; and

and confequently $2\sqrt{^3}\overline{)g} = \frac{2}{1.817,121}$. Therefore the tranfcendental expreffion $2\sqrt{^3}\overline{)g} \times$ the infinite feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c will be $= \frac{2}{1.817,121} \times 1.033,176,708,3 = \frac{2.066,353,416,6}{1.817,121} = 1.137,157,853,8$, &c. Therefore the root x of the propofed equation $x^3 - x = \frac{1}{3}$ will be equal to 1.137,157,853,8, &c. Q. E. I.

167. This value of the root x of the equation $x^3 - x = \frac{1}{3}$ is exact in the firft fix figures of it, to wit, 1.137,15, the more accurate value of the faid root being 1.137,158,164, as may eafily be found by Mr. Raphfon's method of approximation.

EXAMPLE III.

168. Let it be required to find the root of the equation $x^3 - 5x = 4$ by means of the fame tranfcendental expreffion $2\sqrt{^3}\overline{)g} \times$ the infinite feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c.

Here q is $= 5$; t is $= 4$; $\frac{t}{2}$, or g, is $= 2$; $\frac{tt}{4}$, or gg, is $= 4$; q^3 is $= 125$, and $\frac{q^3}{27}$ is $= \frac{125}{27} = 4.629,629,629,629$, &c, which is greater than 4, or $\frac{tt}{4}$. Therefore this equation cannot be refolved by Cardan's rule, but may by means of the tranfcendental expreffion $2\sqrt{^3}\overline{)g} \times$ the infinite feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, in cafe the faid feries is a converging one.

Now, fince $\frac{q^3}{27}$ is $= 4.629,629,629,629$, &c, and $\frac{tt}{4}$, or gg, is $= 4$, we fhall have $\frac{q^3}{27} - \frac{tt}{4}$, or zz, $= 0.629,629,629,629$, &c, which is lefs than 4, or gg, in the proportion of about 6 to 40, which is a pretty large proportion of minority, and much larger than the proportion of zz to gg in either of the former examples; and confequently the faid feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} +$ &c will converge with a greater degree of fwiftnefs than in either of thofe examples. Therefore the faid tranfcendental expreffion $2\sqrt{^3}\overline{)g} \times$ the infinite feries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} +$ &c will be equal to the root x of the equation $x^3 - 5x = 4$. This expreffion may be computed as follows.

169. Since

169. Since gg is $= 4$, and $\frac{q^3}{27} - \frac{rr}{4}$, or zz, is $= 0.629,629,629,629$, &c, we shall have $\frac{zz}{gg} = \frac{0.629,629,629,629}{4} = 0.157,407,407,4$;

and consequently $\frac{z^4}{g^4}$ ($= \frac{zz}{gg} \times \frac{zz}{gg} = 0.157,407,407,4 \times 0.157,407,407,4$) $= 0.024,777,091,9$;

and $\frac{z^6}{g^6}$ ($= \frac{z^4}{g^4} \times \frac{zz}{gg} = 0.024,777,091,9 \times 0.157,407,407,4$) $= 0.003,900,097,7$;

and $\frac{z^8}{g^8}$ ($= \frac{z^6}{g^6} \times \frac{zz}{gg} = 0.003,900,097,7 \times 0.157,407,407,4$) $= 0.000,613,904,2$;

and $\frac{z^{10}}{g^{10}}$ ($= \frac{z^8}{g^8} \times \frac{zz}{gg} = 0.000,613,904,2 \times 0.157,407,407,4$) $= 0.000,096,633,0$;

and $\frac{z^{12}}{g^{12}}$ ($= \frac{z^{10}}{g^{10}} \times \frac{zz}{gg} = 0.000,096,633,0 \times 0.157,407,407,4$) $= 0.000,015,210,7$;

and $\frac{z^{14}}{g^{14}}$ ($= \frac{z^{12}}{g^{12}} \times \frac{zz}{gg} = 0.000,015,210,7 \times 0.157,407,407,4$) $= 0.000,002,394,2$;

and $\frac{z^{16}}{g^{16}}$ ($= \frac{z^{14}}{g^{14}} \times \frac{zz}{gg} = 0.000,002,394,2 \times 0.157,407,407,4$) $= 0.000,000,376,8$;

and $\frac{z^{18}}{g^{18}}$ ($= \frac{z^{16}}{g^{16}} \times \frac{zz}{gg} = 0.000,000,376,8 \times 0.157,407,407,4$) $= 0.000,000,059,3$.

Therefore $\frac{Czz}{gg}$ will be ($= C \times 0.157,407,407,4 = \frac{1}{9} \times 0.157,407,407,4 = \frac{0.157,407,407,4}{9}$) $= 0.017,489,711,9$;

and $\frac{Ez^4}{g^4}$ will be ($= E \times 0.024,777,091,9 = \frac{10}{243} \times 0.024,777,091,9 = \frac{10 \times 0.024,777,091,9}{243} = \frac{0.247,770,919,0}{243}$) $= 0.001,019,633,4$;

and $\frac{Gz^6}{g^6}$ will be ($= G \times 0.003,900,097,7 = \frac{154}{6561} \times 0.003,900,097,7 = \frac{154 \times 0.003,900,097,7}{6561} = \frac{0.600,615,045,8}{6561}$) $= 0.000,091,543,2$;

and $\frac{Iz^8}{g^8}$ will be ($= I \times 0.000,613,904,2 = \frac{935}{59,049} \times 0.000,613,904,2 = \frac{935 \times 0.000,613,904,2}{59,049} = \frac{0.574,000,427,0}{59,049}$) $= 0.000,009,720,7$;

and $\frac{Lz^{10}}{g^{10}}$ will be ($= L \times 0.000,096,633,0 = \frac{55,913}{4,782,969} \times 0.000,096,633,0 = \frac{5.403,040,929,0}{4,782,969}$) $= 0.000,001,129,6$;

and

and $\frac{Nz^{12}}{g^{12}}$ will be $(= N \times 0.000,015,210,7 = \frac{1,179,256}{129,140,163} \times 0.000,015,210,7$
$= \frac{1,179,256 \times 0.000,015,210,7}{129,140,163} = \frac{17.937,309,239,2}{129,140,163}) = 0.000,000,138,8$;
and $\frac{Pz^{14}}{g^{14}}$ will be $(= P \times 0.000,002,394,2 = \frac{8,617,640}{1,162,261,467} \times 0.000,002,394,2$
$= \frac{8,617,640 \times 0.000,002,394,2}{1,162,261,467} = \frac{20.632,353,688,0}{1,162,261,467}) = 0.000,000,017,7$;
and $\frac{Rz^{16}}{g^{16}}$ will be $(= R \times 0.000,000,376,8 = \frac{194,327,782}{31,381,059,609} \times 0.000,000,376,8$
$= \frac{194,327,782 \times 0.000,000,376,8}{31,381,059,609} = \frac{73.222,708,257,6}{31,381,059,609}) = 0.000,000,002,3$;
and $\frac{Tz^{18}}{g^{18}}$ will be $(= T \times 0.000,000,059,3 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,000,$
$059,3 = \frac{13,431,479,050 \times 0.000,000,059,3}{2,541,865,828,329} = \frac{796.486,707,665,0}{2,541,865,828,329}) = 0.000,000,000,3$;
and consequently the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c will be

$$\begin{aligned}
&= 1.000,000,000,0 \\
&+ 0.017,489,711,9 - 0.001,019,633,4 \\
&+ 0.000,091,543,2 - 0.000,009,720,7 \\
&+ 0.000,001,129,6 - 0.000,000,138,8 \\
&+ 0.000,000,017,7 - 0.000,000,002,3 \\
&+ 0.000,000,000,3 - 0.000,000,000,0 \\
\\
&= 1.017,582,402,7 - 0.001,029,495,2 \\
&= 1.016,552,907,5.
\end{aligned}$$

Further, since g is $= 2$, we shall have $\sqrt[3]{g} = \sqrt[3]{2} = 1.259,921,049,8$, and consequently $2\sqrt[3]{g} = 2 \times 1.259,921,049,8 = 2.519,842,099,6$. Therefore the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c will be equal to $2.519,842,099,6 \times 1.016,552,907,5 = 2.561,552,812,7$. Therefore the root of the proposed equation $x^3 - 5x = 4$ is $= 2.561,552,812,7$. Q. E. I.

170. This number 2.561,552,812,7 is true to ten places of figures, and errs only in the last figure, which ought to be an 8 instead of a 7. For the accurate value of x in this equation is $\frac{1+\sqrt{17}}{2}$, or $\frac{1+4.123,105,625,6}{2}$, or $\frac{5.123,105,625,6}{2}$, or 2.561,552,812,8. For, if we substitute $\frac{1+\sqrt{17}}{2}$ instead of x in the compound quantity $x^3 - 5x$, we shall find that the said quantity will be equal to 4, which is the absolute term of the equation $x^3 - 5x = 4$. For, if x is $= \frac{1+\sqrt{17}}{2}$, we shall

shall have x^3 $\left(= \frac{1+3\times 1^2 \times\sqrt{17}+3\times 1\times 17+17\sqrt{17}}{8} = \frac{1+3\sqrt{17}+3\times 17+17\sqrt{17}}{8}\right.$ $= \frac{1+3\sqrt{17}+51+17\sqrt{17}}{8} = \frac{52+20\sqrt{17}}{8} = \frac{13\times 4+5\times 4\sqrt{17}}{4\times 2}\Big) = \frac{13+5\sqrt{17}}{2}$, and $5x$ $\left(= 5 \times \overline{\left|\frac{1+\sqrt{17}}{2}\right.}\right) = \frac{5+5\sqrt{17}}{2}$, and consequently $x^3 - 5x$ $\left(= \frac{13+5\sqrt{17}}{2}\right.$ $\left.\frac{-5-5\sqrt{17}}{2} = \frac{13-5}{2} = \frac{8}{2}\right) = 4.$ Q. E. D.

171. These examples sufficiently prove that the expression $2\sqrt{^3}\,\overline{|g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c (which we derived from the former expression $2\sqrt{^3}\,\overline{|e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Tz^{18}}{e^{18}}$ — &c, by the peculiar train of reasoning used in the investigation set forth in art. 139, 140, 141, &c . . . . . 158) gives the true root of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, or when the said equation cannot be resolved by Cardan's rule; provided that t (though less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) is greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{tt}{4}$ (though less than $\frac{q^3}{27}$) is greater than $\frac{q^3}{54}$.

I will, however, subjoin one more example to the same purpose; which shall be that of the equation $x^3 - 63x = 162$, which both Dr. Wallis and Mr. De Moivre have resolved by extracting what they call the impossible cube-roots of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$. Now this equation may be resolved by the foregoing expression $2\sqrt{^3}\,\overline{|g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, in the manner following.

EXAMPLE IV.

172. Let it be required to find the root of the equation $x^3 - 63x = 162$ by means of the transcendental expression $2\sqrt{^3}\,\overline{|g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c, *ad infinitum.*

Here q is $= 63$; t is $= 162$; $\frac{t}{2}$, or g, is $= 81$; $\frac{tt}{4}$, or gg, is $= 6561$; $\frac{q}{3}$ is $= 21$; and $\frac{q^3}{27}$ is $= 9261$, which is greater than 6561, or $\frac{tt}{4}$. Therefore this equation cannot be resolved by Cardan's rule, but may by the expression $2\sqrt{^3}\,\overline{|g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$ + &c, in case that series is a converging one.

4 Now,

Now, since $\frac{q^3}{27}$ is $=9261$, and $\frac{tt}{4}$, or gg, is $=6561$, we shall have $\frac{q^3}{27}-\frac{tt}{4}$ $(=9261-6561)=2700$, that is, zz will be $=2700$, which is less than 6561, or gg, in the proportion of 100 to $(\frac{6561}{27}$, or$)$ 243. Therefore the series $1+\frac{Czz}{gg}-\frac{Ez^4}{g^4}+\frac{Gz^6}{g^6}-\frac{Iz^8}{g^8}+\frac{Lz^{10}}{g^{10}}-\frac{Nz^{12}}{g^{12}}+$ &c will be a converging series, and the expression $2\sqrt{}^3\, \overline{g} \times$ the said series will be equal to the root of the equation $x^3-63x=162$.

173. Since zz is $=2700$, and gg is $=6561$, we shall have $\frac{zz}{gg}$ $(=\frac{2700}{6561}=\frac{27\times 100}{27\times 243}=\frac{100}{243})=0.411,522,633,7$;

and consequently $\frac{z^4}{g^4}$ $(=\frac{zz}{gg}\times\frac{zz}{gg}=0.411,522,633,7\times\frac{100}{243}=\frac{100\times 0.411,522,633,7}{243}=\frac{41.152,263,370,0}{243})=0.169,350,878,0$;

and $\frac{z^6}{g^6}$ $(=\frac{z^4}{g^4}\times\frac{zz}{gg}=0.169,350,878,0\times\frac{100}{243}=\frac{100\times 0.169,350,878,0}{243}=\frac{16.935,087,800,0}{243})=0.069,691,719,3$;

and $\frac{z^8}{g^8}$ $(=\frac{z^6}{g^6}\times\frac{zz}{gg}=0.069,691,719,3\times\frac{100}{243}=\frac{100\times 0.069,691,719,3}{243}=\frac{6.969,171,930,0}{243})=0.028,679,719,8$;

and $\frac{z^{10}}{g^{10}}$ $(=\frac{z^8}{g^8}\times\frac{zz}{gg}=0.028,679,719,8\times\frac{100}{243}=\frac{100\times 0.028,679,719,8}{243}=\frac{2.867,971,980,0}{243})=0.011,802,353,8$;

and $\frac{z^{12}}{g^{12}}$ $(=\frac{z^{10}}{g^{10}}\times\frac{zz}{gg}=0.011,802,353,8\times\frac{100}{243}=\frac{100\times 0.011,802,353,8}{243}=\frac{1.180,235,380,0}{243})=0.004,445,413,0$;

and $\frac{z^{14}}{g^{14}}$ $(=\frac{z^{12}}{g^{12}}\times\frac{zz}{gg}=0.004,445,413,0\times\frac{100}{243}=\frac{100\times 0.004,445,413,0}{243}=\frac{0.444,541,300,0}{243})=0.001,829,388,0$;

and $\frac{z^{16}}{g^{16}}$ $(=\frac{z^{14}}{g^{14}}\times\frac{zz}{gg}=0.001,829,388,0\times\frac{100}{243}=\frac{100\times 0.001,829,388,0}{243}=\frac{0.182,938,800,0}{243})=0.000,752,834,5$;

and $\frac{z^{18}}{g^{18}}$ $(=\frac{z^{16}}{g^{16}}\times\frac{zz}{gg}=0.000,752,834,5\times\frac{100}{243}=\frac{100\times 0.000,752,834,5}{243}=\frac{0.075,283,450,0}{243})=0.000,309,808,4$.

Therefore $\frac{Czz}{gg}$ will be $(=C\times 0.411,522,633,7=\frac{1}{9}\times 0.411,522,633,7=\frac{0.411,522,633,7}{9})=0.045,724,737,0$;

and

and $\frac{Ez^4}{g^4}$ will be $\left(= E \times 0.169,350,878,0 = \frac{10}{243} \times 0.169,350,878,0 = \frac{10 \times 0.169,350,878,0}{243} = \frac{1.693,508,780,0}{243}\right) = 0.006,969,171,9$;

and $\frac{Gz^6}{g^6}$ will be $\left(= G \times 0.069,691,719,3 = \frac{154}{6561} \times 0.069,691,719,3 = \frac{10.732,524,772,2}{6561}\right) = 0.001,635,806,2$;

and $\frac{Iz^8}{g^8}$ will be $\left(= I \times 0.028,679,719,8 = \frac{935}{59,049} \times 0.028,679,719,8 = \frac{935 \times 0.028,679,719,8}{59,049} = \frac{26.815,538,013,0}{59,049}\right) = 0.000,454,123,4$;

and $\frac{Lz^{10}}{g^{10}}$ will be $\left(= L \times 0.011,802,353,8 = \frac{55,913}{4,782,969} \times 0.011,802,353,8 = \frac{55,913 \times 0.011,802,353,8}{4,782,969} = \frac{659.905,008,019,4}{4,782,969}\right) = 0.000,137,969,7$;

and $\frac{Nz^{12}}{g^{12}}$ will be $\left(= N \times 0.004,445,413,0 = \frac{1,179,256}{129,140,163} \times 0.004,445,413,0 = \frac{1,179,256 \times 0.004,445,413,0}{129,140,163} = \frac{5,242.279,952,728,0}{129,140,163}\right) = 0.000,040,593,7$;

and $\frac{Pz^{14}}{g^{14}}$ will be $\left(= P \times 0.001,829,388,0 = \frac{8,617,640}{1,162,261,467} \times 0.001,829,388,0 = \frac{8,617,640 \times 0.001,829,388,0}{1,162,261,467} = \frac{15765.007,204,320,0}{1,162,261,467}\right) = 0.000,013,564,0$;

and $\frac{Rz^{16}}{g^{16}}$ will be $\left(= R \times 0.000,752,834,5 = \frac{194,327,782}{31,381,059,609} \times 0.000,752,834,5 = \frac{194,327,782 \times 0.000,752,834,5}{31,381,059,609} = \frac{146,296.658,598,079,0}{31,381,059,609}\right) = 0.000,004,661,9$;

and $\frac{Tz^{18}}{g^{18}}$ will be $\left(= T \times 0.000,309,808,4 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,309,808,4 = \frac{13,431,479,050 \times 0.000,309,808,4}{2,541,865,828,329} = \frac{4,161,185.034,114,020,0}{2,541,865,828,329}\right) = 0.000,001,637,0$;

and consequently the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c will be

$$\begin{aligned}
&1.000,000,000,0 \\
&+ 0.045,724,737,0 - 0.006,969,171,9 \\
&+ 0.001,635,806,2 - 0.000,454,123,4 \\
&+ 0.000,137,969,7 - 0.000,040,593,7 \\
&+ 0.000,013,564,0 - 0.000,004,661,9 \\
&+ 0.000,001,637,0 - \&c
\end{aligned}$$

$$\begin{aligned}
&= 1.047,513,713,9 - 0.007,468,550,9 \\
&= 1.040,045,163,0.
\end{aligned}$$

Further, since g is $= 81$, we shall have $\sqrt[3]{g} = \sqrt[3]{81} = 4.326,749$, and consequently $2\sqrt[3]{g}$ $(= 2 \times 4.326,749) = 8.653,498$. Therefore the transcendental

cendental expression $2\sqrt{^3}\,g \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c will be $= 8.653,498 \times 1.040,045,163,0 = 9.000,028,737,9$, and consequently the root of the proposed equation $x^3 - 63x = 162$ is equal to $9.000,028,737,9$. Q. E. I.

174. This number $9.000,028,737,9$ is true to five places of figures, the true value of x in the equation $x^3 - 63x = 162$ being the whole number 9, as will appear by substituting 9 instead of x in the compound quantity $x^3 - 63x$. For, if we suppose x to be equal to 9, we shall have $x^3 = 729$, and $63x$ $(= 63 \times 9) = 567$, and consequently $x^3 - 63x$ $(= 729 - 567) = 162$; and consequently 9 is equal to the value of x in the equation $x^3 - 63x = 162$.

Q. E. D.

A SCHOLIUM.

175. This resolution of the equation $x^3 - 63x = 162$ answers to Dr. Wallis's resolution of it by extracting the cube-roots of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$, in as much as both resolutions are originally derived from Cardan's rule. But the difference between them is, that the method here delivered is intelligible in every step of it, whereas Dr. Wallis's method treats of impossible quantities, or quantities of which no clear idea can be formed, in the whole course of the process by which the value of x is investigated, though it concludes with a result that is intelligible, to wit, that x is equal to the sum of the two impossible quantities $\frac{9}{2} + \frac{1}{2} \times \sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2} \times \sqrt{-3}$, of which quantities the impossible members $+ \frac{1}{2} \times \sqrt{-3}$ and $- \frac{1}{2} \times \sqrt{-3}$ are equal to each other, and are marked with the contrary signs + and —, and therefore (when added together in order to obtain the sum of the said two impossible quantities $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$) will destroy each other, and leave us only the two possible members of the said two quantities, to wit, $\frac{9}{2}$ and $\frac{9}{2}$, of which the sum is the whole number 9. Doctor Wallis's method of finding $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$ to be the cube-roots of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$, is considered by both Professor Saunderson and Mr. De Moivre as only *tentative*, and not likely to succeed in equations of which the roots are incommensurable to unity, which is the case with ninety-nine equations out of a hundred, when the equations are taken at random, and not framed on purpose with rational numbers for their roots. But Mr. De Moivre has supplied this defect, and given *a certain* method of finding the cube-roots of such impossible binomial quantities: but not without the trisection of an angle, or finding (by the help of a table

a table of ſines, or otherwiſe) the coſine of the third part of a circular arc of which the coſine is given; by means of which triſection it is well known (independently both of Cardan's rule, and of Mr. De Moivre's proceſs) that the ſecond caſe of the cubick equation $y^3 - qy = r$ (in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$) may be reſolved. So that Mr. De Moivre's method of doing this buſineſs, though more perfect than Dr. Wallis's, does not ſeem to be of much uſe in the reſolution of theſe equations. And both methods are equally liable to the objection above-mentioned, of exhibiting to our eyes during the whole courſe of the proceſſes, a parcel of algebraïck quantities, of which our underſtandings cannot form any idea; though, by means of the ultimate excluſion of thoſe quantities, the reſults become intelligible and are true. It is by the introduction of ſuch needleſs difficulties and myſteries into algebra (which, for the moſt part, take their riſe from the ſuppoſition of the exiſtence of negative quantities, or quantities leſs than nothing, or of the poſſibility of ſubtracting a greater quantity from a leſſer) that the otherwiſe clear and elegant ſcience of algebra has been clouded and obſcured, and rendered diſguſting to numbers of men of learning, who are poſſeſſed of a juſt taſte for reaſoning, and could therefore, if they pleaſed, make great advances in the mathematical ſciences, but who are apt to complain of this branch of them, and deſpiſe it on that account. And, doubtleſs, they have too much reaſon to do ſo; and to ſay, in the words of the famous French mathematician and philoſopher, Monſieur Des Cartes, in his diſſertation *De Methodo*, page 11—*Algebram verò, ut ſolet doceri, animadverti certis regulis et numerandi formulis ita eſſe contentam, ut videatur potiùs ars quædam confuſa, cujus uſu ingenium quodam modo turbatur et obſcuratur, quam ſcientia, quâ excolatur et perſpicacius reddatur.* If this complaint was juſt in Des Cartes's time, there is certainly much more reaſon for it now.

176. The paſſage above alluded to in Dr. Wallis's Algebra, is in the 48th chapter, pages 179, 180, of the folio edition printed at London in 1685. And Mr. De Moivre's method of extracting the cube-root of an impoſſible binomial quantity, as $81 + \sqrt{-2700}$, or $a + \sqrt{-b}$, is publiſhed in the appendix to the ſecond volume of Profeſſor Saunderſon's Algebra, pages 744, 745, 746, 747. It is very ingenious, and ſhews that author's great ſkill in the uſe and management of algebraïck quantities. See alſo on this ſubject Monſieur Clairaut's *Elémens d'Algébre*, and a paper of Mr. Nicole in the Memoirs of the French Academy of Sciences for the year 1738, pages 99 and 100. See alſo Mac Laurin's Algebra, part 1, the ſupplement to the 14th chapter, pages 127, 128, 129, 130; and the Philoſophical Tranſactions, No. 451.

Another

Another expression of the value of the root of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$, derived from the foregoing expression of it.

177. But there is another expression for the value of the root x of the equation $x^3 - qx = t$ in the case here supposed, which, as it may be derived from the foregoing expression of it, to wit, $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c, ought not, I think, to be omitted. This expression does not consist entirely of an infinite series (as the foregoing expression does), but partly of a finite algebraick expression, and partly of an infinite series; and fewer terms of the infinite series are necessary to be computed and added together in order to obtain the value of the series to any proposed degree of exactness, than of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, contained in the foregoing expression. It is as follows, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c, *ad infinitum*; of which expression the first part, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, is algebraïck; and the latter part, to wit, $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c is transcendental.

The terms of the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c, are taken from the series that is equal to $\overline{1 + \frac{z}{g}}|^{\frac{1}{3}}$, or the cube root of the binomial quantity $1 + \frac{z}{g}$, to wit, the series $1 + \frac{Bz}{g} - \frac{Czz}{gg} + \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} + \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} + \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} + \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} + \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} + \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} + \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} + \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} +$ &c, *ad infinitum*, by beginning with the third term, $\frac{Czz}{gg}$, and taking every fourth term reckoned from it. This mixt expression $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c, may be derived from the foregoing transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c, in the manner following.

The derivation of the mixt expression of the value of the root of the cubick equation $x^3 - qx = t$, given in the preceeding article 177, from the former transcendental expression of it.

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178. By the binomial theorem in the case of roots we have $\sqrt[3]{1 + \frac{z}{g}} =$ the infinite series $1 + \frac{Bz}{g} - \frac{Czz}{gg} + \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} + \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} + \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} + \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} + \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} + \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} + \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} + \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} +$ &c; and, by the residual theorem in the case of roots, we have $\sqrt[3]{1 - \frac{z}{g}} =$ the infinite series $1 - \frac{Bz}{g} - \frac{Czz}{gg} - \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} - \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} - \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} - \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} - \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} - \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} - \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} - \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} -$ &c. Therefore, if we add $\sqrt[3]{1 - \frac{z}{g}}$ to $\sqrt[3]{1 + \frac{z}{g}}$, and add the latter of the two foregoing serieses (which is equal to $\sqrt[3]{1 - \frac{z}{g}}$) to the former series (which is equal to $\sqrt[3]{1 + \frac{z}{g}}$), the sums thereby obtained will be equal to each other; that is, $\sqrt[3]{1 + \frac{z}{g}} + \sqrt[3]{1 - \frac{z}{g}}$ will be equal to the infinite series $2 - \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{12}} - \frac{2Tz^{18}}{g^{18}} -$ &c, and consequently $\frac{1}{2} \times \sqrt[3]{1 + \frac{z}{g}} + \frac{1}{2} \times \sqrt[3]{1 - \frac{z}{g}}$ will be equal to the infinite series $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} - \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} - \frac{Tz^{18}}{g^{18}} -$ &c. Let the infinite series $\frac{2Czz}{gg} + \frac{2Gz^6}{g^6} + \frac{2Lz^{10}}{g^{10}} + \frac{2Pz^{14}}{g^{14}} + \frac{2Tz^{18}}{g^{18}}$ + &c be added to both sides of the last equation; and we shall have $\frac{1}{2} \times \sqrt[3]{1 + \frac{z}{g}} + \frac{1}{2} \times \sqrt[3]{1 - \frac{z}{g}} +$ the infinite series $\frac{2Czz}{gg} + \frac{2Gz^6}{g^6} + \frac{2Lz^{10}}{g^{10}} + \frac{2Pz^{14}}{g^{14}} + \frac{2Tz^{18}}{g^{18}} +$ &c $=$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c. Therefore (multiplying both sides by $2\sqrt[3]{g}$) we shall have $2\sqrt[3]{g} \times \frac{1}{2} \times \sqrt[3]{1 + \frac{z}{g}} + 2\sqrt[3]{g} \times \frac{1}{2} \times \sqrt[3]{1 - \frac{z}{g}} + 2\sqrt[3]{g} \times$ the series $\frac{2Czz}{gg} + \frac{2Gz^6}{g^6} + \frac{2Lz^{10}}{g^{10}} + \frac{2Pz^{14}}{g^{14}} + \frac{2Tz^{18}}{g^{18}}$ + &c *ad infinitum* $= 2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}}$

$-\frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c *ad infinitum*. But $2\sqrt{}^3 \overline{|g} \times \frac{1}{2} \times \sqrt{}^3 \overline{|1 + \frac{z}{g}}$ is $(= \sqrt{}^3 \overline{|g} \times \sqrt{}^3 \overline{|1 + \frac{z}{g}} = \sqrt{}^3 \overline{|g \times \overline{|1 + \frac{z}{g}}}) = \sqrt{}^3 \overline{|g + z}$; and $2\sqrt{}^3 \overline{|g} \times \frac{1}{2} \times \sqrt{}^3 \overline{|1 - \frac{z}{g}}$ is $(= \sqrt{}^3 \overline{|g} \times \sqrt{}^3 \overline{|1 - \frac{z}{g}} = \sqrt{}^3 \overline{|g \times \overline{|1 - \frac{z}{g}}}) = \sqrt{}^3 \overline{|g - z}$; and $2\sqrt{}^3 \overline{|g} \times$ the series $\frac{2Czz}{gg} + \frac{2Gz^6}{g^6} + \frac{2Lz^{10}}{g^{10}} + \frac{2Pz^{14}}{g^{14}} + \frac{2Tz^{18}}{g^{18}}$ + &c is equal to $4\sqrt{}^3 \overline{|g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + &c. Therefore $\sqrt{}^3 \overline{|g + z} + \sqrt{}^3 \overline{|g - z} + 4\sqrt{}^3 \overline{|g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + &c will be $= 2\sqrt{}^3 \overline{|g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}}$ — &c, and consequently will be equal to the root of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$. Q. E. D.

An application of the last expression of the value of the root of the equation $x^3 - qx = t$ to the resolution of the above-mentioned numeral equations $x^3 - 50 = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, and $x^3 - 63x = 162$.

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179. This new expression $\sqrt{}^3 \overline{|g + z} + \sqrt{}^3 \overline{|g - z} + 4\sqrt{}^3 \overline{|g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + &c *ad infinitum* may be applied to the resolution of the four foregoing cubick equations $x^3 - 50x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, and $x^3 - 63x = 162$, in the manner following.

In the first equation $x^3 - 50x = 120$ we have seen above in art. 162, 163, that g is $= 60$, and zz is $= 1029.629,629,629,629$, &c, and $\frac{Czz}{gg}$ is $= 0.031,778,692,2$, and $\frac{Gz^6}{g^6}$ is $= 0.000,549,144,0$, and $\frac{Lz^{10}}{g^{10}}$ is $= 0.000,022,372,1$, and $\frac{Pz^{14}}{g^{14}}$ is $= 0.000.001,160,7$, and $\frac{Tz^{18}}{g^{18}}$ is $= 0.000,000,067,6$, and $\sqrt{}^3 \overline{|g}$ is $= 3.914,867,641,1$. Therefore the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + &c will be $(= 0.031,778,692,2 + 0.000,549,144,0 + 0.000,022,372,1 + 0.000,001,160,7 + 0.000,000,067,6$ + &c$) = 0.032,351,436,6$ &c, and

$4\sqrt[3]{g}$ will be $(= 4 \times 3.914,867,641,1) = 15.659,470,564,4$, and $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c will be $(= 15.659,470,564,4 \times 0.032,351,436,6) = 0.506,606,369,1$.

And z will be $(= \sqrt{1029.629,629,629,629}$, &c$) = 32.087,842,395,9$; and consequently $g + z$ will be $(= 60 + 32.087,842,395,9) = 92.087,842,395,9$, and $g - z$ will be $(= 60.000,000,000,0 - 32.087,842,395,9) = 27.912,157,604,1$, and $\sqrt[3]{g+z}$ will be $(= \sqrt[3]{92.087,842,395,9}) = 4.515,793,760,9$ and $\sqrt[3]{g-z}$ will be $(= \sqrt[3]{27.912,157,604,1}) = 3.033,410,154,2$. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c will be $= 4.515,793,760,9 + 3.033,410,154,2 + 0.506,606,369,1 = 8.055,810,284,2$; and consequently the root of the equation $x^3 - 50x = 120$ will be $= 8.055,810,284,2$. Q. E. I.

This value of x is exact in the first seven figures 8.055,810, its more accurate value being 8.055,810,345,702.

180. In the next equation $x^3 - x = \frac{1}{3}$ we have seen above, in art. 165, 166, that g is $= \frac{1}{6}$, and zz is $= \frac{1}{3 \times 36}$, or $\frac{1}{108}$, and $\frac{zz}{gg}$ is $= \frac{1}{3}$, and $\frac{Czz}{gg}$ is $= 0.037,037,037,0$, and $\frac{Gz^6}{g^6}$ is $= 0.000,869,344,5$ and $\frac{Lz^{10}}{g^{10}}$ is $= 0.000,048,107,0$, and $\frac{Pz^{14}}{g^{14}}$ is $= 0.000,003,390,2$, and $\frac{Tz^{18}}{g^{18}}$ is $= 0.000,000,268,4$, and that $\sqrt[3]{g}$ is $= \frac{1}{1.817,121}$, and $2\sqrt[3]{g}$ is $= \frac{2}{1.817,121}$. Therefore the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c is $(= 0.037,037,037,0 + 0.000,869,344,5 + 0.000,048,107,0 + 0.000,003,390,2 + 0.000,000,268,4 +$ &c$) = 0.037,958,147,1$ and consequently $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c will be $(= 4\sqrt[3]{g} \times 0.037,958,147,1 = \sqrt[3]{g} \times 4 \times 0.037,958,147,1 = \sqrt[3]{g} \times 0.151,832,588,4 = \frac{1}{1.817,121} \times 0.151,832,588,4 = \frac{0.151,832,588,4}{1.817,121}) = 0.083,556,674,7$.

And z will be $(= \sqrt{\frac{1}{108}} = \sqrt{0.009,259,259,2}) = 0.096,225,044,8$, and consequently $g + z$ will be $(= \frac{1}{6} + 0.096,225,044,8 = 0.166,666,666,6 + 0.096,225,044,8) = 0.262,891,711,4$, and $g - z$ will be $(= 0.166,666,666,6 - 0.096,225,044,8) = 0.070,441,621,8$, and $\sqrt[3]{g+z}$ will be $(= \sqrt[3]{0.262,891,711,4}) = 0.640,607,911,4$, and $\sqrt[3]{g-z}$ will be $(= \sqrt[3]{0.070,441,621,8}) = 0.412,993,403,9$. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g}$ $\times$ the

$\times$ the infinite ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + &c will be = 0.640, 607,911,4 + 0.412,993,403,9 + 0.083,556,674,7 = 1.137,157,989,0; and conſequently the root of the equation $x^3 - x = \frac{1}{3}$ will be = 1.137,157,989,0.

Q. E. I.

This value of x is exact in the firſt ſix figures, 1.137,15, its more accurate value being 1.137,158,164.

181. In the third equation $x^3 - 5x = 4$ we have ſeen above, in art. 168, 169, that g is = 2, and $\sqrt[3]{g}$ is = 1.259,921,049,8, and zz is = 0.629,629, 629,629, &c, and $\frac{zz}{gg}$ is = 0.157,407,407,4, and $\frac{Czz}{gg}$ is = 0.017,489,711,9, and $\frac{Gz^6}{g^6}$ is = 0.000,091,543,2, and $\frac{Lz^{10}}{g^{10}}$ is = 0.000,001,129,6, and $\frac{Pz^{14}}{g^{14}}$ is = 0.000,000,017,7, and $\frac{Tz^{18}}{g^{18}}$ is = 0.000,000,000,3; and conſequently the ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + &c will be (= 0.017,489,711,9 + 0.000, 091,543,2 + 0.000,001,129,6 + 0.000,000,017,7 + 0.000,000,000,3 + &c) = 0.017,582,402,7 &c, and $4\sqrt[3]{g} \times$ the ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + &c will be (= 4 $\times$ 1.259,921,049,8 $\times$ 0.017,582,402,7 = 5.039,684,199,2 $\times$ 0.017,582,402,7) = 0.088,609,757,0.
Further z will be (= $\sqrt{}$ 0.629,629,629,629,629,629,629) = 0.793,492, 047,6, and conſequently $g + z$ will be (= 2 + 0.793,492,047,6) = 2.793, 492,047,6, and $g - z$ will be (= 2.000,000,000,0 — 0.793,492,047,6) = 1.206,507,952,4, and $\sqrt[3]{g+z}$ will be (= $\sqrt[3]{}$ 2.793,492,047,6) = 1.408, 366,911,5, and $\sqrt[3]{g-z}$ will be (= $\sqrt[3]{}$ 1.206,507,952,4) = 1.064,576, 143,3. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} \pm \frac{Tz^{18}}{g^{18}}$ + &c will be (= 1.408,366,911,5 + 1.064, 576,143,3 + 0.088,609,757,0) = 2.561,552,811,8; and conſequently the root of the equation $x^3 - 5x = 4$ will be = 2.561,552,811,8. Q. E. I.

This value of the root of the equation $x^3 - 5x = 4$, is exact in the firſt nine figures 2.561,552,81, its more accurate value being 2.561,552,812,8. See above, art. 170.

182. In the fourth equation $x^3 - 63x = 162$ we have ſeen above in art. 172, 173, that g is = 81, and $\sqrt[3]{g}$ is = 4.326,749, and zz is = 2700, and $\frac{zz}{gg}$ is = $\frac{100}{243}$, and $\frac{Czz}{gg}$ is = 0.045,724,737,0, and $\frac{Gz^6}{g^6}$ is = 0.001,635,806,2, and $\frac{Lz^{10}}{g^{10}}$ is

is $=$ 0.000,137,969,7, and $\frac{Pz^{14}}{g^{14}}$ is $=$ 0.000,013,564,0, and $\frac{Tz^{18}}{g^{18}}$ is $=$ 0.000, 001,637,0.

Therefore the ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c will be (= 0.045,724,737,0 + 0.001,635,806,2 + 0.000,137,969,7 + 0.000,013,564,0 + 0.000,001,637,0 + &c) = 0.047,513,713,9, and $4\sqrt[3]{g} \times$ the ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c will be (= $4\sqrt[3]{g} \times$ 0.047,513,713,9 = 4 × 4.326,749 × 0.047,513,713,9 = 17.306,996 × 0.047,513,713,9) = 0.822,319,656,4.

Further z will be (= $\sqrt{2700} = \sqrt{900 \times 3} = \sqrt{900} \times \sqrt{3} = 30 \times \sqrt{3}$ = 30 × 1.732,050,807,5) = 51.961,524,225,0; and conſequently $g + z$ will be (= 81 + 51.961,524,225,0) = 132.961,524,225,0, and $g - z$ will be (= 81.000,000,000,0 — 51.961,524,225,0) = 29.038,475,775,0, and $\sqrt[3]{g+z}$ will be (= $\sqrt[3]{}$ 132.961,524,225,0) = 5.103,976,447,9, and $\sqrt[3]{g-z}$ will be (= $\sqrt[3]{}$ 29.038,475,775,0) = 3.073,674,958,2. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c *ad infinitum* will be = 5.103,976,447,9 + 3.073,674,958,2 + 0.822,319,656,4 = 8.999,971,062,5; and conſequently the root of the equation $x^3 - 63x = 162$ will be = 8.999,971,062,5. Q. E. I.

This number 8.999,971,062,5 is exact in the firſt five figures, 8.9999, the true value of the root x in this equation being the whole number 9. See above, art. 174.

End of the reſolution of the four equations $x^3 - 50x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, *and* $x^3 - 63x = 162$, *by means of the mixt expreſſion* $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ *the infinite ſeries* $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}}$ + *&c.*

A third expreſſion of the value of the root of the equation $x^3 - qx = t$, derived from the expreſſion obtained for it above in art. 139, 140, 141, *&c* . . . . 157.

183. We may alſo derive another expreſſion for the value of the root of the equation $x^3 - qx = t$, when t is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{tt}{4}$ is leſs than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, from the former tranſcendental expreſſion of it, to wit, $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8}$

$\frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c, which was obtained by the inveſtigation ſet forth in art. 139, 140, 141, &c .... 157. This expreſſion is $4\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Wz^{20}}{g^{20}} -$ &c *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$; which (like the ſecond expreſſion of the value of x given above in art. 177) is partly algebraïck, and partly tranſcendental; but in this expreſſion the tranſcendental part, to wit, $4\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Wz^{20}}{g^{20}} -$ &c *ad infinitum*, is greater than the algebraïck part, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, whereas in the ſecond expreſſion before-mentioned, to wit, the expreſſion $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c, the algebraïck part, $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, was greater than the tranſcendental part $4\sqrt[3]{g} \times$ the infinite ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c *ad infinitum.* And, further, in the laſt, or third, expreſſion, to wit, $4\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Wz^{20}}{g^{20}} -$ &c $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$, it is the difference of the tranſcendental part and the algebraïck part, or the exceſs of the former above the latter, that is equal to the root of the equation $x^3 - qx = t$; whereas in the former, or ſecond, expreſſion, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite ſeries $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c *ad infinitum*, it is the ſum of the algebraïck and the tranſcendental parts of the expreſſion that is equal to the root of the ſaid equation. In both expreſſions the indexes of the powers of z and g in the numerators and denominators of the terms of the infinite ſerieſes contained in the tranſcendental parts of them, increaſe continually by 4.

This third expreſſion $4\sqrt[3]{g} \times$ the infinite ſeries $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Vz^{20}}{g^{20}} -$ &c *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$, may be derived from the foregoing tranſcendental expreſſion $2\sqrt[3]{g} \times$ the infinite ſeries $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c *ad infinitum* in the manner following.

The

The Derivation of the third expression of the value of the root of the equation $x^3 - qx = r$, given in the preceeding article 183, from the former transcendental expression of it.

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184. Since $\sqrt[3]{g+z}$ is $\left(= \sqrt[3]{g} \times \sqrt[3]{1+\frac{z}{g}}\right) =$ (by the binomial theorem in the case of roots) $\sqrt[3]{g} \times$ the infinite series $1 + \frac{Bz}{g} - \frac{Czz}{gg} + \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} + \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} + \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} + \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} + \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} + \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} + \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} + \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} + \frac{Vz^{19}}{g^{19}} - \frac{Wz^{20}}{g^{20}} +$ &c, and $\sqrt[3]{g-z}$ is $\left(= \sqrt[3]{g} \times \sqrt[3]{1-\frac{z}{g}}\right) =$ (by the residual theorem in the case of roots) $\sqrt[3]{g} \times$ the infinite series $1 - \frac{Bz}{g} - \frac{Czz}{gg} - \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} - \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} - \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} - \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} - \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} - \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} - \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} - \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} - \frac{Vz^{19}}{g^{19}} - \frac{Wz^{20}}{g^{20}} -$ &c, it follows that $\sqrt[3]{g+z} + \sqrt[3]{g-z}$ will be $= \sqrt[3]{g} \times$ the infinite series $2 - \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} -$ &c. Now the series $2 - \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} -$ &c *ad infinitum* is evidently less than the series $2 - \frac{2Ez^4}{g^4} - \frac{2Iz^8}{g^8} - \frac{2Nz^{12}}{g^{12}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Wz^{20}}{g^{20}} -$ &c *ad infinitum*; and therefore, *à fortiori*, is less than twice that series, or than the series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} -$ &c *ad infinitum*. Therefore $\sqrt[3]{g} \times$ the series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} -$ &c *ad infinitum* will be greater than $\sqrt[3]{g} \times$ the series $2 - \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} -$ &c *ad infinitum*, and consequently will be greater also than $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, which is equal to that quantity. Let $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, and, its equal, the quantity $\sqrt[3]{g} \times$ the infinite series $2 - \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} -$ &c *ad infinitum*, be subtracted from the quantity $\sqrt[3]{g} \times$ the infinite series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} -$ &c *ad infinitum*; and the remainders will

will be equal to each other; that is, $\sqrt[3]{g} \times$ the infinite series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}}$ — &c *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$ will be equal to $\sqrt[3]{g} \times$ the series $2 + \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} + \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} + \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} + \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} + \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}}$ + &c *ad infinitum*, and consequently to $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \frac{Wz^{20}}{g^{20}}$ + &c *ad infinitum*, or to the transcendental expression for the root of the equation $x^3 - qx = t$ discovered by the investigation contained in art. 139, 140, 141, &c . . . . 157. Therefore the expression $\sqrt[3]{g} \times$ the infinite series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}}$ — &c *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$, or the expression $4\sqrt[3]{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Wz^{20}}{g^{20}}$ — &c *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$, will be equal to $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \frac{Wz^{20}}{g^{20}}$ + &c *ad infinitum*, or to the root of the equation $x^3 - qx = t$. Q. E. D.

An application of the last, or third, expression of the value of the root of the equation $x^3 - qx = t$ to the resolution of the above-mentioned numeral equations $x^3 - 50x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, and $x^3 - 63x = 162$.

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185. This last expression, $4\sqrt[3]{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Wz^{20}}{g^{20}}$ — &c *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$, may be applied to the resolution of the foregoing equations $x^3 - 50x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, and $x^3 - 63x = 162$ in the manner following.

In the first equation $x^3 - 50x = 120$ we have seen above, in art. 162, 163, that g is $= 60$, and zz is $= 1029.629,629,629,629$, &c, and $\frac{Ez^4}{g^4}$ is 0.003,366,284,2, and $\frac{Iz^8}{g^8}$ is $=$ 0.000,105,952,9, and $\frac{Nz^{12}}{g^{12}}$ is $=$ 0.000,004,998,2, and $\frac{Rz^{16}}{g^{16}}$ is $=$ 0.000,000,277,2, and that $\sqrt[3]{g}$ is $=$ 3.914,867,641,1. Therefore the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}}$ — &c will be (= 1.000,000,000,0

— 0.003,

2

— 0.003,366,284,2 — 0.000,105,952,9 — 0.000,004,998,2 — 0.000,000, 277,2 — &c = 1.000,000,000,0 — 0.003,477,512,5) = 0.996,522,487,5, and $4\sqrt[3]{g}$ will be (= 4 × 3.914,867,641,1) = 15.659,470,564,4, and $4\sqrt[3]{g}$ × the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}}$ — &c *ad infinitum* will be (= 15.659,470,564,4 × 0.996,522,487,5) = 15.605,014,559,7.

And we have seen above, in art. 179, that $g + z$ is = 92.087,842,395,9, and $g - z$ is = 27.912,157,604,1, and $\sqrt[3]{g+z}$ is = 4.515,793,760,9, and $\sqrt[3]{g-z}$ is = 3.033,410,154,2. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z}$ will be (= 4.515,793,760,9 + 3.033,410,154,2) = 7.549,203,915,1, and $4\sqrt[3]{g}$ × the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}}$ — &c *ad infinitum* — $\sqrt[3]{g+z} - \sqrt[3]{g-z}$ will be (= 15.605,014,559,7 — 7.549,203, 915,1) = 8.055,810,644,6. Therefore the root of the equation $x^3 - 50x = 120$ will be = 8.055,810,644,6. Q. E. I.

This value of x is exact in the first seven figures 8.055,810, its more accurate value being 8.055,810,345,702.

186. In the next equation $x^3 - x = \frac{1}{3}$ we have seen above, in art. 165, 166, that g is = $\frac{1}{6}$, and zz is = $\frac{1}{3 \times 36}$, or $\frac{1}{108}$, and $\frac{zz}{gg}$ is = $\frac{1}{3}$, and $\frac{Ez^4}{g^4}$ is = 0.004,572,473,7, and $\frac{Iz^8}{g^8}$ is = 0.000,195,485,2, and $\frac{Nz^{12}}{g^{12}}$ is = 0.000,012, 526,1, and $\frac{Rz^{16}}{g^{16}}$ is = 0.000,000,943,8 and $\sqrt[3]{g}$ is = $\frac{1}{1.817,121}$. Therefore the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}}$ — &c will be (= 1.000,000,000,0 — 0.004,572,473,7 — 0.000,195,485,2 — 0.000,012,526,1 — 0.000,000, 943,8 — &c = 1.000,000,000,0 — 0.004,781,428,8) = 0.995,218,571,2, and $4\sqrt[3]{g}$ × the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}}$ — &c will be (= $4\sqrt[3]{g}$ × 0.995,218,571,2 = 4 × $\frac{1}{1.817,121}$ × 0.995,218,571,2 = $\frac{4 \times 0.995,218,571,2}{1.817,121}$ = $\frac{3.980,874,284,8}{1.817,121}$) = 2.190,759,055,0.

And we have seen in art. 181 that z is = 0.096,225,044,8, and $g + z$ is = 0.262,891,711,4, and $g - z$ is = 0.070,441,621,8, and $\sqrt[3]{g+z}$ is = 0.640,607,911,4, and $\sqrt[3]{g-z}$ is = 0.412,993,403,9. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z}$ will be (= 0.640,607,911,4 + 0.412,993,403,9) = 1.053,601, 315,3, and $4\sqrt[3]{g}$ × the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}}$ — &c *ad infinitum* — $\sqrt[3]{g+z} - \sqrt[3]{g-z}$ will be (= 2.190,759,055,0 — 1.053,601, 315,3) = 1.137,157,739,1; and consequently the root of the equation $x^3 - x = \frac{1}{3}$ will be = 1.137,157,739,1. Q. E. D.

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This value of x is exact in the first six figures, 1.137,15, its more accurate value being 1.137,158,164.

187. In the third equation $x^3 - 5x = 4$ we have seen above, in art. 168, 169, that g is $= 2$, and $\sqrt[3]{g}$ is $=$ 1.259,921,049,8 and zz is $=$ 0.629,629, 629,629, &c, and $\frac{zz}{gg}$ is $=$ 0.157,407,407,4, and $\frac{Ez^4}{g^4}$ is $=$ 0.001,019,633,4, and $\frac{Iz^8}{g^8}$ is $=$ 0.000,009,720,7, and $\frac{Nz^{12}}{g^{12}}$ is $=$ 0.000,000,138,8, and $\frac{Rz^{16}}{g^{16}}$ is $=$ 0.000,000,002,3. Therefore the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} -$ &c will be ($=$ 1.000,000,000,0 $-$ 0.001,019,633,4 $-$ 0.000,009,720,7 $-$ 0.000,000,138,8 $-$ 0.000,000,002,3 $-$ &c $=$ 1.000,000,000,0 $-$ 0.001, 029,495,2) $=$ 0.998,970,504,8, and $4\sqrt[3]{g} \times$ the said series will be ($= 4\sqrt[3]{g} \times$ 0.998,970,504,8 $= 4 \times$ 1.259,921,049,8 $\times$ 0.998,970,504,8 $=$ 5.039,684,199,2 $\times$ 0.998,970,504,8) $=$ 5.034,495,868,5.

And we have seen in art. 181 that $\sqrt[3]{g+z}$ is $=$ 1.408,366,911,5, and that $\sqrt[3]{g-z}$ is $=$ 1.064,576,143,3. Therefore $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} -$ &c $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$ will be $=$ 5.034, 495,868,5 $-$ 1.408,366,911,5 $-$ 1.064,576,143,3 $=$ 5.034,495,868,5 $-$ 2.472,943,054,8 $=$ 2.561,552,813,7; and consequently the root of the equation $x^3 - 5x = 4$ will be $=$ 2.561,552,813,7. Q. E. I.

This value of x is exact in the first nine figures 2.561,552,81, the more accurate value of it being 2.561,552,812,8. See above, art. 170.

188. In the 4th and last equation $x^3 - 63x = 162$ we have seen above, in art. 172, 173, that g is $= 81$, and $\sqrt[3]{g}$ is $=$ 4.326,749, and zz is $= 2700$, and $\frac{zz}{gg}$ is $= \frac{100}{243}$, and $\frac{Ez^4}{g^4}$ is $=$ 0.006,969,171,9, and $\frac{Iz^8}{g^8}$ is $=$ 0.000,454, 123,4, and $\frac{Nz^{12}}{g^{12}}$ is $=$ 0.000,040,593,7, and $\frac{Rz^{16}}{g^{16}}$ is $=$ 0.000,004,661,9. Therefore the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} -$ &c will be ($=$ 1.000,000,000,0 $-$ 0.006,969,171,9 $-$ 0.000,454,123,4 $-$ 0.000,040,593,7 $-$ 0.000,004,661,9 $-$ &c $=$ 1.000,000,000,0 $-$ 0.007,468,550,9) $=$ 0.992,531,449,1, and $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}}$ $-$ &c *ad infinitum* will be ($= 4\sqrt[3]{g} \times$ 0.992,531,449,1 $= 4 \times$ 4.326,749 $\times$ 0.992,531,449,1 $=$ 17.306,996 $\times$ 0.992,531,449,1) $=$ 17.177,737,819,4.

And we have seen in art. 182, that $\sqrt[3]{g+z}$ is $=$ 5.103,976,447,9, and that $\sqrt[3]{g-z}$ is $=$ 3.073,674,958,2. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z}$ will be ($=$ 5.103,976,447,9 $+$ 3.073,674,958,2) $=$ 8.177,651,406,1. Therefore $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} -$ &c *ad infinitum* $-$

$\sqrt{^3}\,\overline{g+z} - \sqrt{^3}\,\overline{g-z}$ will be $= 17.177,737,819,4 - 8.177,651,406,1 = 9.000,086,413,3$; and consequently the root of the equation $x^3 - 63x = 162$ will be $= 9.000,086,413,3$. Q. E. I.

This value of x is exact in the first five places of figures, 9.0000, the true value of x being the whole number 9.

A short view of the three expressions that have been here obtained for the value of the root of the cubick equation $x^3 - qx = t$.

189. It appears therefore that the root of the cubick equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, will be equal to either of the three following expressions; to wit,

First, to $2\sqrt{^3}\,\overline{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} -$ &c *ad infinitum*; which is an expression wholly transcendental;

And, 2dly, to the expression $\sqrt{^3}\,\overline{g+z} + \sqrt{^3}\,\overline{g-z} + 4\sqrt{^3}\,\overline{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} +$ &c *ad infinitum*; which is an expression partly algebraïck and partly transcendental, and in which the transcendental part is added to the algebraïck part;

And, 3dly, to the expression $4\sqrt{^3}\,\overline{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} -$ &c *ad infinitum* $- \sqrt{^3}\,\overline{g+z} - \sqrt{^3}\,\overline{g-z}$; which is also an expression partly transcendental and partly algebraïck, but of which the greater part is transcendental, and the algebraïck part is subtracted from the transcendental part.

And we may observe that the algebraïck part of the two latter expressions, to wit, the quantity $\sqrt{^3}\,\overline{g+z} + \sqrt{^3}\,\overline{g-z}$, is similar to the algebraïck expression $\sqrt{^3}\,\overline{e+s} + \sqrt{^3}\,\overline{e-s}$, given by Cardan's second rule for the root of the equation $y^3 - qy = r$, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$. See on this subject Monsieur Clairaut's *Elémens d'Algébre*, pages 282, 283, &c, to page 297.

Of

Of the resolution of the equation $qy - y^3 = r$ by means of either of the three foregoing expressions of the value of the root of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$.

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190. The greatest possible magnitude of the absolute term r in the equation $qy - y^3 = r$ is $\frac{2q\sqrt{q}}{3\sqrt{3}}$, which is the value of the compound quantity $qy - y^3$ when y is $= \frac{\sqrt{q}}{\sqrt{3}}$. And, when r is equal to this quantity $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qy - y^3 = r$ will have but one root, to wit, $\frac{\sqrt{q}}{\sqrt{3}}$; and, when r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qy - y^3 = r$ will have two roots, of which the lesser will be less than $\frac{\sqrt{q}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{q}}{\sqrt{3}}$, but less than $\sqrt{q}$. See my Dissertation on the Use of the Negative Sign in Algebra, art. 114, page 92. Therefore, whenever we have an equation of this form, $qy - y^3 = r$, proposed to us to be resolved, the absolute term r of such equation will be always equal to the absolute term t of some equation of the opposite form $x^3 - qx = t$, in which the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and which therefore cannot be resolved by means of the expressions given by Cardan's second rule; and consequently we may suppose the compound quantity $qy - y^3$, which forms the first side of the equation $qy - y^3 = r$, to be equal to the compound quantity $x^3 - qx$, which forms the first side of the equation $x^3 - qx = t$. Now, upon this supposition that $qy - y^3$ is equal to $x^3 - qx$, we may reduce the equation $qy - y^3 = r$ to a quadratick equation, and derive the values of its two roots from the value of x (the only root of the equation $x^3 - qx = t$) by proceeding in the manner following.

Since $qy - y^3$ is $(= r = t) = x^3 - qx$, add y^3 to both sides; and we shall have $qy = x^3 - qx + y^3$. Now let qx be added to both sides; and we shall have $qx + qy = x^3 + y^3$. Therefore $\frac{qx+qy}{x+y}$ will be $= \frac{x^3+y^3}{x+y}$, and consequently q will be $= xx - xy + yy$. Add xy to both sides; and we shall have $q + xy = xx + yy$, and (subtracting yy from both sides) $q + xy - yy = xx$. Further, since q is less than xx, and consequently than the other side of the equation, let it be subtracted from both sides; and we shall have $xy - yy = xx - q$. And, because $\frac{xx}{4}$, or the square of $\frac{x}{2}$, is greater than $\overline{x - y}| \times y$, or than $xx - xy$, and consequently than the other side of the equation, both sides may be subtracted from $\frac{xx}{4}$. Let them be so subtracted; and we shall have $\frac{xx}{4} - xy + yy$ $(= \frac{xx}{4} - xx + q = \frac{xx}{4} - \frac{4xx}{4} + \frac{4q}{4}) = \frac{4q - 3xx}{4}$. Therefore the square-root of $\frac{xx}{4} - xy + yy$ will be $= \frac{\sqrt{4q - 3xx}}{2}$. But the quantity $\frac{xx}{4} - xy + yy$ has two square-

roots,

roots, a greater and a leſſer, to wit, $\frac{x}{2} - y$ and $y - \frac{x}{2}$, according as $\frac{x}{2}$ is greater, or leſs, than y. Therefore for the determination of the greater value of y we ſhall have the equation $y - \frac{x}{2} = \frac{\sqrt{4q - 3xx}}{2}$, and conſequently $y = \frac{x + \sqrt{4q - 3xx}}{2}$; and for the determination of the leſſer value of y we ſhall have the equation $\frac{x}{2} - y = \frac{\sqrt{4q - 3xx}}{2}$, and conſequently $\frac{x}{2} = y + \frac{\sqrt{4q - 3xx}}{2}$, and $y = \frac{x - \sqrt{4q - 3xx}}{2}$. And conſequently the greater root of the cubick equation $qy - y^3 = r$ will be $\frac{x + \sqrt{4q - 3xx}}{2}$, and the leſſer root of the ſame equation will be $\frac{x - \sqrt{4q - 3xx}}{2}$. Therefore, if t, or r, is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or if $\frac{tt}{4}$, or $\frac{rr}{4}$, is leſs than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, we may obtain the values of the two roots of the cubick equation $qy - y^3 = r$ by firſt computing the value of the root x of the oppoſite equation $x^3 - qx = t$ by means of either of the three expreſſions ſet down in the preceeding article 189, and then computing the two quadratick expreſſions $\frac{x + \sqrt{4q - 3xx}}{2}$ and $\frac{x - \sqrt{4q - 3xx}}{2}$.

Q. E. I.

Of the Triſection of a circular arc by means of either of the three foregoing expreſſions for the value of the root of the cubick equation $x^3 - qx = t$, when t is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{tt}{4}$ is leſs than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$.

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191. If a be the radius of a circle, and conſequently $2a$ be its diameter, and k be the chord of any arc in it, and y be the chord of the third part of the leſſer of the two arcs whereof k is the common chord, the relation between the chords k and y will be expreſſed by this equation $3aay - y^3 = aak$, of which the ſaid chord y will be the leſſer root; as is demonſtrated in many books of mathematicks, and, amongſt the reſt in my Diſſertation on the Uſe of the Negative Sign in Algebra, publiſhed in the year 1758, pages 183, 184, art. 220, 221. Now let q be ſubſtituted in this equation inſtead of $3aa$, and let r be ſubſtituted in it inſtead of aak; and the equation $3aay - y^3 = aak$ will thereby be converted into the equation $qy - y^3 = r$, in which $\frac{\sqrt{q}}{\sqrt{3}}$ will repreſent a, or the radius of the circle, and $\frac{2\sqrt{q}}{\sqrt{3}}$ will repreſent $2a$, or the diameter of the circle, and ($\frac{r}{aa}$, or r divided

vided by $\frac{q}{3}$, or $r \times \frac{3}{q}$, or) $\frac{3r}{q}$ will reprefent k, or the given chord of the greater arch, which is to be trifected, or of the third part of which we are to find the chord y; and the chord y will be the leffer root of the faid equation $qy - y^3 = r$. We muft therefore endeavour to find the leffer root of this equation $qy - y^3 = r$. Now it appears from the foregoing art. 190, that, if the abfolute term t of the equation $x^3 - qx = t$ be equal to the abfolute term r of the equation $qy - y^3 = r$, the leffer root of the equation $qy - y^3 = r$ will be equal to the quadratick expreffion $\frac{x - \sqrt{4q - 3xx}}{2}$. Therefore the value of the chord of the arc which is equal to one third part of the leffer of the two arcs of which the given chord k, or $\frac{3r}{q}$, is the common chord, in the circle of which a, or $\frac{\sqrt{q}}{\sqrt{3}}$, is the radius, and $2a$, or $\frac{2\sqrt{q}}{\sqrt{3}}$, is the diameter, will be equal to the quadratick expreffion $\frac{x - \sqrt{4q - 3xx}}{2}$. Therefore, if r, or t, is lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or if aak is lefs than $2a^3$, but greater than $\sqrt{2} \times a^3$, or if k is lefs than ($\frac{2a^3}{aa}$, or) $2a$, but greater than ($\sqrt{2} \times \frac{a^3}{aa}$, or) $\sqrt{2} \times a$, that is, if the given chord k is lefs than the diameter $2a$, but greater than $\sqrt{2} \times a$, or the chord of a quadrantal arc, or of an arc of 90 degrees, or, if the given arc that is to be trifected, is lefs than the femicircumference, but greater than the arch of a quadrant, we may find the value of the chord y by firft finding the root x of the oppofite equation $x^3 - qx = t$, by means of one of the three expreffions fet down in art. 189, and then computing the quadratick expreffion $\frac{x - \sqrt{4q - 3xx}}{2}$.

Q. E. I.

Of the analogy, or harmony, between circular arcs and logarithms, or between the meafures of angles and the meafures of ratios.

192. This application of the expreffions obtained in the foregoing articles to the trifection of a circular arc is an inftance of the analogy, or *harmony* (as Mr. Cotes calls it) that fubfifts between logarithms and circular arcs, or between the meafures of ratios and the meafures of angles. For from the expreffion $\sqrt[3]{e + s} + \sqrt[3]{e - s}$ given by Cardan's fecond rule for the root of the equation $y^3 - qy = r$ in the firft cafe of that equation, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$ (the value of which expreffion is to be obtained by extracting the cube-roots of the given quantities $e + s$ and $e - s$, that is, by trifecting the ratios of $e + s$ to 1 and of $e - s$ to 1) we have derived the three expreffions fet down in art. 189, by either of which we are enabled to find the value of

of the root x of the equation $x^3 - qx = t$, when t is lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, from which value we may afterwards, by computing the quadratick expreffion $\frac{x - \sqrt{4q - 3xx}}{2}$, obtain the value of the leffer root of the equation $qy - y^3 = r$, or the value of the chord of an arc that is equal to the third part of the leffer of the two arcs of which the given quantity $\frac{3r}{q}$, or k, is the common chord, in a circle of which the given quantity $\frac{\sqrt{q}}{\sqrt{3}}$, or a, is the radius; or, in other words, may trifect the faid leffer of the two arcs of which $\frac{3r}{q}$, or k, is the common chord; provided that the faid arc is greater than the arch of a quadrant. And confequently problems that require the trifection of a circular arc, or of an angle, may, by means of the method here explained, be folved by the trifection of a ratio, or by the help of a table of logarithms.

Of the refolution of the cubick equation $x^3 - qx = t$ by means of expreffions derived from Cardan's rules, when t is lefs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is lefs than $\frac{q^3}{54}$.

193. When the abfolute term t of the equation $x^3 - qx = t$, or the abfolute term r of the oppofite equation $qy - y^3 = r$, is lefs than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$, or $\frac{rr}{4}$, is lefs than $\frac{q^3}{54}$, the leffer root of the equation $qy - y^3 = r$ may be found by means of the expreffions inveftigated in the preceeding difcourfe contained in pages 379, 380, 381, &c ..... 440 of this volume of tracts; which expreffions are derived from the expreffion given by Cardan's firft rule for the refolution of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$: and, when the value of the faid leffer root of the equation $qy - y^3 = r$ has been fo obtained, the value of the root x of the equation $x^3 - qx = t$ may be derived from it by computing the quadratick expreffion $\frac{y + \sqrt{4q - 3yy}}{2}$. For, fince $x^3 - qx$ is $(= t = r) = qy - y^3$, we fhall (by adding y^3 to both fides) have $x^3 - qx + y^3 = qy$, and (by adding qx to both fides) $x^3 + y^3 = qx + qy = q \times \overline{x + y}$. Therefore $\frac{x^3 + y^3}{x + y}$ will be $= \frac{q \times \overline{x + y}}{x + y} = q$; that is, $xx - xy + yy$ will be $= q$; and confequently, fubtracting yy (which is lefs than q, and therefore than the other fide of the equation) from both fides, we fhall have $xx - xy = q - yy$, or $xx - yx = q - yy$; and (adding $\frac{yy}{4}$ to both fides) $xx - yx + \frac{yy}{4} = q - yy + \frac{yy}{4} = \frac{4q - 4yy + yy}{4}$

$\frac{4q - 4yy + yy}{4} = \frac{4q - 3yy}{4}$. Therefore $x - \frac{y}{2}$ will be $= \frac{\sqrt{4q - 3yy}}{2}$, and x will be $= \frac{y}{2} + \frac{\sqrt{4q - 3yy}}{2}$, or $\frac{y + \sqrt{4q - 3yy}}{2}$. Q. E. D.

Conclusion concerning both branches of the second or irreducible *case of the cubick equation* $y^3 - qy = r$.

194. It appears therefore that in both the branches of the equation $x^3 - qx = t$, or of the second case of the cubick equation $y^3 - qy = r$, that is, both when t, or r, is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, and when it is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or both when $\frac{tt}{4}$, or $\frac{rr}{4}$, is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, and when it is less than $\frac{q^3}{54}$, the value of the root x, or y, of the equation $x^3 - qx = t$, or $y^3 - qy = r$, may always be found by means either of the expressions derived in the present discourse from Cardan's second rule, given for the resolution of the equation $y^3 - qy = r$ in its first case, or of the expressions derived in the last preceeding discourse in this volume of tracts from Cardan's first rule, given for the resolution of the equation $qy + y^3 = r$, or $y^3 + qy = r$. And thus that which is called the *irreducible* case of the equation $y^3 - qy = r$ is shewn to be always capable of being resolved by means of expressions derived from one or other of Cardan's rules by the help of Sir Isaac Newton's binomial and residual theorems, which therefore seem to form a sort of bridge of communication between the problems belonging to the trisection of a ratio, and the problems belonging to the trisection of an angle, or between the problems depending on, or resolvable by means of, the asymptotick areas of an hyperbola, or a table of logarithms, and the problems depending on, or resolvable by means of, circular arches, or a table of sines and tangents.

A SCHOLIUM.

195. The first person who seems to have discovered the possibility of thus extending Cardan's rules to the second, or *irreducible*, case of the cubick equation $y^3 - qy = r$, was the famous German mathematician and philosopher, of the latter part of the last century, Mr. Leibnitz of Hanover, as is shewn in the scholium at the end of the last preceeding discourse in this volume of tracts, pages 438, 439, 440. But he has done it by the means of what are called *impossible* quantities, or quantities involving the square-roots of negative quantities, which, in the expressions to which his calculations lead him, are of such magnitudes, and so connected by the signs + and —, as ultimately to destroy, or expunge, each other; whereas in this and the preceeding tract I have carefully avoided all mention of these unintelligible mysteries.

6

A

CONJECTURE

CONCERNING THE METHOD BY WHICH

CARDAN's RULES

FOR RESOLVING

The Cubick Equation $x^3 + qx = r$ in all Cases (or in all Magnitudes of the known Quantities q and r)

And the Cubick Equation $x^3 - qx = r$ in the first Case of it (or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$)

were probably discovered by Scipio Ferreus *of* Bononia, *and* Nicholas Tartalea, *or whoever else were the first Inventors of them.*

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

N. B. This Tract was first published in the Philosophical Transactions for the year 1780.

A

CONJECTURE

CONCERNING

CARDAN's RULES.

Art. 1. THERE is nothing more amufing, or more grateful to an inquifitive mind, in the ftudy of the fciences of Geometry or Algebra (for, if we banifh from it the ridiculous myfteries arifing from the fuppofition of *negative* quantities, or quantities *lefs than nothing*, the latter may deferve the name of a *fcience* as well as the former), than to contemplate the methods by which the feveral ingenious and furprifing truths that are delivered in the books that treat of them were firft difcovered. This we are fometimes enabled to do by the authors themfelves to whom we are indebted for thefe difcoveries, who have candidly informed their readers of the feveral fteps, and fometimes of the accidents, by which they have been led to them: but it alfo often happens, that the authors of thefe difcoveries have neglected to give their readers this fatisfaction, and have contented themfelves with either barely delivering the propofitions they have found out, without any demonftrations, or with giving formal and pofitive demonftrations of them, which command indeed the affent of the underftanding to their truth, but afford no clue whereby to difcover the train of reafoning by which they were firft found out; and confequently contribute but little to enable the reader to make fimilar difcoveries himfelf on the like fubjects. This feems to be the cafe with thofe ingenious rules for the refolution of certain cubick equations, which are ufually known by the name of CARDAN's Rules. We are told to make certain fubftitutions of fome quantities for others in thefe equations $x^3 + qx = r$ and $x^3 - qx = r$ (which are the objects of thofe rules), and certain fuppofitions concerning the quantities fo fubftituted; by doing which, we find that thofe equations will be transformed into other equations which will involve the fixth power of the unknown quantity contained in them, but which (though of double the dimenfions of the original equations $x^3 + qx = r$ and $x^3 - qx = r$, from which they were derived) will be more eafy to refolve than thofe equations, becaufe they will contain only the fixth power and the cube of the unknown quantity which is their root, and confequently will be of the fame form as quadratick equations; fo that by refolving them as quadratick equations we may obtain the value of the cube of the unknown quantity which is their root; and afterwards, by extracting the cube-root of the faid value, we

 may

may obtain the value of the ſaid root, or unknown quantity, itſelf; and then, at laſt, by the relation of this laſt root to x, or the root of the original equation (which relation is derived from the ſuppoſitions that have been made in the courſe of the preceeding transformations), we may determine the value of x. And, if we pleaſe to examine the ſeveral ſteps of this proceſs with ſufficient attention, we may perceive, as we go along, that all theſe ſubſtitutions are legitimate and practicable, or are founded upon poſſible ſuppoſitions; though I cannot but obſerve, that the writers on algebra, for the moſt part, have not been ſo kind as to ſhew us that they are ſo. But ſtill the queſtion recurs, "How came "SCIPIO FERREUS, of Bononia (who, as CARDAN tells us, was the firſt in-" ventor of the former of theſe rules, and NICHOLAS TARTALEA, who was the "firſt inventor of the latter of them), or the other perſons, whoever they were, "that invented them, to think of making theſe lucky ſubſtitutions which thus "transform the original cubick equations into equations of the ſixth power, "which contain only the ſixth and third powers of the unknown quantities "which are their roots, and conſequently are of the form of quadratick equa-"tions?" To anſwer this queſtion as well as I can *by conjecture* (for I know of no hiſtorical account of this matter in any book of algebra), and in a manner that appears to me to be *probable*, is the deſign of the following pages.

Art. 2. The moſt probable conjecture concerning the invention of theſe rules, called CARDAN's Rules, by SCIPIO FERREUS, of Bononia, and by NICHOLAS TARTALEA, or whoever elſe were the inventors of them, ſeems to be this: that the ſaid inventors tried a great variety of methods of reducing the three cubick equations of the third claſs, to wit, $x^3 + qx = r$ and $x^3 - qx \doteq r$, and $qx - x^3 = r$ (to ſome one of which all other cubick equations may, by proper ſubſtitutions, be reduced), to a lower degree, or to a more ſimple form, by ſubſtituting various quantities in the ſtead of x, in hopes that ſome of the terms ariſing by ſuch ſubſtitutions might be equal to others of them, and, having contrary ſigns prefixed to them, might deſtroy them, and thereby render the new equation more ſimple and manageable than the old one. And, amongſt other trials, it ſeems natural to imagine, that they would ſubſtitute the ſum or difference of two other quantities inſtead of x, as being the moſt ſimple and obvious ſubſtitutions that could be made. And, by making theſe ſubſtitutions, the above-mentioned rules would of courſe come to be diſcovered, as well as the aforeſaid limitation of them in the reſolution of the equation $x^3 - qx = r$ (which reſtrains the rule to thoſe caſes only in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$), and their utter inutility in all the caſes of the equation $qx - x^3 = r$. This will appear by examining each of theſe equations ſeparately in the following manner.

Of the equation $x^3 + qx = r$.

Art. 3. In the equation $x^3 + qx = r$ the inveſtigator of theſe rules would naturally be inclined to ſubſtitute the difference of two quantities (which we will here

here call y and z, and of which we will ſuppoſe y to be the greater) inſtead of x, rather than their ſum, or would ſuppoſe x to be equal to $y - z$, rather than to $y + z$; becauſe, if he was to ſuppoſe x to be equal to the ſum of the two quantities y and z, and was to ſubſtitute that ſum, or the binomial quantity $y + z$, inſtead of x in the equation $x^3 + qx = r$, it is evident that (as the ſigns of x^3 and qx are, both of them, affirmative) the terms of the new equation, ariſing from ſuch ſubſtitution, would all of them be likewiſe affirmative; and conſequently none of them, though they ſhould happen to be exactly equal to each other, could exterminate each other, and thereby render the new equation more ſimple than the old one, which was the only view with which the ſubſtitution would have been made. He would therefore ſuppoſe x to be equal to $y - z$; and by ſubſtituting this quantity inſtead of x in the original equation $x^3 + qx = r$, he would transform that equation into the following one, to wit,

$$y^3 - 3yyz + 3yzz - z^3 + qy - qz = r,$$
$$\text{or } y^3 - 3yz \times \overline{y - z} - z^3 + q \times \overline{y - z} = r.$$

Now in this equation it is evident that the terms $3yz \times \overline{y - z}$ and $q \times \overline{y - z}$ have contrary ſigns; and therefore, if their co-efficients $3yz$ and q can be ſuppoſed to be equal to each other, thoſe terms will mutually deſtroy each other, and the equation will be reduced to the following ſhort one, $y^3 - z^3 = r$. And, if in this equation we ſubſtitute, inſtead of z, its value $\frac{q}{3y}$, derived from the ſame ſuppoſition of the equality of q and $3yz$, the equation will be $y^3 - \frac{q^3}{27y^3} = r$; and, by multiplying both ſides by y^3, it will be $y^6 - \frac{q^3}{27} = ry^3$; and, by adding $\frac{q^3}{27}$ to both ſides, it will be $y^6 = ry^3 + \frac{q^3}{27}$; and, by ſubtracting ry^3 from both ſides, it will be $y^6 - ry^3 = \frac{q^3}{27}$; which equation, though it riſes to the ſixth power of the unknown quantity y, is evidently of the form of a quadratick equation, and may therefore be reſolved, ſo far as to find the value of the cube of y, in the ſame manner as a quadratick equation; after which it will be poſſible to find the value of y itſelf by the mere extraction of the cube root; and then at laſt, from the relation of y to x (derived from the foregoing ſuppoſitions that $y - z$ was equal to x, and that $3yz$ was equal to q, and conſequently z equal to $\frac{q}{3y}$), we ſhall be able to determine the value of x.

Art. 4. It would therefore remain for the inveſtigator of this method to enquire, whether or no the ſuppoſition "that $3yz$ was equal to q," was a poſſible ſuppoſition; that is, whether it was poſſible (whatever might be the magnitudes of q and r) for two quantities, y and z, to exiſt, whoſe nature ſhould be ſuch that their difference $y - z$ ſhould be equal to the unknown quantity x in the equation $x^3 + qx = r$, and that three times their product ſhould at the ſame time be equal to q, or their ſimple product to the third part of q. And this ſuppoſition he would ſoon find to be always poſſible, whatever may be the magnitudes of q and r; becauſe, if the leſſer quantity z is ſuppoſed to increaſe from *o ad infinitum*, and the greater quantity y is likewiſe ſuppoſed to increaſe with equal ſwiftneſs,

nefs, or to receive equal increments in the fame times, and thereby to preferve their difference $y - z$ always of the fame magnitude, or equal to x, it is evident that the product or rectangle yz will increafe continually at the fame time from *o ad infinitum*, and confequently will pafs fucceffively through all degrees of magnitude, and therefore muft at one point of time during its increafe become equal to $\frac{q}{3}$.

And having thus found this fuppofition of the equality of yz and $\frac{q}{3}$, or of $3yz$ and q, to be always poffible, whatever might be the magnitudes of q and r, our inveftigator would juftly confider his folution of the equation $x^3 + qx = r$ (which was founded on that fuppofition) as legitimate and complete. And thus we fee in what manner it feems probable, that CARDAN's rule for refolving the cubick equation $x^3 + qx = r$ may have been difcovered.

Of the equation $x^3 - qx = r$.

Art. 5. In this fecond equation $x^3 - qx = r$, in which the fecond term qx is fubtracted from the firft, or marked with the fign —, it feems to have been natural for the perfon who invented thefe rules to fubftitute the *fum* as well as the *difference* of two other quantities, y and z, inftead of x, in the terms x^3 and qx, in hopes of fuch an extermination of equal terms, and confequential reduction of the equation to one of a fimpler and more manageable form, as was found to be fo ufeful in the cafe of the former equation $x^3 + qx = r$. We will therefore try both thefe fubftitutions; and, as that of the difference $y - z$ has in the former cafe proved fo fuccefsful, we will begin by that.

Art. 6. Now, by fubftituting the difference $y - z$ inftead of x in the equation $x^3 - qx = r$, we fhall transform it into the following equation, to wit, $y^3 - 3yyz + 3yzz - z^3 - q \times \overline{y - z} = r$, or $y^3 - 3yz \times \overline{y - z} - z^3 - q \times \overline{y - z} = r$; in which the terms $3yz \times \overline{y - z}$ and $q \times \overline{y - z}$ have both of them the fame fign — prefixed to them, and confequently can never exterminate each other, whether $3yz$ be equal or unequal to q. This fubftitution therefore is in this cafe of no ufe.

Art. 7. We will now therefore try the fubftitution of the *fum* of y and z, inftead of their difference, in the equation $x^3 - qx = r$.

Now, if x be fuppofed to be equal to $y + z$, and $y + z$ be fubftituted inftead of it in the equation $x^3 - qx = r$, that equation will be thereby transformed into the following one, to wit,

$$y^3 + 3yyz + 3yzz + z^3 - q \times \overline{y + z} = r,$$

$$\text{or } y^3 + 3yz \times \overline{y + z} + z^3 - q \times \overline{y + z} = r.$$

Now in this equation, the terms $3yz \times \overline{y + z}$ and $q \times \overline{y + z}$ have contrary figns. Confequently, if they can be fuppofed to be equal to each other, they will deftroy each other, and the equation will be thereby reduced to the following

ing short one, $y^3 + z^3 = r$; that is, if $3yz$ and q can be supposed to be equal to each other, or if yz can be supposed to be equal to $\frac{q}{3}$, the equation will be reduced to the short equation $y^3 + z^3 = r$. And, if in this short equation we substitute instead of z its value $\frac{q}{3y}$ (derived from the same supposition of the equality of $3yz$ and q), the equation thence resulting will be $y^3 + \frac{q^3}{27y^3} = r$; and by multiplying both sides by y^3, it will be $y^6 + \frac{q^3}{27} = ry^3$; and, by subtracting y^6 from both sides, it will be $ry^3 - y^6 = \frac{q^3}{27}$; which, though it rises to the sixth power of y, is evidently of the form of a quadratick equation, and consequently may be resolved in the same manner as a quadratick equation, so far as to find the value of y^3, or the cube of the root y; after which it will be possible to find the value of y itself by the mere extraction of the cube root; and, lastly, from the relation of y to x (contained in the two suppositions, that $y + z$ is equal to x, and that $3yz$ is equal to q, and consequently that z is equal $\frac{q}{3y}$) we may determine the value of x.

Art. 8. The only thing, therefore, that would remain for the investigator of these rules to do, in order to know whether the foregoing method of resolving the equation $x^3 - qx = r$ was practicable or not, would be to enquire, whether it was possible in all cases, that is, in all magnitudes of the known quantities q and r, for $3yz$ to be equal to q, or for yz (or the product, or rectangle, of the two quantities y and z, whose sum is equal to x) to be equal to $\frac{q}{3}$; and, if it was not possible in all cases, but only in some, to determine in what cases it *was* possible, or what must be the relation between q and r to make it possible.

Art. 9. Now, in order to determine this question, it would be proper and natural to observe, that the quantity yz, or the product of the multiplication of the two quantities y and z, whose sum is supposed to be equal to x, can never be greater than the square of half that sum, that is, than the square of $\frac{x}{2}$, or than $\frac{xx}{4}$, by El. 2, 5, but may be of any magnitude that does not exceed that square. Therefore, if $\frac{q}{3}$ is greater than $\frac{xx}{4}$, it will be impossible for yz to be equal to it; but, if $\frac{q}{3}$ is either equal to, or less than, $\frac{xx}{4}$, it will be possible for yx to be equal to it; and, if $\frac{q}{3}$ is exactly equal to $\frac{xx}{4}$, z will be exactly equal to y, and each of them equal to one half of x. We must therefore enquire what is the magnitude of x when $\frac{q}{3}$ is equal to $\frac{xx}{4}$. Now, when $\frac{xx}{4}$ is $= \frac{q}{3}$, xx will be $= \frac{4q}{3}$, and $x = \frac{2\sqrt{q}}{\sqrt{3}}$: therefore, when x is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be impossible for

y z

yz to be equal to $\frac{q}{3}$; but when x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be poſſible for yz to be equal to $\frac{q}{3}$.

But when x is $= \frac{2\sqrt{q}}{\sqrt{3}}$, x^3 will be $= \frac{8q\sqrt{q}}{3\sqrt{3}}$, and qx will be $= \frac{2q\sqrt{q}}{\sqrt{3}}$ or $\frac{6q\sqrt{q}}{3\sqrt{3}}$, and conſequently $x^3 - qx$ will be $= \frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{6q\sqrt{q}}{3\sqrt{3}} = \frac{2q\sqrt{q}}{3\sqrt{3}}$.

Therefore, if it be true (as we ſhall preſently ſee that it is) that, while x increaſes from being equal to $\sqrt{q}$ (which is evidently its leaſt poſſible magnitude) to any other magnitude, the compound quantity $x^3 - qx$, or the exceſs of x^3 above qx, will alſo continually increaſe from o (to which it is equal when x is $= \sqrt{q}$, or xx is $= q$) to ſome correſpondent magnitude without ever decreaſing; it will follow that, when x is leſs than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity $x^3 - qx$ will be leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and when x is greater than $\frac{2\sqrt{q}}{\sqrt{q}}$, the compound quantity $x^3 - qx$ will be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and, *è converſo*, if the compound quantity $x^3 - qx$ is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be leſs than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if the compound quantity $x^3 - qx$ is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$. Conſequently, if the compound quantity $x^3 - qx$, or, its equal, the abſolute term r in the equation $x^3 - qx = r$, is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, it will be impoſſible for yz to be equal to $\frac{q}{3}$; but if $x^3 - qx$, or r, is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be poſſible for yz to be equal to $\frac{q}{3}$. Therefore, if $x^3 - qx$, or r, is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, the foregoing method of reſolving the cubick equation $x^3 - qx = r$ will be impracticable; but, if $x^3 - qx = r$, or r, is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be practicable.

Art. 10. It now only remains to be proved, that while x increaſes, from being equal to $\sqrt{q}$, *ad infinitum*, the compound quantity $x^3 - qx$ will likewiſe increaſe from o *ad infinitum*, without ever decreaſing. Now this may be demonſtrated as follows.

Art. 11. It is evident that while x increaſes from being equal to $\sqrt{q}$ *ad infinitum*, both the quantities x^3 and qx will increaſe *ad infinitum* likewiſe. But it does not therefore follow, that the exceſs of x^3 above qx will continually increaſe at the ſame time. This will depend upon the relation of the contemporary increments of x^3 and qx: if the increment of x^3 in any given time is equal to the contemporary increment of qx, the compound quantity $x^3 - qx$ will neither increaſe nor decreaſe, but continue always of the ſame magnitude during the ſaid time, notwithſtanding the increaſe of x; if the former increment is leſs than the

latter, the ſaid compound quantity will decreaſe; and if it is greater, it will increaſe. We muſt therefore enquire whether the increment of x^3 in any given time is greater or leſs than the contemporary increment of qx.

Art. 12. Now, if $\dot{x}$ be put for the increment which x receives in any given time, the increment of x^3 in the ſame time will be the exceſs of $\overline{x + \dot{x}}|^3$ above x^3, that is, the exceſs of $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ above x^3; and the increment of qx in the ſame time will be the exceſs of $q \times \overline{x + \dot{x}}$, or $qx + q\dot{x}$, above qx; that is, the increment of x^3 will be $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, and that of qx will be $q\dot{x}$. Now in the equation $x^3 - qx = r$ it is evident that xx muſt be greater than q; for otherwiſe x^3 would not be greater than qx, as it is ſuppoſed to be. Conſequently, $xx \times \dot{x}$ muſt be greater than $q\dot{x}$; and, *à fortiori*, $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ (which is more than triple of $x^2\dot{x}$) muſt be greater than $q\dot{x}$; that is, the increment of x^3 will be greater than the contemporary increment of qx. Therefore the exceſs of x^3 above qx, or the compound quantity $x^3 - qx$, will increaſe continually, without decreaſing, while x increaſes from $\sqrt{q}$ *ad infinitum*. Q. E. D.

Art. 13. It follows, therefore, upon the whole of theſe enquiries, that, if the compound quantity $x^3 - qx$, or, its equal, the abſolute term r, is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, it will be impoſſible for yz to be equal to $\frac{q}{3}$, and conſequently the foregoing method of reſolving the equation $x^3 - qx = r$ will be impracticable; but, if $x^3 - qx$, or r, is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be poſſible for yz to be equal to $\frac{q}{3}$, and conſequently, the foregoing method of reſolving the equation $x^3 - qx = r$ will be practicable. And thus we ſee in what manner it is probable that CARDAN's rule for reſolving the cubick equation $x^3 - qx = r$ in the firſt caſe of it, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, together with the reſtriction of it to that firſt caſe, may have been diſcovered.

Of the Equation $qx - x^3 = r$.

Art. 14. In the third equation $qx - x^3 = r$ the terms x^3 and qx have different ſigns, as well as in the ſecond equation $x^3 - qx = r$; and therefore it ſeems to have been natural for the inventor of CARDAN's rules to try both the ſubſtitutions of $y - z$ and $y + z$ inſtead of x in this equation, as well as in that ſecond equation, in hopes of an extermination of equal terms that are marked with contrary ſigns, and a conſequent reduction of the equation to another which, though of double the dimenſions of the equation $qx - x^3 = r$, ſhould have been of a ſimpler form, and more eaſy to be reſolved. But it will be found, upon trial, that neither of theſe ſubſtitutions will anſwer the end propoſed.

Art. 15. For, in the firſt place, let us ſuppoſe x to be $= y - z$. Then we ſhall have $x^3 = y^3 - 3yyz + 3yyz - z^3 = y^3 - 3yz \times \overline{y - z} - z^3$, and $qx = q \times \overline{y - z}$ and conſequently $qx - x^3 = q \times \overline{y - z} - y^3 + 3yz \times \overline{y - z} + z^3$. Therefore, $q \times \overline{y - z} - y^3 + 3yz \times \overline{y - z} + z^3$ will be $= r$. Now in this equation it is evident, the terms $q \times \overline{y - z}$ and $3yz \times \overline{y - z}$ have the ſame ſigns, and therefore can never deſtroy each other. Therefore, no ſuch method of reſolving this equation $qx - x^3 = r$ as was found above for reſolving the two former equations $x^3 + qx = r$ and $x^3 - qx = r$, can be obtained by ſubſtituting the difference $y - z$ in it inſtead of x.

Art. 16. We will now try the ſubſtitution of $y + z$ inſtead of x in the terms of this equation.

Now, if x be ſuppoſed to be $= y + z$, we ſhall have $x^3 = y^3 + 3yyz + 3yzz + z^3 = y^3 + 3yz \times \overline{y + z} + z^3$, and $qx = q \times \overline{y + z}$, and conſequently $qx - x^3 = q \times \overline{y + z} - y^3 - 3yz \times \overline{y + z} - z^3$: Therefore, $q \times \overline{y + z} - y^3 - 3yz \times \overline{y + z} - z^3$ will be $= r$.

In this equation it is true indeed that the terms $q \times \overline{y + z}$ and $3yz \times \overline{y - z}$ have different ſigns. But they cannot be equal to each other: for, ſince the three terms y^3 and $3yz \times \overline{y - z}$ and z^3 are all marked with the ſign —, or are to be ſubtracted from the firſt term $q \times \overline{y + z}$, and the remainder is $= r$, it is evident that $q \times \overline{y + z}$ muſt be greater than the ſum of all the three terms y^3, $3yz \times \overline{y + z}$, and z^3, taken together, and therefore, *à fortiori*, greater than $3yz \times \overline{y + z}$ alone. Therefore, no ſuch extermination of equal terms marked with contrary ſigns as took place in the transformed equations derived from the two former equations $x^3 + qx = r$ and $x^3 - qx = r$, can take place in this transformed equation derived from the equation $qx - x^3 = r$ by ſubſtituting $y + z$ in its terms inſtead of x; and conſequently no ſuch method of reſolving the equation $qx - x^3 = r$ as has been found for the reſolution of the equations $x^3 + qx = r$ and $x^3 - qx = r$, can be obtained by means of that ſubſtitution.

Art. 17. Theſe are the methods of inveſtigation by which I conceive it to be probable that CARDAN's rules for the reſolution of the cubick equations $x^3 + qx = r$ and $x^3 - qx = r$, together with the limitation of the rule relating to the latter of thoſe equations, and their inapplicability to the third equation $qx - x^3 = r$, may have been diſcovered by the firſt inventors of them.

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A P P E N D I X

TO THE

T R A C T

CONTAINED IN THE FOREGOING PART OF THIS SECOND VOLUME OF MATHEMATICAL TRACTS, IN PAGES 153, 154, 155, &c, TO 169;

INTITLED,

"A DEMONSTRATION OF SIR ISAAC NEWTON'S BINOMIAL THEOREM IN THE CASE OF INTEGRAL POWERS, OR POWERS OF WHICH THE INDEXES ARE WHOLE NUMBERS:"

Containing an Investigation of the Law by which the co-efficients of the third and fourth and other following terms of the series which is equal to any integral power of a binomial quantity, are derived from the co-efficient of the second term of the said series, grounded on a probable induction from particular examples.

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

Art. 1. IT is shewn in art. 5 of the foregoing tract, pages 155, 156, that, in all integral powers whatsoever of the binomial quantity $a + b$, the literal parts of the terms of the series which is equal to $\overline{a+b}|^m$ (in which the letter m denotes any whole number whatsoever), will always be a^m, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, &c, of which every term is generated from the next before it by the multiplication of the fraction $\frac{b}{a}$. And it is also shewn in art. 6 of the said tract, page 156, that the numeral co-efficient of the first term of the series that is equal to $\overline{a+b}|^m$ must always be 1, or that the first term of the said series will always be a^m, and that the numeral co-efficient of the second term of the said series will always be m, or the index of the power to which $a + b$ is raised, or that the second term of the said series will always be $m \times a^{m-1}b$, to whatever whole number the letter m may be supposed to be equal.

Art. 2. And it is further shewn in the said tract that the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the series which is

equal

equal to $\overline{a+b}|^m$ may always be derived from m, the co-efficient of the fecond term of the faid feries, by the continual multiplication of the following fractions, to wit, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, till we come to the fraction $\frac{m-m}{m+1}$, which is $= 0$, or till the faid feries of fractions is terminated or exhaufted; which fractions, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, are therefore called *the generating fractions* of the co-efficients of the third and other following terms of the feries which is equal to $\overline{a+b}|^m$.

Art. 3. And the method by which it is fhewn in the faid tract that the faid fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, are in all cafes, or when m is fuppofed to reprefent any whole number whatfoever, the generating fractions of the co-efficients of the terms of the feries that is equal to $\overline{a+b}|^m$, or the fractions by which the co-efficients of the third and fourth, and other following terms of the faid feries, are derived from m, the co-efficient of the fecond term $m \times a^{m-1}b$, confifts of the three following parts; to wit, firft, of a demonftration that, if it be true that thefe are the generating fractions of the co-efficients of the third and other following terms of the feries that is equal to $\overline{a+b}|^m$ when m is equal to *any one* whole number whatfoever, it will alfo be true that they will be the generating fractions of the co-efficients of the third and other following terms of the feries that is equal to $\overline{a+b}|^m$ when m is equal to any other whole number greater than the former; and, fecondly, of a proof, by actual trials of the co-efficients of the terms of the feveral feriefes that are equal to $\overline{a+b}|^2$, $\overline{a+b}|^3$, $\overline{a+b}|^4$, and $\overline{a+b}|^5$, that in thefe four feriefes, or when m is equal either to the number 2, or the number 3, or the number 4, or the number 5, the faid fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, and $\frac{m-5}{6}$, are the generating fractions by which the co-efficients of the third and other following terms of the faid feriefes (that are equal to $\overline{a+b}|^2$, $\overline{a+b}|^3$, $\overline{a+b}|^4$, and $\overline{a+b}|^5$) are derived from the co-efficients of the fecond terms of the faid feriefes refpectively; and, thirdly, of a conclufion evidently following from the former two propofitions, to wit, that, to whatever whole number the index m be fuppofed to be equal, it will always be true that the faid fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, will be the generating fractions by the continual multiplication of which the co-efficients of the third and other following terms of the feries that is equal to $\overline{a+b}|^m$ will be derived from m, the co-efficient of the fecond term. Thefe reafonings I take to be juft and clear, and fuch as muft give every reader full fatisfaction as to the truth of the propofition, or conclufion, obtained by means of them.

Art. 4.

Art. 4. But it may be aſked, " How came it to be ſuſpected that the fractions " $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, were the generating fractions by which the " co-efficients of the third and other following terms of the ſeries that is equal " to $\overline{a+b}|^m$ are derived from m, the co-efficient of the ſecond term, in any " one value of the index m, ſince it is by no means apparent from the mere " inſpection of the terms of the ſerieſes that are equal to $\overline{a+b}|^m$ when m is " equal to the ſmall numbers 2, 3, 4, and 5?"

This is a very natural and reaſonable queſtion, and well deſerves to be conſidered; more eſpecially if we recollect that Dr. Wallis informs us that he had ſought for theſe generating fractions without being able to diſcover them. And till a perſon had firſt ſuſpected, and then found upon trial, that theſe are the generating fractions of the co-efficients of the terms of the ſeries that is equal to $\overline{a+b}|^m$ in ſome of the lower values of m, he could never think of ſhewing, in the method above deſcribed, that the ſame generating fractions would enable us to find the co-efficients of the terms of the like ſerieſes in all other integral values of m.

Art. 5. Now, in anſwer to this queſtion it may be obſerved, that theſe fractions will occur to our notice as the generating fractions of the co-efficients of the third and other following terms of the ſerieſes that are equal to $\overline{a+b}|^m$ in ſome of the lower values of m, if we divide the ſeveral ſucceſſive terms of theſe ſerieſes by the terms next before them, in order to diſcover the generating fractions by which they are derived one from another, and then reduce the generating fractions ſo obtained to their loweſt denominations. Thus, for example, the ſixth power of the binomial quantity $a+b$ is $= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; of which terms if we divide the ſecond term $6a^5b$ by the firſt term a^6, the quotient will be $\frac{6a^5b}{a^6}$, or $6 \times \frac{b}{a}$; and, if we divide in like manner the third term by the ſecond, and the fourth term by the third, and the fifth term by the fourth, and the ſixth term by the fifth, and the ſeventh, or laſt, term by the ſixth, the quotients will be $\frac{15a^4b^2}{6a^5b}$, or $\frac{15}{6} \times \frac{b}{a}$, and $\frac{20a^3b^3}{15a^4b^2}$, or $\frac{20}{15} \times \frac{b}{a}$, and $\frac{15a^2b^4}{20a^3b^3}$, or $\frac{15}{20} \times \frac{b}{a}$, and $\frac{6ab^5}{15a^2b^4}$, or $\frac{6}{15} \times \frac{b}{a}$, and $\frac{b^6}{6ab^5}$, or $\frac{1}{6} \times \frac{b}{a}$. And conſequently, if we multiply the firſt term a^6 by the quotient $6 \times \frac{b}{a}$, we ſhall thereby produce the ſecond term $6a^5b$; and, if we multiply the ſecond term $6a^5b$ by $\frac{15}{6} \times \frac{b}{a}$, we ſhall thereby produce the third term $15a^4b^2$; and, if we multiply the third term $15a^4b^2$ by $\frac{20}{15} \times \frac{b}{a}$, we ſhall thereby produce the fourth term $20a^3b^3$; and, if we multiply the fourth term $20a^3b^3$ by $\frac{15}{20} \times \frac{b}{a}$, we ſhall thereby produce the fifth term $15a^2b^4$; and, if

if we multiply the fifth term $15a^2b^4$ by $\frac{6}{15} \times \frac{b}{a}$, we ſhall thereby produce the ſixth term $6ab^5$; and, if we multiply the ſixth term $6ab^5$ by $\frac{1}{6} \times \frac{b}{a}$, we ſhall thereby produce the ſeventh, or laſt, term b^6. Therefore the generating fractions, by the continual multiplication of which the ſecond, and other following terms of the ſeries $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$ (which is equal to $\overline{a+b}|^6$) are derived from the firſt term a^6, are $\frac{6}{1} \times \frac{b}{a}$, $\frac{15}{6} \times \frac{b}{a}$, $\frac{20}{15} \times \frac{b}{a}$, $\frac{15}{20} \times \frac{b}{a}$, $\frac{6}{15} \times \frac{b}{a}$, and $\frac{1}{6} \times \frac{b}{a}$; and conſequently the generating fractions, by the continual multiplication of which the numeral co-efficients of the ſecond and other following terms of the ſaid ſeries $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$ (independently of the literal parts of the ſaid terms) are derived from 1, the numeral co-efficient of the firſt term a^6, will be $\frac{6}{1}$, $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$. Now let theſe co-efficients be reduced to their loweſt terms; and they will then be $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{1}{6}$, of which the numerator of the firſt term $\frac{6}{1}$ is the index 6 of the power of the binomial quantity to which the ſaid ſeries of terms is equal, and the numerators of the following fractions $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{1}{6}$ are derived from the ſaid index 6 by the continual ſubtraction of 1, and the denominators of the ſaid fractions are the natural numbers 1, 2, 3, 4, 5, 6, which begin with an unit, and increaſe by the continual addition of 1. This obſervation on the increaſe and decreaſe of the denominators and numerators of the fractions $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$, and their derivation from the index 6, or $\frac{6}{1}$, in the caſe of the ſeries which is equal to $\overline{a+b}|^6$, is ſufficient to have induced the perſon who ſhould have made it, to conjecture, that poſſibly, when the index m was equal to any other number (ſuch as 5, or 4, or 7, or 8), the generating fractions whereby the numeral co-efficients of the third and fourth and other following terms of the ſeries that was equal to $\overline{a+b}|^m$, were derived from the numeral co-efficient m of the ſecond term, and from each other, might likewiſe, when properly reduced, be found to conſiſt of numerators and denominators that did in like manner decreaſe from the index m by the continual ſubtraction of an unit, and increaſe from 1 by the continual addition of an unit; or, in other words, might be equal to $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c. And this conjecture might have produced a trial whether this rule took place in ſome particular examples, and more eſpecially in the ſerieſes that were equal to $\overline{a+b}|^2$, $\overline{a+b}|^3$, $\overline{a+b}|^4$, $\overline{a+b}|^5$, $\overline{a+b}|^6$, $\overline{a+b}|^7$, and $\overline{a+b}|^8$, and perhaps a few more of the lower integral powers of $a+b$; after which trials, and the ſucceſs that would have attended them, it would have become ſo highly probable that the ſame rule would take

take place in the feriefes that were equal to any other integral powers of $a + b$, that it would have been almoft impoffible to doubt of it. And then it would have been natural to endeavour to find fome general demonftration of the truth of the rule in all integral powers of the binomial quantity $a + b$ whatfoever, which might have led to fuch a demonftration as that which is given in the tract above-mentioned, which is contained in pages 153, 154, 155, &c .... 169, of this volume.

N. B. This method of difcovering (by a conjecture grounded on a trial or two, in fome particular examples), that the generating fractions by which the co-efficients of the third and fourth and other following terms of the feries that is equal to $\overline{a + b}|^m$ (or any integral power of the binomial quantity $a + b$) are derived from m (the index of the power to which the faid binomial quantity is raifed) or from the co-efficient of the fecond term of the faid feries (which is equal to the faid index) are $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, is fuggefted by profeffor Saunderfon, in the fecond volume of his Algebra, in the chapter on the binomial theorem, where the reader will find a good explanation and illuftration of the faid celebrated theorem, by a variety of examples, both in the cafe of integral powers and in the cafe of roots, and other fractional powers, and even in the cafe of negative powers, and of powers that are both fractional and negative; but no demonftration of it in any cafe, not even in that of integral and affirmative powers.

F I N I S.